Journal of Mathematical Extension
Vol. 16, No. 7, (2022) (8)1-25
URL: https://doi.org/10.30495/JME.2022.1659
ISSN: 1735-8299
Original Research Paper

Well-Posedness and Stability of Time-Dependent Impulsive Neutral Stochastic Partial Integrodifferential Equations with Fractional Brownian Motion and Poisson Jumps

K. Ramkumar
PSG College of Arts & Science

K. Ravikumar*
PSG College of Arts & Science

E. M. Elsayed
King Abdulaziz University

Abstract. In this paper, we discuss the existence, uniqueness and stability of mild solutions of time-dependent impulsive neutral stochastic partial integrodifferential equations with fractional Brownian motion and Poisson jumps. The existence of mild solutions for the equations are discussed by means of semigroup theory and theory of resolvent operator. Next, certain sufficient conditions and results are obtained by using the method of successive approximation and Bihari's inequality. Finally, an example is provided to illustrate our results.

AMS Subject Classification: 60G15; 60H15; 60H20; 93B05.
Keywords and Phrases: Resolvent operator, Stochastic partial differential equations, Poisson jumps, Fractional Brownian motion, Successive approximation, Bihari's inequality.

Received: April 2020; Accepted: March 2021
*Corresponding Author

1 Introduction

Stochastic differential equations occur in many areas of science and engineering have attained much attention in the past decades. The partial integrodifferential equations has wide applications in the field of electrical, mechanical and so on. For abstract model of partial integrodifferential equations with resolvent operators, see for instance [1, 9, 10]. The deterministic model often fluctuate due to noise. Under this circumstance, we move the deterministic model problems to stochastic model problems, for more details reader may refer [7, 9, 11]. There are only few works on existence, uniqueness and stability of stochastic differential systems have been established [1, 2, 14, 22]. The stochastic systems with resolvent operators has occur in different applications such as heat equation, viscoelasticity and many other physical phenomena, see for instance [12]. The study of existence, uniqueness and stability of stochastic functional differential with resolvent operator is an unprocessed issue and it is also the motivation of this paper.

As a generalization of the classical Brownian motion, fractional Brownian motion heavily depends on a parameter $\mathtt{H} \in (0, 1)$ called as the Hurst index [17]. When $\mathtt{H} = \frac{1}{2}$ the fractional Brownian motion is a standard Brownian motion. When $\mathtt{H} \neq \frac{1}{2}$ the fractional Brownian motion is not a semimartingale see Biagini et al [5], we can not use the classical ito theory to construct a stochastic calculus with respect to fractional Brownian motion. Especially, when $\mathtt{H} > \frac{1}{2}$, fractional Brownian motion has a long range dependence. This property makes this process as a useful driving noise in models appeared in telecommunications networks, finance, and other fields. Since some physical phenomena are naturally modelled by stochastic partial differential equations, the randomness can be described by a fractional Brownian motion, it is important to study the existence, uniqueness and stability of infinite dimensional equations with a fractional Brownian motion. Many studies of the solutions of stochastic equations in an infinte dimensional space with a fractional Brownian motion have been emerged recently, see [3, 6, 8, 19].

In addition, stochastic functional differential equations with Poisson jumps have become very popular in modeling the phenomena arising in the fields of economics, medicine, biology and so on. Moreover, many practical systems (such as sudden price variations [jumps] due to market crashes, earthquakes and epidemics ect.) may undergo some jump type stochastic perturbations. The sample paths of such systems are not continuous. Therefore, it is more appropriate to consider stochastic processes with jumps to describe such models. These jump models are generally based on Poisson random measure, and has the sample paths which are right continuous and have left limits. In recent years, stochastic evolution equations with Poisson jumps have been studied by many authors, [1, 4, 15, 20].

Based on the above discussion, in this paper, we are interested to study the existence and stability results for time-dependent impulsive neutral stochastic partial integrodifferential equations with fractional Brownian motion and Poisson jumps of the form

$$\begin{aligned} d\left[x(t)+g(t,x_t)\right] &= A(t)\left[x(t)+g(t,x_t)\right]dt+\bigg[\int_0^t \Theta(t-s)\big[x(s) \\ &+ g(s,x_s)\big]ds+f(t,x_t)\bigg]dt+\sigma(t)d\mathtt{B}^{\mathtt{H}}(t) \\ &+ \int_{\mathcal{U}} h(t,x_t,u)\widetilde{N}(dt,du), \quad t\neq t_k, \quad t\in[0,T], \\ \Delta x(t_k) &= x(t_k^+)-x(t_k^-)=I_k(x(t_k)), \quad t=t_k, \quad k=1,2,...m, \\ x(t) &= \varphi\in\mathcal{D}^b_{\mathscr{B}_0}((-\infty,0];\mathcal{X}), \end{aligned} \tag{1}$$

where $A(t)$ is the linear operators generates a linear evolution systems $\{\mathcal{R}(t,s),t\geq 0\}$ on $\mathcal{X}$, and $\Theta(t-s)$, $t\in[0,T]$ is a closed linear operator on $\mathcal{X}$ with domain $\mathcal{D}(\Theta)\supset\mathcal{D}(A)$ which is independent of t. $\mathtt{B}^{\mathtt{H}}$ is a fractional Brownian motion on a real and separable Hilbert space $\mathcal{Y}$. Let $\mathbb{R}^+=[0,\infty)$ and let the functions $g,f:\mathbb{R}^+\times\hat{\mathcal{D}}\to\mathcal{X}$, $\sigma:\mathbb{R}^+\to\mathcal{L}(\mathcal{Y},\mathcal{X})$ and $h:\mathbb{R}^+\times\hat{\mathcal{D}}\times\mathcal{U}\to\mathcal{X}$ are appropriate functions. Here $\hat{\mathcal{D}}=\mathcal{D}((-\infty,0];\mathcal{X})$ denotes the family of all right piecwise continuous functions with left-hand limit φ from $(-\infty,0]$ to $\mathcal{X}$. The phase space $\mathcal{D}((-\infty,0];\mathcal{X})$ is assumed to be equipped with the norm

$$\|\varphi\|_t = \sup_{-\infty<\theta\leq 0}|\varphi(\theta)|.$$

We also assume that $\mathcal{D}^b_{\mathscr{B}_0}((-\infty,0];\mathcal{X})$ denotes the family of all almost surely bounded, $\Im_0$-measurable, $\hat{\mathcal{D}}$-valued random variables. Furthermore, the fixed moments of time t_k satisfy $0<t_1<\cdots<t_m<T$, where

$x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. And $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ represents the jump in the state x at time t_k with I_k determining the size of the jump. Further, let $\mathcal{B}_T$ be a Banach space of all $\Im_t$- adapted processes $\varphi(t,w)$ which are almost surely continuous in t for fixed $w \in \Omega$ with norm defined for any $\varphi \in \mathcal{B}_T$ by

$$\|\varphi\|_{\mathcal{B}_T} = \left(\sup_{0\le t\le T} \mathbf{E}\, \|\varphi\|_t^2\right)^{1/2}.$$

2 Wiener Process and Deterministic Integrodifferential Equations

2.1 Wiener Process

In this section we introduce the fractional Brownian motion as well as the Wiener integral with respect to it. We also need to establish some important results which will be needed throughout the paper. So, first let $(\Omega, \Im, \mathbb{P})$ be a complete probability space. Let $\mathcal{X}$, $\mathcal{Y}$ be real separable Hilbert spaces and $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ be the space of bounded linear operators mapping $\mathcal{Y}$ into $\mathcal{X}$, and let $Q \in \mathcal{L}(\mathcal{Y}, \mathcal{Y})$ be a nonnegative self-adjoint operator. By $\mathcal{L}_Q^0$ we denote the space of all $\gamma \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that $\gamma Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator and the norm is given by

$$|\gamma|^2_{\mathcal{L}_Q^0(\mathcal{Y},\mathcal{X})} = \left|\gamma Q^{\frac{1}{2}}\right|^2_{HS} = \mathrm{tr}(\gamma Q \gamma^*).$$

Then γ being a Q-Hilbert-Schmidt operator maps from $\mathcal{Y}$ into $\mathcal{X}$.

