Journal of Mathematical Extension
Vol. 15, No. 4, (2021) (17)1-15
URL: https://doi.org/10.30495/JME.2021.1660
ISSN: 1735-8299
Original Research Paper

ON Class of Subalgebras of Bounded BCK-Algebras

H. HARIZAVI
Shahid Chamran University of Ahvaz

Abstract. In this paper, for any two elements y, u of a BCK-algebra X, we assign a subset of X, denoted by $S_y(u)$, and investigate some related properties. We show that $S_y(u)$ is a subalgebra of X for all $y, u \in X$. Using these subalgebras, we characterize the involutive BCK-algebras, and give a necessary and sufficient condition for a bounded BCK-algebra to be a commutative BCK-chain. Finally, we show that the set of all subalgebras $S_y(u)$ forms a bounded distributive lattice.

AMS Subject Classification: 03G10; 03G25
Keywords and Phrases: BCK-algebra, commutative BCK-chain, implicative BCK-algebra. bounded distributive lattice

1 Introduction

The notion of I-algebras was introduced as a generalization of set-theoretic difference and propositional calculi in [1]. In the same year, the BCK-algebras as a generalization of I-algebras; and the BCI-algebras as a generalization of BCK-algebras were introduced in [2]. These algebras are two important classes of logical algebras. Commutative BCK-algebras are an important class of BCK-algebras, which forms a class of the lower semilattice [8, 9]. Other important types of BCK-algebras

Received: April 2020; Accepted: March 2021

are implicative and positive implicative which introduced by K. Iseki (1975). It is proved that a BCK-algebra is implicative and if and only if it is commutative and positive implicative. The concept of an ideal in a BCK-algebra was introduced in [5, 4]. One important type of ideals is commutative, which has a close relationship with commutative BCK-algebras, in the sense that a BCK-algebra X is commutative if and only if every ideal of X is commutative [6].

It is well known that every initial set of a BCK-algebra is a subalgebra. But a subalgebra is not necessarily an initial set. In this paper, we introduce and study a new kind of subalgebras different from the initial sets. For this purpose, we assign a subset of X, denoted by $S_y(u)$, for any two elements y, u of a BCK-algebra X and investigate some related properties. We show that $S_y(u)$ is a subalgebra of X for all $y, u \in X$. Also, in a commutative BCK-algebra, we give a necessary and sufficient condition for $S_y(u)$ to be an ideal. Moreover, we prove that a bounded BCK-algebra X is a commutative BCK-chain if and only if every $S_y(u)$ is an initial set of X. We show that the set of all such subsets forms a bounded distributive lattice. Finally, assuming L(y,X) denote the set of all $S_y(u)$ where $u \in X$, we prove that $S_y(u)$ is the least element of L(y,X) with property $A(u) \subseteq S_y(u)$.

2 Preliminaries

In this section, we review some definitions and known results, which will be used in this paper. The reader is referred to [10, 7] for more details.

Definition 2.1. *By a BCK-algebra we mean an algebra* $(X; *, 0)$ *of type* $(2, 0)$ *satisfying the following axioms: for all* $x, y, z \in X$*,*
BCK-1: $((x * y) * (x * z)) * (z * y) = 0$,
BCK-2 : $(x * (x * y)) * y = 0$,
BCK-3 : $x * x = 0$,
BCK-4 : $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$,
BCK-5 : $0 * x = 0$.

For brevity, we often write X instead of $(X; *, 0)$ for a BCK-algebra. In any BCK-algebra X, one can define a partial order $\leq$ by putting $x \leq y$ if and only if $x * y = 0$.

In any BCK-algebra X, the followings hold:
(a_1) $x * 0 = x$,
(a_2) $x * y \leq x$,
(a_3) $(x * y) * z = (x * z) * y$,
(a_4) $x \leq y$ *implies* $x * z \leq y * z$ *and* $z * y \leq z * x$,
(a_5) $(x * z) * (y * z) \leq x * y$,
(a_6) $x * (x * (x * y)) = x * y$,
(a_7) $x * (x * y) \leq y$,
(a_8) $x * y \leq z \Leftrightarrow x * z \leq y$,
for any $x, y, z \in X$.

