Journal of Mathematical Extension
Vol. 6, No. 2, (2012), 75-90

Solving Integro-Differential Equations by a Semi-Analytic Method

M. Fardi*
Islamic Azad University, Najafabad Branch

M. Ghasemi
Shahrekord University

F. Hemati Boroujeni
Islamic Azad University, Boroujen Branch

Abstract. In this paper, we propose a method to obtain approximate solutions to Fredholm integral-differential equations by employing the homotopy analysis method (HAM). The HAM gives the possibility to increase convergence region and rate of series solution. we show that the adomian decomposition method (ADM) cannot give better results than the present method. Five examples are presented to illustrate convergence and accuracy of the method to the solution. Also, we compute the absolute error to show that obtained results have reasonable accuracy.

AMS Subject Classification: 45J05; 65T60
Keywords and Phrases: Integro-differential equations, Homotopy analysis method

1. Introduction

The homotopy analysis method have been used for many years for solving mathematical problems. This new method has been presented by Liao ([1]) and applied to nonlinear oscillators with discontinuities ([2-4]), heat transfer ([5,6]), boundary layer flows ([7-9]), chaotic dynamical systems ([10]), systems of ODEs ([11]), delay differential equation ([12]),

Received: August 2011; Accepted: December 2012
*Corresponding author

ordinary differential equations ([13]), Glauert-jet problem ([14]), chaotic dynamical systems ([15]) and strongly nonlinear oscillatory system ([16]). Consider the following Feredholm integro-Differential equation

$$\sum_{j=0}^{m} P_j(t)u^{(j)}(t) = F(t) + \int_a^b K(t,s)G(u(s),...,u^{(m)}(s))ds, \quad (1)$$
$$\frac{\partial^i u(t)}{\partial t^i}|_{t=a} = \lambda_i, i = 0,1,...,m_1,$$
$$\frac{\partial^i u(t)}{\partial t^i}|_{t=b} = \lambda_i, i = 0,1,...,m_2.$$

where $u(t) : [a,b] \to \Re$ is the unknown function. where $K(t,s)$ and $P_j(t)$, $j = 0,1,2,...,m$ are known functions.

In this paper, we propose an analytical method to solve the Feredholm's Integro-Differential Equations. Comparisons are made between ADM and the proposed method. It is demonstrated that the solutions obtained by the ADM are special cases of the present method. For the purpose, we first give the following definition and theorems.

Definition 1.1. *Let ϕ be a function of the homotopy parameter q, then*

$$D_n(\phi) = \tfrac{1}{n!}\tfrac{\partial^n \phi}{\partial q^n}\Big|_{q=0},$$

is called the nth-order homotopy-derivative of ϕ, where $n \geqslant 0$ is an integer.

Theorem 1.2. *For homotopy-series*

$$\phi = \sum_{k=0}^{\infty} u_k q^k,$$

it holds the recurrence formulas

$$D_0(e^{\phi}) = e^{u_0},$$

$$D_m(e^{\phi}) = \sum_{k=0}^{m-1}(1 - \frac{k}{m})D_k(e^{\phi})D_{m-k}(\phi),$$

where $m \geqslant 1$ is integer.

Proof. See [17]. $\square$

Theorem 1.3. *For homotopy-series*

$$\phi = \sum_{k=0}^{\infty} u_k q^k,$$

it holds the recurrence formulas

$$D_0(\sin(\phi)) = \sin(u_0), \ D_0(\cos(\phi)) = \cos(u_0),$$

$$D_m(\sin(\phi)) = \sum_{k=0}^{m-1}(1-\frac{k}{m})D_k(\cos(\phi))D_{m-k}(\phi),$$
$$D_m(\cos(\phi)) = \sum_{k=0}^{m-1}(1-\frac{k}{m})D_k(\sin(\phi))D_{m-k}(\phi),$$

where $m \geqslant 1$ is integer.