Definition 2.1. A two-sided one-dimensional fractional Brownian motion with Hurst parameter $\mathtt{H} \in (0,1)$ is a continuous centered Gaussian process $\beta^{\mathtt{H}} = \left\{\beta^{\mathtt{H}}(t), t \in \mathbb{R}\right\}$ with the covariance function

$$R_{\mathtt{H}(t,s)} = \mathbf{E}\left[\beta^{\mathtt{H}}(t)\beta^{\mathtt{H}}(s)\right] = \frac{1}{2}\left(|t|^{2\mathtt{H}} + |s|^{2\mathtt{H}} - |t-s|^{2\mathtt{H}}\right), \quad t, s \in \mathbb{R}.$$

Now, let us introduce Wiener integral with respect to the one-dimensional fractional Brownian motion $\beta^{\mathtt{H}}$. Fix $b > 0$. The notation Φ is denoted by the linear space of $\mathbb{R}$-valued step functions on $[0,b]$, that is $\varphi \in \Phi$ if

$$\varphi(t) \quad = \quad \sum_{i=1}^{n-1} \mathscr{X}_i \mathscr{V}_{[t_i,t_{i+1})}(t), \quad t \in [0,b],$$

where $0 = t_1 < t_2 < ... < t_n = b$ and $\mathscr{X}_i \in \mathbb{R}$.
Define the Wiener integral of $\varphi \in \Phi$ with respect to $\beta^{\mathtt{H}}$ by

$$\int_0^b \varphi(s) d\beta^{\mathtt{H}}(s) \quad = \quad \sum_{i=1}^{n-1} \mathscr{X}_i(\beta^H(t_{i+1}) - \beta^{\mathtt{H}}(t_i)).$$

Now, let $\mathcal{H}$ be the Hilbert space that consist of closure functions Φ with respect to the scalar product

$$\left\langle \mathscr{V}_{[0,t]}, \mathscr{V}_{[0,s]} \right\rangle_{\mathcal{H}} \quad = \quad R_{\mathtt{H}}(t,s).$$

Then, we have

$$\varphi \quad = \quad \sum_{i=1}^{n-1} \mathscr{X}_i \mathscr{V}_{[t_i,t_{i+1})} \mapsto \int_0^b \varphi(s) d\beta^{\mathtt{H}}(s).$$

The above mapping is an isometry between Φ and the linear space span $\left\{\beta^{\mathtt{H}}, t \in [0,b]\right\}$, which can be extended to an isometry between $\mathscr{H}$ and the first Wiener chaos of the fractional Brownian motion $\overline{span}^{\mathcal{L}^2(\Omega)}\left\{\beta^{\mathtt{H}},\right.$ $\left.t \in [0,b]\right\}$ (see [21]). Denote by $\beta^{\mathtt{H}}(\varphi)$ the image of φ by this isometry. At this point in time, we present an explicit expression of this integral. Let $K_{\mathtt{H}}(t,s)$ be the kernel given by

$$K_{\mathtt{H}}(t,s) \quad = \quad C_{\mathtt{H}} s^{\frac{1}{2}-\mathtt{H}} \int_s^t (\tau - s)^{\mathtt{H}-\frac{3}{2}} \tau^{\mathtt{H}-\frac{1}{2}} d\tau, \qquad \text{for} \quad t > s,$$

Here, $C_{\mathtt{H}} = \sqrt{\frac{\mathtt{H}(2\mathtt{H}-1)}{B(2-2\mathtt{H},\mathtt{H}-\frac{1}{2})}}$ with B is the Beta function. It is easy to see that

$$\frac{\partial K_{\mathtt{H}}(t,s)}{\partial t} \quad = \quad c_{\mathtt{H}}(\frac{t}{s})^{\mathtt{H}-\frac{1}{2}}(t-s)^{\mathtt{H}-\frac{3}{2}}.$$

Let us consider the operator $K_{\mathtt{H}}^{*}:\Phi\to\mathcal{L}^{2}([0,T])$ defined by

$$(K_{\mathtt{H}}^{*}\varphi)(s) \quad = \quad \int_{s}^{t}\varphi(t)\frac{\partial K}{\partial t}(t,s)(dt).$$

And then

$$(K_{\mathtt{H}}^{*}\mathscr{V}_{[0,t]})(s) \quad = \quad K_{\mathtt{H}}(t,s)\mathscr{V}_{[0,t]}(s).$$

The isometry $K_{\mathtt{H}}^{*}$ between Φ and $\mathcal{L}^{2}([0,b])$ can be extended to $\mathscr{H}$. Now we consider $\mathcal{W}=\{\mathcal{W},t\in[0,b]\}$, defined by

$$\mathcal{W}(t) \quad = \quad \beta^{\mathtt{H}}((K_{\mathtt{H}}^{*})^{-1}\mathscr{V}_{[0,t]}).$$

Then $\mathcal{W}$ is a Wiener process and

$$\beta^{\mathtt{H}}(t) \quad = \quad \int_{0}^{t}K_{\mathtt{H}}(t,s)d\mathcal{W}(s).$$

Also

$$\int_{0}^{b}\varphi(s)d\beta^{\mathtt{H}}(s) \quad = \quad \int_{0}^{b}(K_{\mathtt{H}}^{*}\varphi)(t)d\mathcal{W}(t).$$

for any $\varphi\in\mathscr{H}$ iff $K_{\mathtt{H}}^{*}\varphi\in\mathcal{L}^{2}([0,b])$. Moreover, let $\mathcal{L}^{2}_{\mathscr{H}}([0,b])=\big\{\varphi\in\mathscr{H},K_{\mathtt{H}}^{*}\varphi\in\mathcal{L}^{2}([0,b])\big\}$. when $\mathtt{H}>\frac{1}{2}$. we have $\mathcal{L}^{\frac{1}{\mathtt{H}}}([0,b])\subset\mathcal{L}^{2}_{\mathscr{H}}([0,b])$.

Lemma 2.2. *[18] For $\varphi\in\mathcal{L}^{\frac{1}{\mathtt{H}}}([0,b])$,*

$$\mathtt{H}(2\mathtt{H}-1)\int_{0}^{b}\int_{0}^{b}|\varphi(v)|\,|\varphi(\tau)|\,|v-\tau|^{2\mathtt{H}-2}\,dvd\tau \quad \leq \quad c_{\mathtt{H}}\,\|\varphi\|^{2}_{\mathcal{L}^{\frac{1}{\mathtt{H}}}([0,b])}.$$

Let $\big\{\beta_{n}^{\mathtt{H}}(t)\big\}_{n\in\mathbb{N}}$ be a sequence of two-side one dimensional standard fractional Brownian motion mutually independent on $(\Omega,\Im,\mathbb{P})$. Consider the following series

$$\sum_{n=1}^{\infty}\beta_{n}^{\mathtt{H}}(t)e_{n},\quad t\geq 0,$$

where $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis in $\mathcal{Y}$, the series does not necessarily converge in the space $\mathcal{Y}$. Therefore, we consider a $\mathcal{Y}$-valued stochastic process $B_Q^{\mathtt{H}}(t)$ given by the following series:

$$B_Q^{\mathtt{H}}(t) \quad = \quad \sum_{n=1}^{\infty} \beta_n^{\mathtt{H}}(t) Q^{\frac{1}{2}} e_n, \quad t \geq 0.$$