A subset A of a BCK-algebra X is called:
(i) *subalgebra* of X if it is closed under $*$, multiplication of X, i.e., $x * y \in A$ for all $x, y \in A$;
(ii) *ideal* of X if it satisfies (i) $0 \in A$ and (ii) $x, y * x \in A$ imply $y \in A$ for all $x, y \in X$.

A BCK-algebra X is called:
(i) *chain* if $x \leq y$ or $y \leq x$ for all $x, y \in X$;
(ii) *bounded* if it has the greatest element (denoted by 1). For any $x \in X$, we denote $1 * x$ by Nx;
(iii) *commutative* if it satisfies the condition: $x * (x * y) = y * (y * x)$ for all $x \in X$. In this case, $x * (x * y)$ (and $y * (y * x)$) is the greatest lower bound of x and y with respect to BCK-order $\leq$, and we denote it by $x \wedge y$;
(iv) *positive implicative* if it satisfies the condition: $(x * y) * z = (x * z) * (y * z)$;
(v) *implicative* if it satisfies the condition: $x * (y * x) = x$ for all $x, y \in X$.

Let X be a commutative BCK-algebra and $A \subseteq X$. Then the set

$$Ann(A) := \{x \in X | x \wedge a = 0 \text{ for all a} \in A\}$$

is called the *annihilator* of A.

Note that in a bounded BCK-algebra the property NNx=x is not true in general. An element $x \in X$ is called involution if it satisfies NNx=x, and a bounded BCK-algebra X is called involutive if every element $x \in X$ is involutive.

Theorem 2.2. *[10] Let X be a BCK-algebra. Then*

(i) X is commutative if and only if $x*(x*y) \leq y*(y*x)$ for all $x, y \in X$;

(ii) X is positive implicative if and only if $(x*y)*y = x*y$ for any $x, y \in X$;

(iii) X is implicative if and only if it is commutative and positive implicative.

A partial ordered set P is said to be *lattice* if for any two elements $x, y \in P$ there exist the greatest lower bound of x, y (denoted by $x \wedge y$) and the least upper bound of x, y (denoted by $x \vee y$).

A BCK-algebra X is called a BCK-lattice if it with respect to its BCK-ordering forms a lattice.

Theorem 2.3. *[7] In any bounded commutative BCK-algebra X, the followings hold: for all $x, y \in X$,*
(b_1) $NNx = x$,
(b_2) $Nx*Ny = y*x$,
(b_3) $Nx \vee Ny = N(x \wedge y)$ and $Nx \wedge Ny = N(x \vee y)$.

Theorem 2.4. *[7] Every bounded commutative BCK-algebra is a commutative BCK-lattice with $x \wedge y = y*(y*x)$ and $x \vee y = N(Nx \wedge Ny)$.*

Theorem 2.5. *[10] Let X be a commutative BCK-lattice. Then the following identities hold: for any $x, y, z \in X$,*
(c_1) $x*(y \vee z) = (x*y) \wedge (x*z)$,
(c_2) $x*(y \wedge z) = (x*y) \vee (x*z)$,
(c_3) $(x \vee y)*z = (x*z) \vee (y*z)$.

3 On Class of Subalgebras of BCK-algebras

In this section, we introduce the special subsets of bounded BCK-algebras and investigate some related properties.

Definition 3.1. *For any two elements y, u of a bounded BCK-algebra X, we denote*

$$S_y(u) := \{x \in X \mid (NNy)*u \leq y*x\}.$$

By (a_7), $0 \in S_y(u)$ for any $y, u \in X$.

The following proposition shows that $S_y(u)$ is a generalization of the annihilator.