Proof. See [17]. $\square$

2. Main Results

2.1 Analysis of the Method for the Feredholm Integro-Differential Equations

From (4) we define the nonlinear operator

$$\begin{aligned} N(S(t;q),q) &= \sum_{j=0}^{m} P_j(t)\frac{\partial^j S(t;q)}{\partial t^j} - F(t) \\ &- \int_a^b K(t,s)G(S(s;q),...,\frac{\partial^m S(s;q)}{\partial s^m})ds, \end{aligned} \tag{2}$$

and we choose the auxiliary linear operator as follows

$$L(S(t;q)) = \frac{\partial^m S(t;q)}{\partial t^m},$$

where $q \in [0,1]$ is an embedding parameter; $S(t;q)$, is real function of t and q, respectively. Let $\hbar$ denote a nonzero auxiliary parameter. Also, assume u_0 denote the initial guess of the exact solution $u(t)$.
We construct the zero-order deformation equation

$$(1-q)L[S(t;q)-u_0] = q\hbar N(S(t;q),q), \tag{3}$$

subject to the boundary conditions

$$\frac{\partial^i S(t;q)}{\partial t^i}|_{t=a} = \lambda_i, i = 0,1,...,m_1,$$
$$\frac{\partial^i S(t;q)}{\partial t^i}|_{t=b} = \lambda_i, i = 0,1,...,m_2.$$

Using Taylor's theorem, we expand $S(t;q)$ in the power series of q as follows

$$S(t;q) = u_0 + \sum_{j=1}^{\infty} u_j(t)q^j, \tag{4}$$

where

$$u_j(t) = D_j(S(t;q)).$$

Note that (2.) contains an auxiliary parameter $\hbar$. Assuming that is correctly chosen so that (2.) is convergent at $q = 1$, we have the series solution

$$u(t) = S(t;1) = u_0 + \sum_{j=1}^{\infty} u_j(t),$$

Operating on both sides of (2), we have the so-called nth-order deformation equation

$$L[u_n(t) - \chi_n u_{n-1}(t)] = \hbar R_n(u_0, u_1, ..., u_{n-1}, t),$$

$$\frac{\partial^i u_{n-1}(t)}{\partial t^i}|_{t=a} = 0, i = 0,1,...,m_1,$$
$$\frac{\partial^i u_{n-1}(t)}{\partial t^i}|_{t=b} = 0, i = 0,1,...,m_2.$$

where

$$\chi_n = \begin{cases} 0, & n \leqslant 1, \\ 1, & otherwise. \end{cases}$$

and

$$R_n(u_0, ..., u_{n-1}, t) \quad = \quad D_{n-1}(N(u_0 + \sum_{j=1}^{\infty} u_j(t)q^j)).$$

Then, we have

$$\begin{aligned} R_n &= \sum_{j=0}^{m} P_j(t)u_{n-1}^{(j)} - F(t)(1-\chi_n) \\ &- \int_a^b K(t,s)D_{n-1}(G(S(s;q), ..., \frac{\partial^m S(s;q)}{\partial s^m}))ds. \end{aligned} \qquad (5)$$

We gain u_n $(n = 1, 2, 3, ...)$, successively. At the Mth-order approximation we have

$$u(t) \approx U_M(t, \hbar) = u_0 + \sum_{j=1}^{M} u_j(t).$$

2.2 Convergence of Method and Comparison to ADM

Theorem 2.2.1. *If the series solution*

$$u_0(t) + \sum_{j=1}^{\infty} u_j(t),$$

converges then it is an exact solution of of (4).

Proof. If the series solution:

$$u_0(t) + \sum_{j=1}^{\infty} u_j(t),$$

is convergent, then:

$$\lim_{j\to\infty} u_j(t) = 0. \tag{6}$$

Using (6), we obtain:

$$\lim_{m\to\infty} \sum_{n=1}^{m} L\big[u_n(t) - \chi_n u_{n-1}(t)\big] = \lim_{m\to\infty} u_m(t) = 0.$$

Since $\hbar \neq 0$, we deduce:

$$\sum_{n=1}^{\infty} R_n(u_0, u_1, ..., u_{n-1}, t) = 0.$$

Now, from (2), it conclude:

$$\begin{aligned}\sum_{n=1}^{\infty} R_n &= \sum_{n=1}^{\infty}\sum_{j=0}^{m} P_j(t)u_{n-1}^{(j)} - \sum_{n=1}^{\infty} F(t)(1-\chi_n) \\ &- \int_a^b K(t,s)\sum_{n=1}^{\infty} D_{n-1}(G(S(s;q),..., \frac{\partial^m S(s;q)}{\partial s^m}))ds] = 0. \end{aligned} \tag{7}$$