Moreover, if Q is a non-negative self-adjoint trace class operator, then this series converges in the space $\mathcal{Y}$, that is, it holds that $B_Q^{\mathtt{H}}(t) \in \mathcal{L}^2(\Omega, \mathcal{Y})$, and $B_Q^{\mathtt{H}}(t)$ is a $\mathcal{Y}$-valued Q-cylindrical fractional Brownian motion with covariance operator. For example, if $\{\lambda_n\}_{n\in\mathbb{N}}$ is a bounded sequence of non-negative real numbers such that $Qe_n = \lambda_n e_n$, then if Q is a nuclear operator in $\mathcal{Y}$, then the stochastic process

$$B_Q^{\mathtt{H}}(t) \quad = \quad \sum_{n=1}^{\infty} \beta_n^{\mathtt{H}}(t) Q^{\frac{1}{2}} e_n = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n^{\mathtt{H}}(t) e_n, \quad t \geq 0,$$

is well-defined as a $\mathcal{Y}$-valued Q-cylindrical fractional Browian motion. Let $\varphi : [0, b] \to \mathcal{L}_Q^0(\mathcal{Y}, \mathcal{X})$ be such that

$$\sum_{n=1}^{\infty} \left\| K_{\mathtt{H}}^*(\varphi Q^{\frac{1}{2}} e_n) \right\|_{\mathcal{L}^2[0,b];\mathcal{X}} < \infty. \tag{2}$$

Definition 2.3. Let $\varphi(s), s \in [0, b]$ be a function with values in $\mathcal{L}_Q^0(\mathcal{Y}, \mathcal{X})$. Then the Wiener integral of φ with respect to $B_Q^{\mathtt{H}}$ is defined by

$$\begin{aligned} \int_0^t \varphi(s) dB_Q^{\mathtt{H}}(s) \quad &= \quad \sum_{n=1}^{\infty} \int_0^t \varphi(s) Q^{\frac{1}{2}} e_n d\beta_n^{\mathtt{H}} \\ &= \quad \sum_{n=1}^{\infty} \int_0^t \left(K_{\mathtt{H}}^*(\varphi Q^{\frac{1}{2}} e_n) \right)(s) d\mathcal{W}(s), \ t \geq 0. \end{aligned}$$

Note that if

$$\sum_{n=1}^{\infty} \left\| \varphi Q^{\frac{1}{2}} e_n \right\|_{\mathcal{L}^2[0,b];\mathcal{X}} < \infty, \tag{3}$$

then certainly (3) holds, which follows directly from $\mathcal{L}^{\frac{1}{\mathtt{H}}}([0, e]) \subset \mathcal{L}^2_{\mathscr{H}}([0, e])$.

Lemma 2.4. *For any $\varphi : [0,b] \mapsto \mathcal{L}_Q^0(\mathcal{Y},\mathcal{X})$ satisfies $\int_0^T \|\varphi\|^2_{\mathcal{L}_Q^0} ds < \infty$ then the above sum in (4) is well defined as a $\mathcal{X}$-valued random variable and we have*

$$\mathbf{E}\left\|\int_\beta^\alpha \varphi(s)dB_Q^{\mathtt{H}}(s)\right\|^2 \leq c\mathtt{H}(2\mathtt{H}-1)(\alpha-\beta)^{2\mathtt{H}-1}\int_\beta^\alpha \|\varphi\|^2_{\mathcal{L}_Q^0} ds.$$

2.2 Poisson Process

Let $\mathcal{X}$ be a separable Hilbert space and let $\mathcal{B}_\sigma(\mathcal{X})$ denotes the Borel σ-algebra of $\mathcal{X}$. Let $p(t)$, $t \geq 0$ be an $\mathcal{X}$-valued, σ-finite stationary $\Im_t$-adapted Poisson point process on $(\Omega, \Im, \mathbb{P})$. The counting random meaure N_p defined by $N_p((t_1,t_2]\times\Lambda)(w) = \sum_{t_1<s\leq t_2} I_\Lambda(p(s))$ for any $\Lambda \in \mathcal{B}_\sigma(\mathcal{X})$ is called the Poisson random measure associated to the Poisson jump proces p. Then we define the measure $\widetilde{N}(dt,du) = N_p(dt,du) - dtv(du)$, where λ is the characteristic measure on $\mathcal{H}$, which is called the compensated Poisson random measure associated to the Poisson point process p. For a main source for the material on Poisson process and random measure we refer the reader to [13]. For a Borel set $\mathcal{U} \in \mathcal{B}_\sigma(\mathcal{X}-[0])$, we denote by $p^2([0,T]\times\mathcal{U};\mathcal{X})$ the space of all predicable mapping $h : [0,T]\times\mathcal{U}\times\Omega \to \mathcal{X}$ for which

$$\int_0^T\int_{\mathcal{U}} \mathbf{E}\,\|h(t,u)\|^2\, dt\lambda(du) < \infty.$$

We may then define the $\mathcal{X}$-valued stochastic integral

$$\int_0^T\int_{\mathcal{U}} h(t,u)\widetilde{N}(dt,du)$$

, which is a centered square-integrable martingale [16].

2.3 Partial Integrodifferential Equations

Let us recall some fundamental results needed to establish our results. The resolvent operator plays an important role in the study of the existence of solutions and to given a variation of constant formula for linear systems. However, need to know when the linear system (4) has a resolvent operator. For more details on resolvent operator, refer [10].

(H1) $A(t)$ generates a strongly continuous semigroup of evolution operators.

(H2) Suppose $\mathcal{Y}$ is a Banach space formed from $\mathcal{D}(A)$ equipped with the graph norm. $A(t)$ and $\Theta(t,s)$ are in the set of bounded linear operators fro $\mathcal{Y}$ to $\mathcal{X}$, $\Theta(\mathcal{Y},\mathcal{X})$ for $0 \le t \le T$ and $0 \le s \le t \le T$, respectively. $A(t)$ and $\Theta(t,s)$ are continuous on $0 \le t \le T$ and $0 \le s \le t \le T$, respectively, into $\mathcal{L}(\mathcal{Y},\mathcal{X})$.

To obtain the results, consider the integrodifferential abstract Cauchy problem

$$\begin{aligned} dx(t) &= \left[A(t)x(t) + \int_0^t \Theta(t,s)x(s)ds\right] dt, \quad 0 \le s \le t \le T, \\ x(0) &= x_0 \in \mathcal{X}. \end{aligned} \tag{4}$$

Definition 2.5. A resolvent operator for equation (4) is a bounded linear operator valued function $\mathcal{R}(t,s) \in \mathcal{L}(\mathcal{X})$ for $0 \le s \le t \le T$, satisfying the following properties:
1 $\mathcal{R}(t,t) = I$ and $|\mathcal{R}(t,s)| \le Me^{\beta(t-s)}, \quad t,s \in [0,T], \quad M$ and β are constant.
2 $\mathcal{R}(t,s)$ is strongly continuously in s and t on $\mathcal{X}$.
3 For $y \in \mathcal{X}$, $\mathcal{R}(t,s)y$ is continuously differentiable in s and t, and for $0 \le s \le t \le T$,

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{R}(t,s)y &= A(t)\mathcal{R}(t,s)y + \int_s^t \Theta(t,r)\mathcal{R}(r,s)ydr, \\ \frac{\partial}{\partial s}\mathcal{R}(t,s)y &= -\mathcal{R}(t,s)A(s)y - \int_s^t \mathcal{R}(r,s)\Theta(t,r)ydr, \end{aligned}$$

with $\frac{\partial}{\partial t}\mathcal{R}(t,s)y$ and $\frac{\partial}{\partial s}\mathcal{R}(t,s)y$ are strongly continuous on $0 \le s \le t \le T$. Here $\mathcal{R}(t,s)$ can be extracted from the evolution operator of the generator $A(t)$.
For the family $\{A(t) : 0 \le t \le T\}$ of linear operators, the following restrictions are imposed:

(A1) The domain $\mathcal{D}(A)$ of $\{A(t) : 0 \le t \le T\}$ is dense in $\mathcal{X}$ and independent of t, $A(t)$ is closed linear operator.