Proposition 3.2. *If X is a bounded commutative BCK-algebra, then for all $y \in X$, $S_y(0) = Ann(y)$.*

Proof. Observe that: $x \in S_y(0) \Leftrightarrow NNy \leq y * x \Leftrightarrow y \leq y * x \Leftrightarrow y * x = y \Leftrightarrow y * (y * x) = 0 \Leftrightarrow x \wedge y = 0 \Leftrightarrow x \in Ann(y)$. □

Proposition 3.3. *Let X be a bounded BCK-algebra. Then the followings hold, for any $y, u \in X$*
(i) $0 \in S_y(u)$.
(ii) If $y \leq u$, then $S_y(u) = X$.
(iii) If in addition X is involutive, then $S_y(u) = X$ implies $y \leq u$.

Proof.(i) Let $y, u \in X$. Then, using $(a2)$ and (a_7), we get $NNy * u \leq NNy \leq y = y * 0$. This implies $0 \in S_y(u)$.

(ii) Let $y \leq u$. Then, by (a_4), $NNy \leq NNu$. But $NNu \leq u$. Hence $NNy \leq u$ and so $(NNy) * u = 0 \leq y * x$ for any $x \in X$. This implies $X \subseteq S_y(u)$ and so $S_y(u) = X$.

(iii) Let $S_y(u) = X$. Then $1 \in S_y(u)$ and so $(NNy) * u \leq y * 1 = 0$. Thus $(NNy) * u = 0$ and so, since X is involutive, $y * u = 0$, that is, $y \leq u$. □

The following example shows that the involutive condition in Proposition 3.3(iii) is necessary.

Example 3.4. Let $X = \{0, a, b, 1\}$. Define the operation $*$ on X by the following table:

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
1	1	a	a	0

Then X is a BCK-algebra, but it is not involutive because $NNb = 1 * (1 * b) = a$. Since $NNb * a = 0$, it follows that $S_b(a) = X$, but $b \not\leq a$.

Proposition 3.5. *Let X be a bounded BCK-algebra. Then the followings hold, for any $y, u, v \in X$*
(i) if $u \leq v$, then $S_y(u) \subseteq S_y(v)$.
(ii) if in addition X is commutative and $u, v \leq y$, then $S_y(u) \subseteq S_y(v)$ implies $u \leq v$.

Proof. (i) Let $u \leq v$. Then $(NNy) * v \leq (NNy) * u$. Now, assume that $x \in S_y(u)$. Then $(NNy) * u \leq y * x$ and so $(NNy) * v \leq y * x$. Hence $x \in S_y(v)$.

(ii) Let $S_y(u) \subseteq S_y(v)$. Then it follows from $u \in S_y(u)$ that $u \in S_y(v)$ and so $NNy*v \leq y*u$. But, since X is commutative, we have $NNy = y$. Thus $y * v \leq y * u$ and so by (a_4), we get $y * (y * u) \leq y * (y * v)$. Thus, using the commutatively of X, we obtain $y \wedge u \leq y \wedge v$ and so from $u, v \leq y$, we conclude $u \leq v$. □

The commutative property in Proposition 3.5 is necessary as shown in the following example.

Example 3.6. Let $X = \{0, a, b, c, 1\}$. Define the operation $*$ on X by the following table:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
1	1	1	1	1	0

Then $(X; *, 0)$ is a BCK-algebra, but it is not commutative because $1 * (1 * a) = 0 \neq a = a * (a * 1)$. By simple calculation, we have $S_c(b) = \{0, a, b\} \subseteq X = S_c(a)$ but $b \nleq a$.

The relationship between $S_y(u)$ and the initial set $A(u)$ is introduced in the following.

Proposition 3.7. *Let X be a bounded BCK-algebra and $y, u \in X$. Then the followings hold:*
(i) $A(u) \subseteq S_y(u)$, in which $A(u) = \{x \in X \mid x \leq u\}$.
(ii) $S_y(u) = A(z)$ if and only if z is the maximum of $S_y(u)$.

Proof. (i) Let $x \in A(u)$. Then $x \leq u$ and so $(NNy) * u \leq (NNy) * x \leq y * x$. Therefore $x \in S_y(u)$.

(ii) If $S_y(u) = A(z)$, then clearly the result holds.