If the series solution

$$u(t) = u_0(t) + \sum_{j=1}^{\infty} u_j(t),$$

is convergent, then the series

$$\sum_{n=1}^{\infty} D_{n-1}(G(S(s;q),..., \frac{\partial^m S(s;q)}{\partial s^m})),$$

will converge to $G(u(s),...,u^{(m-1)}(s),u^{(m)}(s))$ (see [18]).
Now, by using (7) we have:

$$\sum_{j=0}^{m} P_j(t)u^{(j)}(t) = F(t) + \int_a^b K(t,s)G(u(s),...,u^{(m-1)}(s),u^{(m)}(s))ds. \tag{8}$$

This completes the proof. □

Remark 2.2.2. *The valid region of $\hbar$ for convergence of series solution $u_0(t) + \sum_{j=1}^{\infty} u_j(t)$, can be found although approximately by plotting the curves of unknown quantities versus $\hbar$. Let $t_0 \in [a, b]$, then $U_M(t_0, \hbar)$, is function of $\hbar$. In accordance with $\hbar$-curve of $U_M(t_0, \hbar)$, we can find the valid region of $\hbar$ **[1]**.*

Theorem 2.2.3. *(Comparison to ADM) If $\hbar = -1$ and $L[u_0(t)] = F(t)$, the present method will be converted to ADM.*

Proof. See [1]. □

3. Test Examples

In this section, we solve five test problems to demonstrate the accurate nature of the proposed method. The validity of the method is based on assumption that the series (2.) converges at $q = 1$.
There is the convergence-control parameter $\hbar$ which guarantees that this assumption can be satisfied. We need to concentrate on the convergence of the obtained results by properly choosing $\hbar$.

Example 3.1. Consider the following nonlinear integro differential equation

$$\begin{cases} u''(t) = 2 - \frac{sin(1)}{2}(t^2+1) - \int_0^1 z(t^2+1)\cos(u(z))dz, \\ u(0) = 0, \quad u'(0) = 0. \end{cases}$$

The exact solution of this problem is $u(t) = t^2$ ([18]).
We choose $u_0(t) = 0$ as initial approximation guess. We study the influence of $\hbar$ on the convergence of $U_6'(0.5, \hbar)$. We can investigate the influence of $\hbar$ on the convergence region of $U_6'(0.5, \hbar)$ by means of $\hbar$-curve as shown in Fig. 1. From Fig. 1, the convergence region of $U_6'(0.5, \hbar)$ is $[-1.2, -0.5]$. The Error function $|U_M(t, \hbar) - u(t)|$ with M = 6 has been plotted for different $\hbar$ in Fig. 2.

Example 3.2. Consider the following linear integro differential equation

$$\begin{cases} u''(t) = t - \sin(t) - \int_0^{\frac{\pi}{2}} tzu(z)dz, \\ u(0) = 0, \quad u'(0) = 1. \end{cases}$$

The exact solution of this problem is $u(t) = \sin(t)$ ([18]).
We choose $u_0(t) = t$ as initial approximation guess. The curve of $U_6(\frac{\pi}{2}, \hbar)$ is plotted in Fig. 3 to determine the valid region of $\hbar$. As shown in Fig. 3, the series solutions of $U_6(\frac{\pi}{2}, \hbar)$ converge at $[-1.05, -0.55]$. The results presented in Table 1 clearly show the good accuracy of present method.

Example 3.3. Consider the following nonlinear integro differential equation

$$\begin{cases} u'(t) = 1 - e^{-1} + \int_0^1 e^{-u'(t)} dt, \\ u(0) = 0. \end{cases}$$

The exact solution of this problem is $u(t) = t$ ([19]).
Let us choose $u_0(t) = 0$ as initial approximation guess. Fig. 4 shows the $\hbar$-curve obtained from the $\frac{\partial U_6}{\partial t}$ at $t = 0.5$. From this figure, the valid values of $\hbar$ fall in the range $[-0.45, -0.35]$. The Error function $|U_M(t, \hbar) - u(t)|$ with $M = 6$ has been plotted for different value of $\hbar$ in Fig. 5.