(A2) For each $t \in [0, T]$, the resolvent $\mathcal{R}(\zeta, A(t))$ exists for all ζ with $Re\zeta \leq 0$ and there exists $K > 0$

$$\|\mathcal{R}(\zeta, A(t))\| \quad \leq \quad \frac{K}{(|\zeta| + 1)}.$$

(A3) There exists $0 < \delta \leq 1$ and $K > 0$ such that

$$\left\|(A(t) - A(s))A^{-1}(\tau)\right\| \quad \leq \quad K\,|t - s|^{\delta}\,, \quad t, s, \tau \in [0, T].$$

(A4) For each $t \in [0, T]$ and some $\{A(t) : 0 \leq t \leq T\}$ generates a unique linear evolution system called linear evolution operator.

Definition 2.6. A two parameter family of bounded linear operators $\mathcal{R}(t, s)$, $0 \leq s \leq t \leq T$, on $\mathcal{X}$ is called an evolution system if the following two conditions hold:
1 $\mathcal{R}(s, s) = I$, $\mathcal{R}(t, r)\mathcal{R}(r, s) = \mathcal{R}(t, s)$, $\quad 0 \leq s \leq \tau \leq t \leq T$.
2 $(t, s) \to \mathcal{R}(t, s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

Lemma 2.7. *Let $\{A(t), t \in [0, T]\}$ be a family of linear operators satisfying $(\mathbf{A1}) - (\mathbf{A4})$. If $\{\mathcal{R}(t, s), 0 \leq s \leq t \leq T\}$ is the linear evolution system generated by $\{A(t), t \in [0, T]\}$, then $\{\mathcal{R}(t, s), 0 \leq s \leq t \leq T\}$ is a compact operator whenever $t - s > 0$.*

Definition 2.8. A stochastic process $\{x(t), t \in (-\infty, T]\}$, is called a mild solution of the equation (1.1) if
(i) $x(t)$ is $\Im_t$-adapted,
(ii) $x(t)$ satisfies the integral equation

$$\begin{aligned}
x(t) &= \varphi(t), \quad t \in (-\infty, 0] \\
x(t) &= \mathcal{R}(t)\left[\varphi(0) + g(0, \varphi)\right] - g(t, x_t) + \int_0^t \mathcal{R}(t, s) f(s, x_s) ds \\
&+ \int_0^t \mathcal{R}(t, s)\sigma(s) d\mathtt{B}^{\mathtt{H}}(s) + \int_0^t \int_{\mathcal{U}} \mathcal{R}(t, s) h(s, x_s, u)\widetilde{N}(ds, du) \\
&+ \sum_{0 < t_k < t} \mathcal{R}(t - t_k) I_k(x(t_k)).
\end{aligned} \tag{5}$$

3 Existence and Uniqueness

In this section, the existence and uniqueness of mild solution of the system (1) are discussed and worked under the following assumptions:

(H1) There exists a resolvent operator $\mathcal{R}(t,s)$ which is compact and continuous in the uniform operator topology for $t > s$. Further, there exists a constant $M > 0$ such that $\|\mathcal{R}(t,s)\| < M$, for all $t \in [0,T]$.

(H2) The functions f and h satisfy the following conditions. For each $x, y \in \hat{\mathcal{D}}$ and for all $t \in [0,T]$ such that

$$i)\ \|f(t,x_t) - f(t,y_t)\|^2 \leq K\left(\|x-y\|_t^2\right).$$

$$ii)\ \int_0^t \int_{\mathcal{U}} \|h(t,x_t,v) - h(t,y_t,u)\|^2 v(du)ds \vee$$

$$\left(\int_0^t \int_{\mathcal{U}} \|h(t,x_t,u) - h(t,y_t,u)\|^4 v(du)ds\right)^{1/2} \leq K\left(\|x-y\|_t^2\right).$$

$$iii)\ \left(\int_0^t \int_{\mathcal{U}} \|h(t,x_t,u) - h(t,y_t,u)\|^4 v(du)ds\right)^{1/2} \leq K\|x\|_t^2\, ds.$$

where $K(\cdot)$ is a concave non-decreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$, such that $K(0) = 0$, $K(u) > 0$, for $u > 0$ and $\displaystyle\int_{0^+} \frac{du}{K(u)} = \infty$.

(H3) Assuming that there exists a positive number L_g such that $L_g < \frac{1}{12}$, for any $x, y \in \hat{\mathcal{D}}$ and for all $t \in [0,T]$ such that

$$\|g(t,x_t) - g(t,y_t)\|^2 \quad \leq \quad L_g \|x-y\|_t^2,$$

(H4) The function $I_k \in \mathscr{C}(\mathcal{X},\mathcal{X})$ and there exists some constant h_k such that

$$\|I_k(x(t_k)) - I_k(y(t_k))\|^2 \leq h_k \|x-y\|_t^2, \quad x, y \in \hat{\mathcal{D}}, \quad k = 1, 2, ...m,$$

(H5) The function $\sigma : [0,T] \to \mathcal{L}_Q^0(\mathcal{Y},\mathcal{X})$ satisfies that exists a positive constant L such that

$$\|\sigma(s)\|_{\mathcal{L}_2^0}^2 \quad \leq \quad L \quad \text{uniformly in} \quad [0,T].$$

(H6) For all $t \in [0,T]$, it follows that $g(t,0), f(t,0)$ and $h(t,0,u) \in \mathcal{L}^2$, for $k = 1,2,...m$ such that

$$\|g(t,0)\|^2 \vee \|f(t,0)\|^2 \vee \|h(t,0,u)\|^2 \vee \|I_k(0)\|^2 \leq k_0,$$

where $k_0 > 0$ is a constant.

Let us now introduce the successive approximation to equation (5) as follows

$$x^n(t) \;=\; \begin{cases} \varphi(t), & \text{for } \; t \in (-\infty, o], \\ \\ \mathcal{R}(t)\varphi(0), & t \in [0,T], \quad \text{for} \quad n = 0, \end{cases}$$

$$\begin{aligned} x^n(t) &= \mathcal{R}(t)\left[\varphi(0) + g(0,\varphi)\right] - g(t, x_t^n) + \int_0^t \mathcal{R}(t-s) f(s, x_s^{n-1}) ds \\ &+ \int_0^t \mathcal{R}(t-s)\sigma(s) d\mathtt{B}^{\mathtt{H}}(s) + \int_0^t \int_{\mathcal{U}} \mathcal{R}(t-s) h(s, x_s^{n-1}, u) \tilde{N}(ds, du) \\ &+ \sum_{0<t_k<t} \mathcal{R}(t-t_k) I_k(x^{n-1}(t_k)), \qquad \text{a.s} \quad t \in [0,T], \end{aligned} \tag{6}$$

with an arbitrary non-negative initial approximation $x^0 \in \mathcal{B}_T$.