Conversely, assume that $z \in S_y(u)$ is the maximum of $S_y(u)$. Then for all $x \in S_y(u)$, $x \leq z$. From this follows that $S_y(u) \subseteq A(z)$. Now, let $x \in A(z)$. Then $x \leq z$ and so $y * z \leq y * x$. On the other hand, from $z \in S_y(u)$, we have $(NNy) * u \leq y * z$. Thus $(NNy) * u \leq y * x$ which yields $x \in S_y(u)$. Therefore $A(z) \subseteq S_y(u)$ and so $S_y(u) = A(z)$. □

Note that $S_y(u)$ is not necessary to be contained in $A(u)$. Consider Example 3.6, routine calculations show that $S_c(a) = X \nsubseteq \{0, a\} = A(a)$.

Theorem 3.8. *Let X be a bounded BCK-algebra. Then $S_y(u)$ is a subalgebra of X for any $y, u \in X$.*

Proof. Obviously, $0 \in S_y(u)$. Let $x, z \in S_y(u)$. Then by (a_4), it follows from $x * z \leq x$ that

$$(NNy) * u \leq (NNy) * x \leq (NNy) * (x * z) \leq y * (x * z).$$

This implies that $x * z \in S_y(u)$. Therefore $S_y(u)$ is a subalgebra of X. □

Proposition 3.9. *Let X be a bounded BCK-algebra. Then the followings hold: for all $y, u, v \in X$,*
(i) $S_y(u * (u * v)) \subseteq S_y(u) \cap S_y(v)$.
(ii) If in addition X is commutative, then $S_y(u \wedge v) = S_y(u) \cap S_y(v)$.

Proof. (i) The proof is straightforward by using (a_7) and Proposition 3.5(i).

(ii) Since X is commutative, $u \wedge v = u * (u * v)$. Then by (i), we only need to prove that $S_y(u) \cap S_y(v) \subseteq S_y(u \wedge v)$. Let $x \in S_y(u) \cap S_y(v)$. From $x \in S_y(u)$, we get $(NNy) * u \leq y * x$. We note that, since X is commutative, we have $NNy = y$. Thus $y * u \leq y * x$ and so $y * (y * x) \leq y * (y * u)$, that is, $x \wedge y \leq u \wedge y$. Similarly, from $x \in S_y(v)$, we have $x \wedge y \leq v \wedge y$. Therefore $x \wedge y \leq (u \wedge y) \wedge (v \wedge y) = y \wedge (u \wedge v)$, that is, $y * (y * x) \leq y * (y * (u \wedge v))$. Thus, using (a_4), we conclude $y*(y*(y*(u\wedge v))) \leq y*(y*(y*x))$ and so, by (a_6), we get $y*(u\wedge v) \leq y*x$. From this follows that $NNy*(u\wedge v) \leq y*x$ and so $x \in S_y(u\wedge v)$. Hence $S_y(u) \cap S_y(v) \subseteq S_y(u \wedge v)$ and so the proof is completed. □

Proposition 3.10. *Let X be a bounded commutative BCK-algebra and $u, v \in X$. Then the following hold:*
(i) $S_y(u) = S_y(v)$ if and only if $y * u = y * v$.
(ii) If $u, v \leq y$, then $S_y(u) = S_y(v) \Leftrightarrow u = v$.

Proof. (i) Let $S_y(u) = S_y(v)$. Since $u \in S_y(u)$, we get $u \in S_y(v)$ and so $(NNy) * v \leq y * u$. Thus by the commutatively of X, we get $y * v \leq y * u$. Similarly, from $v \in S_y(v)$ we obtain $y * u \leq y * v$, therefore $y * u = y * v$.

Conversely, assume that $y * u = y * v$. Then by the commutatively of X, $x \in S_y(u)$ if and only if $y * u \leq y * x$ if and only if $y * v \leq y * x$ if and only if $x \in S_y(v)$. Therefore $S_y(u) = S_y(v)$.

(ii) Let $S_y(u) = S_y(v)$. From (i), we have $y * u = y * v$ and so $y * (y * u) = y * (y * v)$. Hence by the commutatively of X, $u \wedge y = v \wedge y$, and so from $u, v \leq y$, we conclude $u = v$.