Example 3.4. Consider the following nonlinear integro differential equation

$$\begin{cases} u'(t) = \frac{5}{4} - \frac{x^2}{3} + \int_0^1 (x^2 - t) u^2(t) dt, \\ u(0) = 0. \end{cases}$$

The exact solution of this problem is $u(t) = t$ ([20]).
Let us choose $u_0(t) = 0$ as initial approximation guess. In Fig. 6, $\hbar$-curve of $U_8(0.4, \hbar)$ has been plotted, as we see the valid region of $\hbar$ is $[-1.2, -0.3]$. The numerical solution obtained from the present method is much more accurate than the numerical solution given by the ADM, as shown in Table 2.

Example 3.5. Consider the following linear integro differential equation

$$\begin{cases} u''(t) + xu'(t) - xu(x) = e^t - 2\sin(t) - \int_{-1}^1 \sin(t) e^{-z} u(z) dz, \\ u(0) = 1, \quad u'(0) = 1. \end{cases}$$

The exact solution of this problem is $u(t) = e^t$ ([21]).
We choose $u_0(t) = 1 + t$ as initial approximation guess. To find the valid

region of $\hbar$, the $\hbar$-curve given by the $\frac{\partial U_5}{\partial t}$ at $t = -0.5$ is drawn in Fig. 7, which indicates that the valid region of $\hbar$ is about $[-1.2, -0.4]$. The convergence region of the solution given by ADM is $t \in [-0.5, 0.5]$, as shown in Fig. 8. When $\hbar = -1, -0.9, -0.7$, we obtain an approximate solution which is much more accurate than the solution given by the ADM as shown in Table 3.

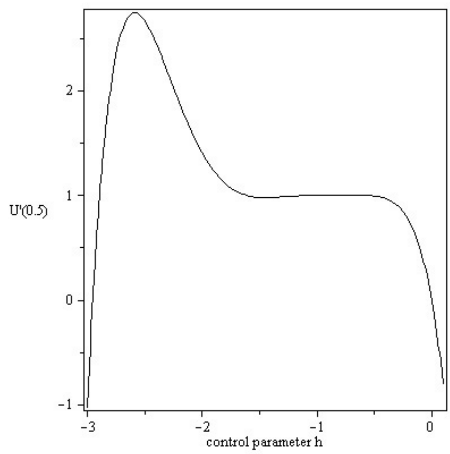

Figure 1: The $\hbar$-curve of $U_6'(0.5, \hbar)$ (Example 3.1).

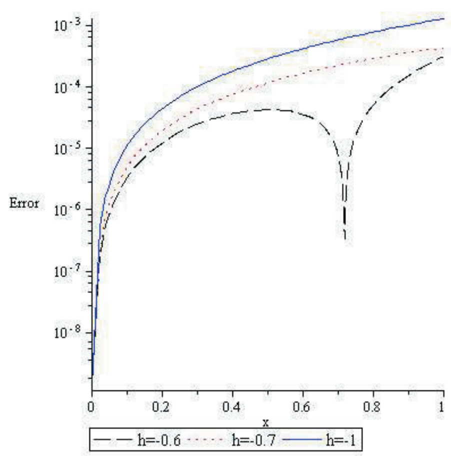

Figure 2: The error with $\hbar = -0.6$, $\hbar = -0.7$ and $\hbar = -1$ (Example 3.1).

Table 1: Absolute error (Example 3.2).

t	u_{exact}	$\hbar=-1$	$\hbar=-0.8$	$\hbar=-0.7$	ADM
$\pi.10^{-1}$	0.3090169944	4.9E-6	$2.6E-8$	$2.9E-7$	$5.4E-6$
$\pi.9^{-1}$	0.3420201433	6.8E-6	$3.6E-8$	$4.0E-7$	$7.4E-6$
$\pi.8^{-1}$	0.3826834325	9.6E-6	$4.9E-8$	$5.6E-7$	$1.1E-5$
$\pi.7^{-1}$	0.4338837393	1.4E-5	$7.1E-8$	$8.1E-7$	$1.5E-5$
$\pi.6^{-1}$	0.5000000000	2.2E-5	$1.0E-7$	$1.2E-6$	$2.5E-5$
$\pi.5^{-1}$	0.5877852524	3.9E-5	$1.7E-7$	$1.9E-6$	$4.3E-5$
$\pi.4^{-1}$	0.7071067810	7.7E-5	$2.7E-7$	$3.1E-6$	$8.4E-5$
$\pi.3^{-1}$	0.8660254040	1.8E-4	$3.8E-7$	$4.3E-6$	$2.0E-4$
$\pi.2^{-1}$	1.0000000000	6.2E-4	$1.3E-6$	$1.5E-5$	$6.8E-4$