Theorem 3.1. *Let the assumptions* $(\mathbf{H1}) - (\mathbf{H6})$ *hold. Then the system (1) has unique mild solution* $x(t)$ *in* $\mathcal{B}_T$ *and*

$$\mathbf{E}\left\{ \sup_{0 \leq t \leq T} \|x^n(t) - x(t)\|^2 \right\} \to 0, \quad as \;\; n \to \infty$$

where $\{x^n(t)\}_{n \geq 1}$ *are the successive approximations (6).*

Proof.: The proof will be split into the following steps:
Step 1: For all $t \in (-\infty, T]$, the sequence $x^n(t)$, $n \geq 1 \in \mathcal{B}_T$ is bounded. Let $x^0 \in \mathcal{B}_T$ be a fixed initial approximation to (6). To begin with the assumptions $(\mathbf{H1}) - (\mathbf{H6})$ and observing that $\|\mathcal{R}(t,s)\| \leq M$ for some

$M \geq 1$ and for all $t \in [0, T]$. Then for any $n \geq 1$, we have

$$
\begin{aligned}
&\|x^n(t)\|^2 \\
&\leq 6M^2 \mathbf{E}\,\|\varphi(0) + g(0,\varphi)\|^2 + 12\mathbf{E}\big[\, \|g(t, x_t^n) - g(t,0)\|^2 + \|g(t,0)\|^2 \,\big] \\
&\quad + 12M^2 T \mathbf{E} \int_0^t \left[\left\| f(s, x_s^{n-1}) - f(s,0) \right\|^2 + \|f(s,0)\|^2 \right] ds \\
&\quad + 6M^2 c\mathtt{H}(2\mathtt{H}-1)T^{2\mathtt{H}-1} \mathbf{E} \int_0^t \|\sigma(s)\|^2_{\mathcal{L}_2^0}\, ds \\
&\quad + 12M^2 \mathbf{E} \int_0^t \int_{\mathcal{U}} \left[\left\| h(s, x_s^{n-1}, u) - h(s,0,u) \right\|^2 + \|h(s,0,u)\|^2 \right] ds \\
&\quad + 6M^2 \mathbf{E} \Big(\int_0^t \int_{\mathcal{U}} \left\| h(s, x_s^{n-1}, u) \right\|^4 v(du) ds \Big)^{\frac{1}{2}} \\
&\quad + 12M^2 m \mathbf{E} \sum_{k=1}^m \left[\left\| I_k(x^{n-1}(t_k)) - I_k(0) \right\|^2 + \|I_k(0)\|^2 \right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|x^n(t)\|^2_{\mathcal{B}} \;\leq\; & \frac{Q_1}{1-12L_g} + \frac{6M^2(2T+3)}{1-12L_g} \mathbf{E} \int_0^t K\left(\left\| x^{n-1} \right\|^2_{\mathcal{B}} \right) ds \\
+ \; & \frac{12M^2 m}{1-12L_g} \sum_{k=1}^m h_k \left\{ \mathbf{E} \left\| x^{n-1} \right\|^2_{\mathcal{B}} \right\}.
\end{aligned}
$$

where,

$$
\begin{aligned}
Q_1 = 12M^2 &\left[\mathbf{E}\,\|\varphi(0)\|^2 + L_g \mathbf{E}\,\|\varphi\|_0^2 + \frac{1}{2} c\mathtt{H}(2\mathtt{H}-1)T^{2\mathtt{H}} L \right] \\
&\qquad + 12\Big[\left(1 + M^2 T(T+1) + M^2 m \sum_{k=1}^m h_k \right) \Big] k_0.
\end{aligned}
$$

Given that $K(\cdot)$ is concave and $K(0) = 0$, we can find positive constants a and b such that

$$
K(u) \;\leq\; a + bu, \qquad \text{for all} \quad u \geq 0.
$$

Then,

$$\begin{aligned}\mathbf{E}\left\|x^n(t)\right\|_{\mathcal{B}}^2 &\leq Q_2 + \frac{6M^2(2T+3)b}{1-12L_g}\int_0^t \mathbf{E}\left\|x^{n-1}\right\|_s^2 ds \\ &+ \frac{12M^2m}{1-12L_g}\sum_{k=1}^m h_k\left\{\mathbf{E}\left\|x^{n-1}\right\|_{\mathcal{B}}^2\right\}, \quad n=1,2,...\end{aligned}$$

where $Q_2 = \frac{Q_1}{1-12L_g} + \frac{6M^2(2T+3)Ta}{1-12L_g}$.

Since,

$$\mathbf{E}\left\|x^0(t)\right\|_{\mathcal{B}}^2 \leq M^2\mathbf{E}\left\|\varphi(0)\right\|^2 = Q_3 < \infty. \tag{7}$$

Thus,

$$\mathbf{E}\left\|x^n(t)\right\|^2 < \infty, \quad \text{for all} \quad n=1,2,... \quad \text{and} \quad t\in[0,T]. \tag{8}$$

This proves the boundedness of $\{x^n(t), n\in\mathbb{N}\}$.
Step 2: The sequence $\{x^n(t)\}$, $n\geq 1$ is a Cauchy sequence.
Let us next show that $\{x^n(t)\}$ is Cauchy sequence in $\mathcal{B}_T$. For this consider,

$$\begin{aligned}\mathbf{E}\left\|x^{n+1}(t)-x^n(t)\right\|^2 &\leq 4L_g\mathbf{E}\left\|x^{n+1}-x^n\right\|_t^2 \\ &+ 4M^2(T+2)\int_0^t K\left(\mathbf{E}\left\|x^n-x^{n-1}\right\|_s^2\right)ds \\ &+ 4M^2m\sum_{k=1}^m h_k\mathbf{E}\left\{\left\|x^n-x^{n-1}\right\|_t^2\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{E}\left\|x^{n+1}(t)-x^n(t)\right\|^2 &\leq \frac{4M^2(T+2)}{1-4L_g}\int_0^t K\left(\mathbf{E}\left\|x^n-x^{n-1}\right\|_s^2\right)ds \\ &+ \frac{4M^2m\sum_{k=1}^m h_k}{1-4L_g}\mathbf{E}\left\{\left\|x^n-x^{n-1}\right\|_t^2\right\}.\end{aligned} \tag{9}$$

Set

$$\Psi_n(t) = \sup_{t\in[0,T]}\mathbf{E}\left\|x^{n+1}-x^n\right\|_t^2. \tag{10}$$

Then, we have in the view of (9),

$$\begin{aligned}\Psi_n(t) &\leq \frac{4M^2(T+2)}{1-4L_g}\int_0^t K\left(\Psi_{n-1}(s)\right)ds \\ &+ \frac{4M^2 m\sum_{k=1}^m h_k}{1-4L_g}\Psi_{n-1}(t), \quad 0\leq t\leq T. \qquad (11)\end{aligned}$$

Choose $T_1 \in [0,T)$ such that

$$C_1\int_0^t K\left(\Psi_{n-1}(s)\right)ds \quad \leq \quad C_1\Psi_{n-1}(s)ds, \; n=1,2,.. \;\; 0\leq t\leq T_1.$$

Moreover,

$$\begin{aligned}\left\|x^1(t)-x^0(t)\right\|^2 &= \Big\|\mathcal{R}(t)g(0,\varphi)-\Big[g(t,x_t^1)-g(t,x_t^0)\Big]-g(t,x_t^0) \\ &+ \int_0^t \mathcal{R}(t-s)f(s,x_s^0)ds+\int_0^t \mathcal{R}(t-s)\sigma(s)d\mathtt{B}^{\mathtt{H}}(s) \\ &+ \int_0^t\int_{\mathcal{U}} \mathcal{R}(t-s)h(s,x_s^0,u)\tilde{N}(ds,du) \\ &+ \sum_{0<t_k<t}\mathcal{R}(t-t_k)I_k(x^0(t_k))\Big\|^2.\end{aligned}$$

Then, we get

$$\begin{aligned}\mathbf{E}\left\|x^1(t)-x^0(t)\right\|_t^2 &\leq Q_4+\frac{14L_g+14M^2m\sum_{k=1}^m h_k}{1-7L_g}\mathbf{E}\left\|x^0\right\|_t^2 \\ &+ \frac{7M^2(2T+3)}{1-7L_g}\int_0^t K\left(\mathbf{E}\left\|x^0\right\|_s^2\right)ds.\end{aligned}$$