Conversely, it is obvious. □

Using Propositions 3.5 and 3.10(ii), we have the following result:

Corollary 3.11. *Let X be a bounded commutative BCK-algebra and $y \in X$. Then for any $u, v \in A(y)$,*

$$S_y(u) = S_y(v) \Leftrightarrow u = v.$$

The following example shows that the commutative property of X in Corollary 3.11 is necessary.

Example 3.12. Let $X = \{0, a, b, c\}$. Define the operation $*$ on X by the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	c	0

Then $(X; *, 0)$ is a BCK-algebra, but it is not commutative because $c * (c * a) = 0 \neq a = a * (a * c)$. Routine calculations show that $S_c(a) = S_c(b) = \{0, a, b\}$, which does not imply $a = b$.

Next, we characterize the involutive BCK-algebras.

Proposition 3.13. *Let X be a bounded BCK-algebra. Then the following are equivalent:*
(i) X is involutive.
(ii) $(\forall u, v \in X)$ $S_1(u) = S_1(v)$ implies $u = v$.

Proof. (i)$\Rightarrow$ (ii) Let $u, v \in X$ be such that $S_1(u) = S_1(v)$. From $u \in S_1(u)$, we get $u \in S_1(v)$ and so $NN1 * v \leq Nu$, that is, $Nv \leq Nu$. Thus using (a_4), we have $NNu \leq NNv$ and so by (i), we conclude $u \leq v$. Similarly, we can show that $v \leq u$. Therefore $u = v$.

(ii) Let u be an arbitrary element of X and assume that $x \in S_1(u)$. Then $NN1 * u \leq Nx$, and so, since $NN1 = 1$, we get $Nu \leq Nx$. Hence by (a_6), $N(NNu) \leq Nx$, that is $x \in S_1(NNu)$, hence $S_1(u) \subseteq S_1(NNu)$. The reverse inclusion follows from $NNu \leq u$ and Proposition 3.5(i). Thus $S_1(u) = S_1(NNu)$ and so by (ii), $NNu = u$. Therefore X is involutive. □

The following theorem provides a property for a bounded BCK-algebra to be a commutative chain.

Theorem 3.14. *Let X be a bounded BCK-algebra. Then the following are equivalent:*
(i) X is a commutative BCK-chain.
(ii) $S_y(u) = A(u)$ for any $y, u \in X$ with $y \not\leq u$.

Proof. (i)$\Rightarrow$ (ii)) Let $y, u \in X$ be such that $y \not\leq u$ and let $z \in S_y(u)$. Since X is a BCK-chain, $y \leq z$ or $z < y$. If $y \leq z$, then from $z \in S_y(u)$ we get $(NNy) * u \leq y * z = 0$ and so, by the commutatively of X, we obtain $y*u = 0$, that is, $y \leq u$, which is a contradiction with assumption $y \not\leq u$. Thus $z < y$. We assert that $z \leq u$. If not, then $u < z$. Thus by (a_4), we obtain $(NNy)*z \leq (NNy)*u$ and so by the commutatively of X, we have

$$y * z \leq y * u. \tag{1}$$

On the other hand, by the commutatively of X, from $z \in S_y(u)$, we get

$$y * u \leq y * z. \tag{2}$$

From (1) and (2), we get $y * z = y * u$. Thus $y * (y * z) = y * (y * u)$ and so, we conclude

$$y \wedge z = y \wedge u. \tag{3}$$

Now, from $y \not\leq u$, we get $u < y$, which implies $y \wedge u = u$. Also, from $z \leq y$, we have $y \wedge z = z$. Thus, by (3), we get $z = u$, which is a contradiction with assumption $u < z$. Hence $z \leq u$ and so $z \in A(u)$. We have shown that $S_y(u) \subseteq A(u)$. But by Proposition 3.7(i), $A(u) \subseteq S_y(u)$. Therefore $S_y(u) = A(u)$.