Table 2: Absolute error (Example 3.4).

t	u_{exact}	$\hbar=-1.0$	$\hbar=-0.8$	$\hbar=-1.1$	ADM
0	0.0	0.0	0.0	0.0	0.0
0.2	0.2	5.1E-3	$5.0E-4$	$5.7E-4$	$5.9E-3$
0.4	0.4	9.6E-3	$9.5E-4$	$1.1E-3$	$1.1E-3$
0.6	0.6	1.3E-2	$1.2E-3$	$1.4E-3$	$1.5E-3$
0.8	0.8	1.5E-2	$1.3E-3$	$1.5E-3$	$1.6E-3$
1.0	1.0	1.4E-2	$1.4E-3$	$1.6E-3$	$1.5E-3$

Table 3: Absolute error (Example 3.5).

t	u_{exact}	$\hbar=-1$	$\hbar=-0.9$	$\hbar=-0.7$	ADM
-1	0.3678794412	2.7E-4	$4.9E-6$	$6.0E-6$	*divergent*
-0.8	0.4493289641	7.7E-5	$6.4E-6$	$1.5E-5$	*divergent*
-0.6	0.5488116361	1.7E-5	$2.2E-6$	$4.9E-6$	*divergent*
-0.4	0.6703200460	5.0E-6	$5.2E-7$	$5.2E-6$	$3.7E-2$
-0.2	0.8187307531	3.6E-7	$1.9E-6$	$4.5E-6$	$4.5E-3$
0	1	0	$1.7E-7$	$1.6E-7$	0
0.2	1.221402758	5.6e-6	$2.0E-6$	$6.7E-6$	$4.0e-3$
0.4	1.491824698	1.9E-6	$6.4E-6$	$2.6E-5$	$2.9E-2$
0.6	1.822118800	2.2E-5	$3.0E-6$	$3.3E-5$	$8.8E-2$
0.8	2.225540928	7.3E-5	$6.5E-7$	$3.1E-5$	*divergent*
1	2.718281828	1.7E-4	$7.4E-7$	$2.6E-5$	*divergent*

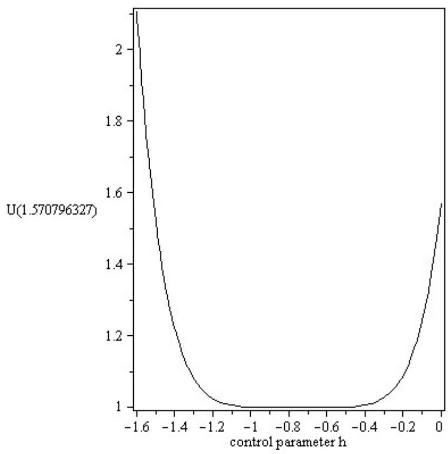

Figure 3: The $\hbar$-curve of $U_6(\frac{\pi}{2}, \hbar)$ (Example 3.2).

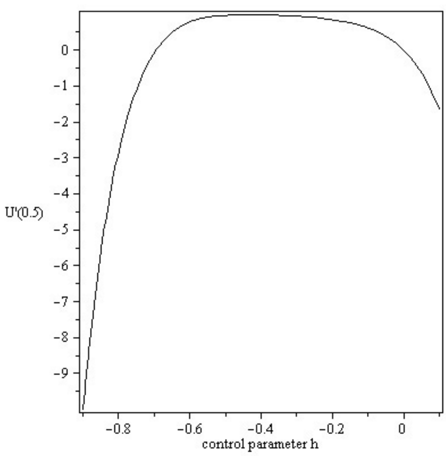

Figure 4: The $\hbar$-curve of $U_6'(0.5, \hbar)$ (Example 3.3).