If we take the supremum over t, and use (8), we get

$$\begin{aligned}&\Psi_0(t)=\sup_{t\in[0,T]}\mathbf{E}\left\|x^1-x^0\right\|_t^2 \\ &\leq Q_5+\frac{7M^2(2T+3)}{1-7L_g}\int_0^t K(Q_3)ds \\ &\leq Q_6. \qquad (12)\end{aligned}$$

Now, for $n = 1$ in (11) we get

$$\Psi_1(t) \leq C_1 \int_0^t K(\Psi_0(s))\, ds + C_2\Psi_0(t), \quad 0 \leq t \leq T_1.$$

where $C_1 = \frac{4M^2(T+2)}{1-4L_g}$ and $C_2 = \frac{4M^2 m \sum_{k=1}^m h_k}{1-4L_g}$.
Therefore,

$$\begin{aligned}
\Psi_1(t) &\leq C_1 \int_0^t K(\Psi_0(s))\, ds + C_2\Psi_0(t) \\
&\leq C_1 \int_0^t Q_6 ds + C_2 Q_6 \\
&\leq (C_1 + C_2)\, T_1 Q_6.
\end{aligned}$$

Now, for $n = 2$ in (11), we get

$$\begin{aligned}
\Psi_2(t) &\leq C_1 \int_0^t K(\Psi_1(s))\, ds + C_2\Psi_1(t) \\
&\leq C_1 \int_0^t (C_1 + C_2)\, sQ_6 ds + C_2(C_1 + C_2)T_1 Q_6 \\
&\leq (C_1 + C_2)^2\, \frac{T_1^2}{2!} Q_6.
\end{aligned}$$

Thus by applying mathematical induction in (11) and using the above work we get

$$\Psi_n(t) \leq \frac{(C_1 + C_2)^n\, T_1^n}{n!} Q_6, \quad n \geq 0, \quad t \in [0, T_1].$$

Note that for any $m > n \geq 0$, we have,

$$\begin{aligned}
\sup_{t\in[0,T_1]} \mathbf{E}\, \|x^m(t) - x^n(t)\|^2 &\leq \sum_{r=n}^{+\infty} \sup_{t\in[0,T_1]} \mathbf{E}\, \left\|x^{r+1} - x^r\right\|_t^2 \\
&\leq \sum_{r=n}^{+\infty} \frac{(C_1 + C_2)^r T_1^r}{r!} Q_6 \\
&\to 0 \text{ as } n \to \infty. \qquad (13)
\end{aligned}$$

This shows that $\{x^n\}$ is Cauchy in $\mathcal{B}_T$. The Borel-Cantelli Lemma shows that as $n \to \infty$, $x^n(t) \to x(t)$ uniformly in t on $[0, T_1]$. By iteration, the existence of solution of (1) on $[0, T]$ can be obtained.
Step 3: Next, we prove the uniqueness of the solution (5). Let $x_1, x_2 \in \mathcal{B}_T$ be two solutions to (5) on some interval $(-\infty, T]$. Then, for $t \in (-\infty, 0]$, the uniqueness is obvious and for $0 \le t \le T$, we have

$$\begin{aligned} \mathbf{E}\,\|x_1(t) - x_2(t)\|^2 &\le 4\Big[L_g + M^2 m \sum_{k=1}^{m} h_k\Big] \mathbf{E}\,\|x_1 - x_2\|_t^2 \\ &+ 4M^2(T+1) \int_0^t K\left(\mathbf{E}\,\|x_1 - x_2\|_s^2\right) ds. \end{aligned}$$

Thus,

$$\mathbf{E}\,\|x_1(t) - x_2(t)\|_t^2 \le \frac{4M^2(T+1)}{1 - Q_7} \int_0^t K\left(\mathbf{E}\,\|x_1 - x_2\|_s^2\right) ds.$$

where, $4\big[L_g + M^2 m \sum_{k=1}^{m} h_k\big]$.
Thus, Bihari's inequality yields that

$$\sup_{t \in [0,T]} \mathbf{E}\,\|x_1(t) - x_2(t)\|_t^2 = 0, \quad 0 \le t \le T.$$

Thus, $x_1(t) = x_2(t)$, for all $0 \le t \le T$. Therefore, for all $-\infty < t \le T$, $x_1(t) = x_2(t)$ a.s. This completes the proof. $\square$

4 Stability

In this section, we study stability through the continuous dependence on initial values.

Definition 4.1. . A mild solution $u(t)$ of the system (1) with inital value ϕ is said to be stable in the mean square if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mathbf{E}\,\|x - \hat{x}\|_{\mathcal{B}}^2 \le \epsilon, \quad \text{whenever} \quad \mathbf{E}\left\|\phi - \hat{\phi}\right\|_{\mathcal{B}}^2 \le \delta$$

where $\hat{x}(t)$ is another mild solution of the system (1) with initial $\hat{\phi}$.

Theorem 4.2. *Let $x(t)$ and $y(t)$ be the mild solution of the system (1) with initial values φ_1 and φ_2 respectively. If the assumption of Theorem 3.1 are satisfied, then the mild solution of the system (1) is stable in the mean square.*

Proof. Let $x(t)$ and $y(t)$ be the mild solutions of equation (1) with initial values φ_1 and φ_2 respectively. Then for $0 \leq t \leq T$,

$$\begin{aligned}
x(t) - y(t) &= \mathcal{R}(t)\Big[[\varphi_1(0) - \varphi_2(0)] + [g(0, \varphi_1) - g(0, \varphi_2)]\Big] \\
&- \Big[g(t, x_t) - g(t, y_t)\Big] \\
&+ \int_0^t \mathcal{R}(t-s)\,[f(s, x_s) - f(s, y_s)]\,ds \\
&+ \int_0^t \int_{\mathcal{U}} \mathcal{R}(t-s)\,[h(s, x_s, u) - h(s, y_s, u)]\,\tilde{N}(ds, du) \\
&+ \sum_{0<t_k<t} \mathcal{R}(t-s)\,[I_k(x(t_k)) - I_k(y(t_k))]\,.
\end{aligned}$$

So, estimating as before, we get

$$\begin{aligned}
\mathbf{E}\,\|x - y\|^2 &\leq 6M^2\,[1 + L_g]\,\mathbf{E}\,\|\varphi_1 - \varphi_2\|^2 \\
&+ 6M^2\,[T+1] \int_0^t K\left(\mathbf{E}\,\|x - y\|_s^2\right) ds \\
&+ 6\left[L_g + M^2 m \sum_{k=1}^{m} h_k\right] \mathbf{E}\,\|x - y\|_t^2\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{E}\,\|x - y\|_t^2 &\leq \frac{6M^2(1 + L_g)}{1 - 6(L_g + M^2 m \sum_{k=1}^{m} h_k)}\mathbf{E}\,\|\varphi_1 - \varphi_2\|^2 \\
&+ \frac{6M^2(T+1)}{1 - 6(L_g + M^2 m \sum_{k=1}^{m} h_k)} \int_0^t K\left(\mathbf{E}\,\|x - y\|_s^2\right) ds.
\end{aligned}$$

Let $K_1(u) = \frac{6M^2(T+1)}{1-6(L_g+M^2 m \sum_{k=1}^{m} h_k)} K(u)$, where K is a concave increasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $K(0) = 0$, $K(u) > 0$ for $u > 0$ and $\int_{0_+} \frac{du}{K(u)} = +\infty$. Then, $K_1(u)$ is concave from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that