(ii)⇒(i) Assume that $x, y \in X$. Obviously, $y \leq y * (y * x)$ or $y \not\leq y * (y * x)$. If $y \leq y * (y * x)$, then by (a_2), we get $y * (y * x) = y$ and so $y * (y * (y * x)) = y * y = 0$. But, by (a_6), $y * (y * (y * x)) = y * x$. Hence $y * x = 0$, that is, $y \leq x$. If $y \not\leq y * (y * x)$, then by (ii), $S_y(y*(y*x)) = A(y*(y*x))$. Using $(a_3), (a_6)$ and axiom $(BCK\text{-}1)$, we get

$$(NNy) * (y * (y * x)) = N(y * (y * x)) * Ny \leq y * (y * (y * x)) = y * x. \quad (4)$$

This implies that $x \in S_y(y * (y * x))$ and so by (ii), $x \in A(y * (y * x))$, that is, $x \leq y * (y * x)$. On the other hand, $y * (y * x) \leq y$. Thus $x \leq y$. Up to now, we have shown that X is a BCK-chain. To prove the commutatively of X, assume that $x, y \in X$. Since X is a BCK-chain, without loss the generality, we may assume that $x * (x * y) \leq y * (y * x)$. We assert that $y * (y * x) \leq x * (x * y)$. If not, then

$$y * (y * x) \not\leq x * (x * y). \quad (5)$$

Since $y*(y*x) \leq x$, it follows from (5) that $x \not\leq x*(x*y)$. Hence by (ii), we have $S_x(x * (x * y)) = A(x * (x * y))$. Similar to the argument of (4), we have $y \in S_x(x*(x*y))$. Thus $y \in A(x*(x*y))$ and so $y \leq x*(x*y)$. On the other hand, $y * (y * x) \leq y$. Hence $y * (y * x) \leq x * (x * y)$, which is a contradiction with (5). Thus $y * (y * x) \leq x * (x * y)$ and so $y * (y * x) = x * (x * y)$. Therefore X is commutative and the proof is completed. □

Let X be a commutative BCK-lattice. For any element $y \in X$, we denote $L(y, X) := \{S_y(u) \mid u \in X\}$ and define operations $\bigtriangledown$ and $\bigtriangleup$ on $L(y, X)$ as follows: for any $u, v \in X$,

$$S_y(u) \bigtriangledown S_y(v) := S_y(u \wedge v) \quad ; \quad S_y(u) \bigtriangleup S_y(v) := S_y(u \vee v). \quad (6)$$

Theorem 3.15. *Let X be a commutative BCK-lattice and let operations $\bigtriangledown$ and $\bigtriangleup$ are defined as (6). Then $(L(y, X); \bigtriangledown, \bigtriangleup)$ is a bounded distributive lattice.*

Proof. Let $S_y(u), S_y(v) \in L(y, X)$. Obviously, by Proposition 3.9(ii), $S_y(u \wedge s)$ is the infimum of $S_y(u)$ and $S_y(v)$. Since $u, v \leq u \vee v$, from Proposition 3.5(i), we get $S_y(u), S_y(v) \subseteq S_y(u \vee v)$. Now let $S_y(z) \in L(y, X)$ be such that $S_y(u), S_y(v) \subseteq S_y(z)$. Then from $u \in S_y(u)$, we have $u \in S_y(z)$ and so $(NNy) * z \leq y * u$. Similarly, $(NNy) * z \leq y * v$. Hence $(NNy)*z \leq (y*u) \wedge (y*v)$, and so, using Theorem 2.5(c_1), we get $(NNy)*z \leq y*(u \vee v)$. This implies $u \vee v \in S_y(z)$. Then by Proposition 3.7(i), $S_y(u \vee s) \subseteq S_y(z)$. Hence $S_y(u \vee v)$ is the supremum of $S_y(u)$ and $S_y(v)$. Therefore $(L(y, X); \triangledown, \triangle)$ is a lattice. By Proposition 3.5, $S_y(0)$ and $S_y(1) = X$ are the least element and greatest upper of $L(y, X)$ respectively, and consequently X is bounded. It remains to prove that $L(y, X)$ is distributive. For this, by (6) and the distributivity of X, it is easily seen that