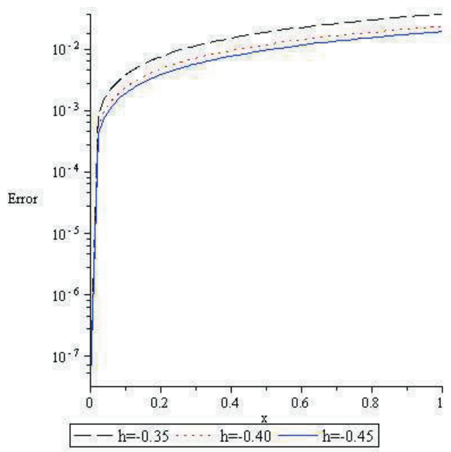

Figure 5: The error with $\hbar = -0.35$, $\hbar = -0.4$ and $\hbar = -0.45$ (Example 3.3).

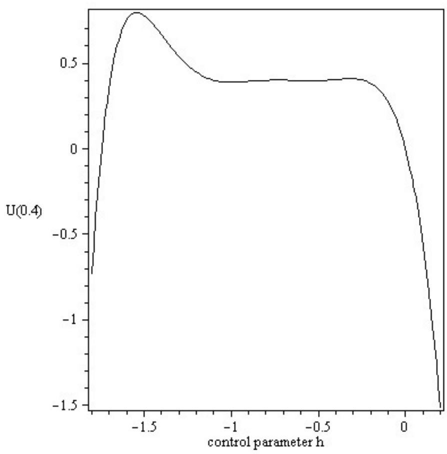

Figure 6: The $\hbar$-curve of $U_8(0.4, \hbar)$ (Example 3.4).

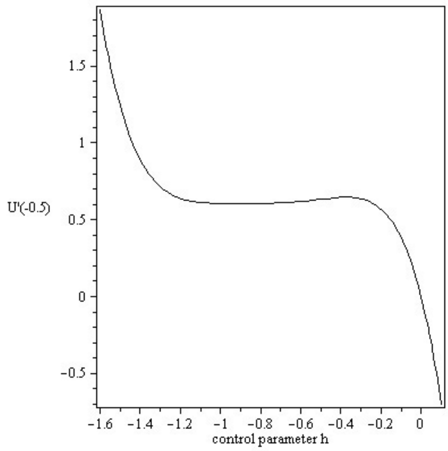

Figure 7: The $\hbar$-curve of $U_5^{'}(-0.5,\hbar)$ (Example 3.5).

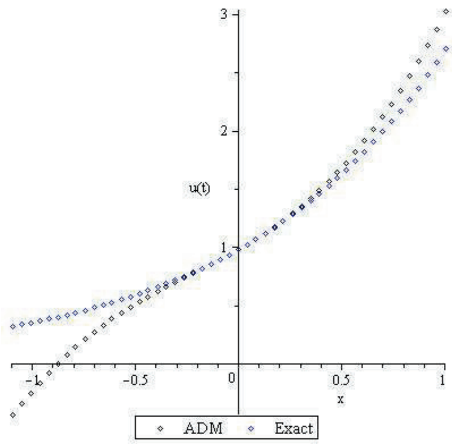

Figure 8: Comparison of the exact result with the 5th-order approximation given by ADM (Example 3.5).

4. Conclusion

In this paper, an semi-analytical method was proposed for solving Feredholm Integro-Differential Equations. The efficiency of this method is demonstrated by solving five examples. We have illustrated that the ADM cannot give better results than the present method. In fact, the ADM are only the especial case of the present method.

References

[1] S. J. Liao, Beyond Perturbation: *Introduction to Homotopy Analysis Method,* Chapman & Hall/ CRC Press, Boca Raton 2003.

[2] S. J. Liao and A. T. Chwang, Application of homotopy analysis method in nonlinear oscillations, *ASME J. of Appl. Mech.*, 65 (1998), 914-922.

[3] S. J. Liao, An analytic approximate technique for free oscillations of positively damped systems with algebraically decaying amplitude, *Int. J. Non-Linear Mech.*, 38 (2003), 1173-1183.

[4] S. J. Liao, An analytic approximate approach for free oscillations of self-excited systems, *Int. J. Non-Linear Mech.*, 39 (2004), 271-280.