$K(0) = 0$, $K_1(u) \geq K(u)$ for $0 \leq u \leq 1$ and $\int_{0+} \frac{du}{K(u)} = +\infty$. Now for any $\epsilon > 0$, $\epsilon_1 = \frac{1}{2}\epsilon$, we have $\lim_{s\to 0} \int_s^{\epsilon_1} \frac{du}{K_1(u)} = \infty$. Then, there is a positive constant $\delta < \epsilon_1$, such that $\int_\delta^{\epsilon_1} \frac{du}{K_1(u)} \geq T$.
Let

$$\begin{aligned}
u_0 &= \frac{6M^2(1+L_g)}{1-6(L_g+M^2 m\sum_{k=1}^m h_k)} \mathbf{E} \left\|\varphi_1 - \varphi_2\right\|^2, \\
u(t) &= \mathbf{E} \left\|u - v\right\|_{\mathcal{B}}^2, \qquad v(t) = 1,
\end{aligned}$$

when $u_0 \leq \delta \leq \epsilon_1$. Then from corollary 2.1 in [2], we deduce that

$$\int_{u_0}^{\epsilon_1} \frac{du}{K_1(u)} \quad \geq \quad \int_\delta^{\epsilon_1} \frac{du}{K_1(u)} \geq T = \int_0^T v(t)ds.$$

It follows, for any $t \in [0, T]$, the estimate $u(t) \leq \epsilon_1$ hold. This completes the proof. $\square$

Remark 4.3. If $m = 0$ in (1), then the system behaves as stochastic partial neutral functional integrodifferential equations with infinite delays and poisson jumps of the form:

$$\begin{aligned}
d\left[x(t) + g(t, x_t)\right] &= A(t)\left[x(t) + g(t, x_t)\right]dt + \left[\int_0^t \Theta(t-s)\big[x(s)\right. \\
&+ \left. g(s, x_s)\big]ds + f(t, x_t)\right]dt + \sigma(t)d\mathtt{B}^{\mathtt{H}}(t) \\
&+ \int_{\mathcal{U}} h(t, x_t, u)\widetilde{N}(dt, du), \quad t \neq t_k, \quad t \in [0, T], \\
x(t) &= \varphi \in \mathcal{D}^b_{\mathscr{B}_0}((-\infty, 0]; \mathcal{X}), \qquad (14)
\end{aligned}$$

By applying Theorem 3.1, under the hypotheses (**H1**) $-$ (**H3**), (**H5**) the system (14) guarantees the existence and uniqueness of the mild solution.

Remark 4.4. If the system (14) satisfies the Remark 4.3, then by Theorem 4.2, the mild solution of the system (14) is stable in mean square.

5 Application

Example 1: Consider the following impulsive neutral stochastic partial integrodifferential equations with fractional Brownian motion and Poisson jumps of the form

$$\begin{aligned}
d\left[u(t,\zeta)+\hat{g}(t,u(t-h),\zeta)\right] &= \frac{\partial^2}{\partial\zeta^2}\left[u(t,\zeta)+\hat{g}(t,u(t-h),\zeta)\right]dt \\
&\quad + \int_0^t \hat{\Theta}(t-s)\frac{\partial^2}{\partial\zeta^2}\left[u(s,\zeta)+\hat{g}(s,u(t-h),\zeta)\right]ds \\
&\quad + \hat{f}(t,u(t-h),\zeta)dt+\hat{\sigma}(t)d\mathtt{B}^{\mathtt{H}}(t) \\
&\quad + \int_{\mathcal{U}} \hat{h}(t,u(t-h),v,\zeta)\widetilde{N}(ds,dv), \quad 0\le\zeta\le\pi, \quad t\in[0,T], \\
u(t,0) &= u(t,\pi)=0, \quad t\in[0,T], \\
\Delta u(t_k) &= (1+b_k)u(\zeta(t_k)), \quad t=t_k, \quad k=1,2,...m, \\
u(t,\zeta) &= \varphi(0,\zeta), \quad \theta\in(-\infty,0], \quad 0\le\zeta\le\pi, \qquad (15)
\end{aligned}$$

Let $\mathcal{X}=\mathcal{L}^2([0,\pi])$. To rewrite (15) into the form (1), define $A:\mathcal{X}\to\mathcal{X}$ by $Az=z''$ with domain $\mathcal{D}(A)=\{z\in\mathcal{X}, z, z'$ are absolutely continuous $z''\in\mathcal{X},\ z(0)=z(\pi)=0\}$. Then, A generates a strongly continuous semigroup $\mathcal{R}(t)$ on $\mathcal{X}$, thus (**H1**) is true. Moreover, the operator A can be expressed as

$$Az = \sum_{n=1}^{\infty} n^2 < z, z_n > z_n, \quad z\in\mathcal{D}(A),$$

where $z_n(s)=\sqrt{\frac{2}{\pi}sin(ns)}$, $n=1,2,...$, is orthonormal set of eigenvectors of A.

In addition, it follows that $\mathcal{R}(t)$ is compact for every $t>0$ and

$$\|\mathcal{R}(t)\| \le e^{-t}, \quad t\ge 0.$$

Now, we define an operator $A(t):\mathcal{D}(A)\subset\mathcal{X}\to\mathcal{X}$ by

$$A(t)x(\zeta) = Ax(\zeta)+b(t,\zeta)x(\zeta).$$

Let $b(\cdot)$ be continuous and $b(t,\zeta) \leq -\gamma(\gamma > 0)$, for every $t \in \mathbb{R}$. Then, the system

$$\begin{cases} u^{'}(t) = A(t)u(t), \quad t \geq s, \\ u(s) = x \in \mathcal{X}, \end{cases}$$

has an associated evolution family, given by

$$\mathcal{R}(t,s)x(\zeta) \quad = \quad \left|\mathcal{R}(t-s)e^{\int_s^t b(s,\zeta)ds}x\right|(\zeta).$$

From the above expression, it follows that $\mathcal{R}(t,s)$ is a compact operator and for every $t, s \in [0,T]$, with $t > s$

$$\|\mathcal{R}(t,x)\| \quad = \quad e^{-(1+\gamma)(t-s)}.$$

Thus, assuming that $\hat{g}, \hat{f} : [0,T] \times \hat{\mathcal{D}} \to \mathcal{X}$, $\hat{h} : [0,T] \times \hat{\mathcal{D}} \times \mathcal{U} \to \mathcal{X}$ and $\sigma : [0,T] \times \hat{\mathcal{D}} \to \mathcal{L}_2^0(\mathcal{Y},\mathcal{X})$ by $g(t,z)(\cdot) = \hat{g}(t,z)(\cdot)$, $f(t,z)(\cdot) = \hat{f}(t,z)(\cdot)$, $h(t,z,u)(\cdot) = \hat{h}(t,z,u)(\cdot)$, $\sigma(t) = \hat{\sigma}$ and $I_k(z(t_k)) = (1+b_k)u(z(t_k)), k = 1,2,...,m$, then, the system (15) can be rewriter as the abstract form as the system (1). Further, all the conditions of Theorem 3.1 have been fulfilled. So, we can conclude that the system (15) has a unique mild solution.

Example 2: We conclude this work an example of the form

$$\begin{aligned}
&d[u(t,x) + \int_0^\pi b(y,x)u(tsint,y)dy] \\
&= \left[\frac{\partial^2}{\partial x^2}\Theta(t\hat{-}s)\big[u(t,x) + \int_0^\pi b(y,x)u(tsint,y)dy\big] + \hat{f}(t,u(tsint,x))\right]dt \\
&\quad + \hat{\sigma}(t)d\mathtt{B}^{\mathtt{H}}(t) + \int_{\mathcal{U}} \hat{h}(t,u(t-h),v,\zeta)\widetilde{N}(ds,dv),\ 0 \leq \zeta \leq \pi,\ t \in [0,T], \\
&u(t_k^+) - u(t_k^-) = (1+b_k)u(x(t_t)) \\
&u(t,0) = u(t,\pi) = 0 \\
&u(t,x) = \Phi(t,x),\ 0 \leq x \leq \pi,\ -\infty < t \leq 0 \qquad (16)
\end{aligned}$$

Let $\mathcal{X} = \mathcal{L}^2([0,\pi])$ and $\mathcal{Y} = \mathbb{R}^1$, the real number σ is the magnitude of continuous noise, $dw(t)$ is a standard one dimension Brownian motion, $\Phi \in \mathcal{D}_{B_0}^b((-\infty,0],\mathcal{X})$, $b_k \geq 0$ for $k = 1,2,...,m$ and $\sum_{k=1}^m b_k < \infty$.