$$\begin{aligned} S_y(z) \triangledown ((S_y(u) \triangle S_y(v)) &= S_y(z \wedge (u \vee v)) \\ &= S_y((z \wedge u) \vee (z \wedge v)) \\ &= S_y(z \wedge u) \triangle S_y(z \wedge v) \\ &= (S_y(z) \triangledown S_y(u)) \triangle (S_y(z) \triangledown S_y(v)), \end{aligned}$$

for any $y, z, u, v \in X$. Therefore $(L(y, X); \triangledown, \triangle)$ is a bounded distributive lattice. □

The subset $S_y(u)$ is not necessary to be an ideal even if X be a commutative BCK-chain as shown in the following example.

Example 3.16. Let $X = \{0, a, 1\}$. Define the operation $*$ on X by the following table:

$*$	0	a	1
0	0	0	0
a	a	0	0
1	1	a	0

Then $(X; *, 0)$ is a commutative BCK-chain. Routine calculations show that $S_1(a) = \{0, a\}$ which is not an ideal of X because $1 * a = a \in S_1(a)$ but $1 \notin S_1(a)$.

Next, we give a property for $S_y(u)$ to be an ideal.

Proposition 3.17. *Let X be a commutative BCK-chain. Then the following are equivalent:*
(i) X is implicative.
(ii) For any $y, u \in X$, $S_y(u)$ is an ideal of X.

Proof. (i)⇒(ii) Let $y, u \in X$. If $y \leq u$, then by Proposition 3.3(ii), $S_y(u) = X$ and so clearly $S_y(u)$ is an ideal of X. Now, assume that $y \nleq u$. Then by Theorem 3.14, $S_y(u) = A(u)$. Hence it suffices to show that $A(u)$ is an ideal of X. Assume that $x, y * x \in A(u)$. Then $x \leq u$ and $y * x \leq u$ and so $x * u = 0$ and $(y * x) * u = 0$. By Theorem 2.2(iii), X is a positive implicative and so, we get $(y * u) * u = y * u$. Using (a_5) and (a_3), we have

$$y * u = (y * u) * 0 = (y * u) * (x * u) = ((y * u) * u) * (x * u) \\ \leq (y * u) * x = (y * x) * u = 0.$$

Thus $y * u = 0$, that is, $y \in A(u)$. Therefore $A(u)$ is an ideal of X.

(ii)⇒(i) By Theorem 2.2(iii), it suffices to show that X is a positive implicative BCK-algebra. Let $x, y \in X$. Then by (ii), $A(x)$ is an ideal of X. Taking, $z := y * (y * x) \leq x$ and $w := y * ((y * x) * x)$, it follows from $y * (y * x) \leq x$ that $z \in A(x)$. Also, we have

$$\begin{aligned} w * z &= (y * ((y * x) * x)) * (y * (y * x)) \\ &\leq (y * x) * ((y * x) * x) && \text{by axiom (BCK-1)} \\ &\leq y * (y * x) && \text{by } (a_5) \\ &\leq x \in A(x). && \text{by } (a_7) \end{aligned}$$

Hence $w * z \in A(x)$ and so from $z \in A(x)$ and the fact that $A(x)$ is an ideal of X, we conclude $w \in A(x)$, that is, $y * ((y * x) * x) \in A(x)$. Thus $y * ((y * x) * x) \leq x$ and so by (a_8), $y * x \leq (y * x) * x$. On the other hand, $(y * x) * x \leq y * x$. Therefore $(y * x) * x = y * x$, and so by Theorem 2.2(ii), X is a positive implicative BCK-algebra. □

Proposition 3.18. *Let X be an involutive BCK-algebra. Then, $S_y(u)$ is the least element of $L(y, X)$ with property $A(u) \subseteq S_y(u)$ for any $y, u \in X$.*

Proof. By Proposition 3.7(i), the property $A(u) \subseteq S_y(u)$ holds. Now, assume that $A(u) \subseteq S_y(v)$ for some $v \in X$. If $x \in S_y(u)$, then $(NNy) *$