[5] S. J. Liao and A. Campo, Analytic solutions of the temperature distri bution in Blasius viscous flow problems, *J. of Fluid Mech.*, 453 (2002), 411-425.

[6] C. Wang et al, On the explicit analytic solution of cheng-chang equation, *Int. J. Heat and Mass Transfer*, 46 (2003), 1855-1860.

[7] S. J. Liao, A kind of approximate solution technique which does not depend upon small parameters (II): an application in fluid mechanics, *Int. J. of Non-Linear Mech.*, 32 (1997), 815-822.

[8] S. J. Liao, An explicit, totally analytic approximation of Blasius viscous flow problems, *Int. J. of Non-Linear Mech.*, 34 (1999), 759-778.

[9] S. J. Liao, An analytic approximation of the drag coe?cient for the viscous flow past a sphere, *Int. J. of Non-Linear Mech.*, 37 (2002), 1-18.

[10] F. M. Allan, Construction of analytic solution to chaotic dynamical systems using the Homotopy analysis method, *Chaos Solitons Fractals,* 39 (2009), 1744-1752.

[11] A. S. Bataineh, M. S. M. Noorani, and I. Hashim, Solving systems of ODEs by homotopy analysis method, *Commun. Nonlinear Sci. Numer. Simul*, 13 (2008), 2060-2070.

[12] A. K. Alomari, M. S. M. Noorani, and R. Nazar, Solution of delay differential equation by means of homotopy analysis method, *Acta Applicandae Mathematicae,* 108 (2009), 395-412.

[13] F. T. Akyildiz, K. Vajravelu, and S. J. Liao, A new method for homoclinic solutions of ordinary differential equations, *Chaos Solitons Fractals,* 39 (2009), 1073-1082.

[14] Y. Bouremel, Explicit series solution for the Glauert-jet problem by means of the homotopy analysis method, *Commun. Nonlinear. Sci. Numer. Simulat*, 12 (2007), 714-724.

[15] F. M. Allan, Construction of analytic solution to chaotic dynamical systems using the Homotopy analysis method, *Chaos Solitons Fractals,* 39 (2009), 1744-1752.

[16] A. Golbabai, M. Fardi, and K. Sayevand, Application of optimal homotopy asymptotic method for solving strongly nonlinear oscillatory system, *Mathematical and Computer Modelling*, in press.

[17] S. J. Liao, Notes on the homotopy analysis method: some definitions and theorems, *Commun. Nonlinear Sci. Numer. Simul*, 14 (2009), 983-997.

[18] Y. Cherruault, *Convergence of Adomians method,* Kybernetes, 18 (1989), 31-38.

[19] S. R. Seyed Alizadeh, G. G. Domairry, and S. Karimpour, An Approximation of the Analytical Solution of the Linear and Nonlinear Integro-Differential Equations by Homotopy Perturbation Method, *Acta Appl Math.*, 104 (2008), 355-366.

[20] M. Ghasemi, M. Tavassoli Kajani, and E. Babolian, Application of He's homotopy perturbation method to nonlinear integro-differential equations, *Applied Mathematics and Computation*, 188 (2007), 538-548.

[21] A. Shahsavaran and A. Shahsavaran, Application of Lagrange Interpolation for Nonlinear Integro Differential Equations, *Applied Mathematical Sciences*, 18 (2012), 887-892.

[22] K. Maleknejad and Y. Mahmoudi, Taylor polynomial solution of high-order nonlinear VolterraFredholm integro-differential equations, *Applied Mathematics and Computation*, 145 (2003), 641-653.

Mojtaba Fardi
Department of Mathematics
M.Sc. of Mathematics
Islamic Azad University, Najafabad Branch
Najafabad, Iran
E-mail: fardi_ mojtaba@yahoo.com

Mehdi Ghasemi
Department of Applied Mathematics
Assistant Professor of Mathematics
Faculty of Mathematical Science
Shahrekord University
P.O. Box 115
Shahrekord, Iran
E-mail: meh_ ghasemi@yahoo.com

Farshad Hemati Boroujeni
Department of Mathematics
M.Sc. of Mathematics
Islamic Azad University, Boroujen Branch
Boroujen, Iran
E-mail: farshad_86@yahoo.com