$u \leq y*x$ and so by the involutivity of X, $y*u \leq y*x$. Thus, using (a_8), we get $y*(y*x) \leq u$, that is, $y*(y*x) \in A(u)$. Hence $y*(y*x) \in S_y(v)$ which yields $y*v \leq y*(y*(y*x))$. Then by (a_6), we get $y*v \leq y*x$ and consequently, $x \in S_y(v)$. Therefore $S_y(u) \subseteq S_y(v)$ and so the proof is completed. □

The converse of Proposition 3.18 is false as shown in the following example.

Example 3.19. Let $X = \{0, a, 1\}$. Define the operation $*$ on X by the following table:

$*$	0	a	1
0	0	0	0
a	a	0	0
1	1	1	0

Then $(X; *, 0)$ is a BCK-chain. Routine calculations show that

$$S_y(u) = X \; for \; all \; u \in X \; and \; y = 0, a;$$

$$A(1) = S_1(1) = X \nsubseteq \{0, a\} = S_1(0) = S_1(a)$$

Therefore X satisfies Proposition 3.18 but it is not involutive because $0 = NNa \neq a$.

In the following, we show that $S_y(u)$ inherits all properties of a commutative BCK-lattice.

Proposition 3.20. *If X is a commutative BCK-lattice, then so is $S_y(u)$ for all $y, u \in X$.*

Proof. By Theorem 3.8, $S_y(u)$ is a subalgebra and so it is a commutative BCK-algebra. Let $x, z \in S_y(u)$. Then $(NNy)*u \leq y*x$. On the other hand, it follows from $x \wedge z \leq x$ that $y*x \leq y*(x \wedge z)$. Therefore $(NNy)*u \leq y*(x \wedge z)$, which yields $x \wedge z \in S_y(u)$. Hence $S_y(u)$ is closed under $\wedge$. Also, using (a_8), since $y*u \leq y*x$, we get $y*(y*x) \leq u$ and so $x \wedge y \leq u$. Similarly, from $y*u \leq y*z$, we conclude $z \wedge y \leq u$. Thus $(x \wedge y) \vee (z \wedge y) \leq u$ and so by the distributivity of X, we obtain $(x \vee z) \wedge y \leq u$, that is, $y*(y*(x \vee z)) \leq u$. Hence, using (a_8), we get $y*u \leq y*(x \vee z)$ and so by $NNy \leq y$, we get $(NNy)*u \leq y*(x \vee z)$ which yields $x \vee z \in S_y(u)$. Therefore $S_y(u)$ is closed under $\vee$. Summarizing the previous results, we conclude that $S_y(u)$ is a commutative BCK-lattice. □

Conclusion and future work

It is well known that every initial set of a BCK-algebra is a subalgebra. But a subalgebra is not necessarily an initial set. In this paper, we have introduced and studied a new kind of subalgebras different from the initial sets. For this purpose, we have assigned a subset of X, denoted by $S_y(u)$, for any two elements y, u of a BCK-algebra X and have investigated some related properties. We have shown that $S_y(u)$ is a subalgebra of X for all $y, u \in X$. Moreover, we have proved that a bounded BCK-algebra X is a commutative BCK-chain if and only if every $S_y(u)$ is an initial set of X. Finally, assuming L(y,X) denote the set of all $S_y(u)$ where $u \in X$, we have proved that $S_y(u)$ is the least element of L(y,X) with property $A(u) \subseteq S_y(u)$.

Our future work is to introduce and study this kind of subalgebras in logical algebraic structures such as (pseudo)BCH-algebras, (pseudo)BE-algebras, (pseudo)CI-algebras and etc.

Acknowledgements
The author would like to thank the referees for the careful reading and valuable comments which helped to improve the manuscript.

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Habib Harizavi
Associate Professor of Mathematics
Department of Mathematics
Shahid Chamran University of Ahvaz
Ahvaz, Iran
E-mail: harizavi@scu.ac.ir