Define A an operator on $\mathcal{X}$ by $Au = \frac{\partial^2 u}{\partial x^2}$ with the domain

$$\mathcal{D}(A) = \left\{ u \in \mathcal{X} | u \text{ and } \frac{\partial u}{\partial x} \text{ are absolutely continuous, } u^{'}, u(0) = u(\pi) = 0 \right\}.$$

It is well known that A generates a strongly continuous semigroup $\mathcal{R}(t)$ which is compact, analytic and self adjoint. Moreover, the operator A can be expressed as

$$Au = \sum_{n=1}^{\infty} n^2 < u, u_n > u_n, \ u \in \mathcal{D}(A),$$

where $u_n(\zeta) = (\frac{2}{\pi})^{\frac{1}{2}} \sin(n\zeta)$, $n = 1, 2, ...$, is the orthonormal set of eigenvectors of A, and

$$\mathcal{R}(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} < u, u_n > u_n, \ u \in \mathcal{X}.$$

We assume that the following condition hold:
(1) The function b is measurable and

$$\int_0^{\pi} \int_0^{\pi} b^2(y, x) dy dx < \infty.$$

(2) Let the function $\frac{\partial}{\partial t} b(y, x)$ be measurable, let $b(y, 0) = b(y, \pi)$, and let

$$L_g = \left[\int_0^{\pi} \int_0^{\pi} (\frac{\partial}{\partial t} b(y, x))^2 dy dx \right]^{\frac{1}{2}} < \infty.$$

Assuming that conditions (1) and (2) are verified, then the problem (16) can be modeled as the abstract impulsive neutral stochastic partial integrodifferential equations with fractional Brownian motion and Poisson jumps of the form (1), as follows

$$g(t, x_t) = \int_0^{\pi} b(y, x) u(tsint, y) dy, \ f(t, x_t) = \hat{f}(t, u(tsint, x)),$$
$$\sigma(t) = \hat{\sigma}(t), \ \int_{\mathcal{U}} h(t, u(t-h), v, \zeta) = \int_{\mathcal{U}} \hat{h}(t, u(t-h), v, \zeta).$$

Acknowledgements
The authors sincerely thank the anonymous reviewer for his careful reading, constructive comments and fruitful suggestions to improve the quality of the manuscript.

References

[1] A. Anguraj and K. Ravikumar, Existence and stability results for impulsive stochastic functional integrodifferential equations with Poisson jumps, *Journal of Applied Nonlinear Dynamics*, 8(3) (2019), 407-417.

[2] A. Anguraj and A. Vinodkumar, Existence, uniqueness and stability results of impulsive stochastic semilinear neutral functional differential equations with infinite delays, *Electronic Journal of Qualitive Theory of Differential Equations*, 67 (2009), 1-13.

[3] B. Boufoussi and S. Hajji, Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space, *Statistics & Probability Letters*, 82(8) (2012), 1549-1558.

[4] B. Boufoussi and S. Hajji, Successive approximation of neutral functional stochastic differential equations with jumps, *Statistics & Probability Letters*, 80(5-6) (2010), 324-332.

[5] F. Biagini, Y. Hu, B. Oksendal and T. Zhang, *Stochastic calculus for Fractional Brownian motion and Applications*, Springer Science & Business Media, (2008).

[6] T. Caraballo, M. A. Diop, and A. A. Ndiaye, Asymptotic behaviour of neutral stochastic partial functional integrodifferential equations driven by a fraction Brownian motion, *J. Nonlinear Sci. Appl*, 7(8) (2014), 407-421.

[7] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, (1992).

[8] K. Dhanalakshmi and P. Balasubramaniam, Stability result of higher-order fractional neutral stochastic differential system with

infinite delay driven by Poisson jumps and Rosenblatt process, *Stochastic Analysis and Applications*, 38 (2019) 352–372.

[9] M. A. Diop, K. Ezzinbi and M. Lo, Existence and uniqueness of mild solutions to some neutral stochastic partial functional integrodifferential equations non-Lipschitz coefficients, *International Journal of Mathematics and Mathematical Sciences*, 4(1) (2012), 1-12.

[10] R. C. Grimmer, Resolvent opeators for integral equations in a Banach space, *Transactions of the American Mathematical Society*, 273 (1982), 333-349.

[11] T. E. Govindan, Stability of mild solutions of stochastic evolution equations with variable delay, *Stochastic Analysis and Applications*, 21 (2003), 1059-1077.

[12] D. N. Keck and M. A. McKibben, Abstract semilinear stochastic Ito-Volterta integrodifferential equations, *International Journal of Stochastic Analysis*, (2006).

[13] J. C. F. Kingman, *Poisson Processes and Poisson Rendom Measure*, Oxford Univ. Press, Oxford, (1993).

[14] A. Lin, Y. Ren and N. Xia, On neutral impulsive stochastic integrodifferential equations with infinite delays via fractional operators, *Mathematical and Computer Modelling*, 51 (2010), 413-424.

[15] J. Luo and T. Taniguchi, The existence and uniqueness for non-lipschitz stochastic neutral delay evolution equations driven by Poisson jumps, *Stoch. Dyn.*, 9(1) (2009), 627-63.

[16] J. Luo and K. Liu, Stability of infinite dimensional stochastic evolution equations with memory and Markovian jumps, *Stochastic processes Appl.*, 118 (2014), 864-895.

[17] B. Mandelbrot and V. Ness, Fractional Brownian motion, fraction noises and applications, *SIAM Reviews*, 10(4) (1968), 422-437.

[18] D. Nualart, *The Mallivavin Calculus and Related Topics*, Berlin: Springer, (2006).

[19] K. Ramkumar, K. Ravikumar and S. Varshini, Fractional neutral stochastic differential equations with Caputo fractional derivative: Fractional Brownian motion, Poisson jumps, and optimal control. *Stochastic Analysis and Applications*, 39(1) (2021), 157-176.

[20] J. Q. Tam, H. L. Wang and Y. F. Guo, Existence and uniqueness of solution to neutral stochastic functional differential equations with Poisson jumps, *Abstract and Applied Analysis*, (2012), 1-20.

[21] S. Tindel, C. A. Tudor and F. Viens, Stochastic evolution equations with fractional Brownian motion, *Probability Theory and Related Fields*, 127(2) (2003), 186-204.

[22] Xiao-Li Ding, Cao-Labora Daniel and Juan J. Nieto, A new generalized gronwall inequality with a double singularity and its applications to fractional stochastic differential equations, *Stochastic Analysis and Applications*, 37(6) (2019), 1042-1056.

Kasinathan Ramkumar
Assistant Professor of Mathematics
Department of Mathematics
PSG College of Arts & Science
Coimbatore, 641 014, India.
E-mail: ramkumarkpsg@gmail.com

Kasinathan Ravikumar
Associate Professor of Mathematics
Department of Mathematics
PSG College of Arts & Science
Coimbatore, 641 014, India.
E-mail: ravikumarkpsg@gmail.com

Elsayed Mohammed Elsayed
Full Professor of Mathematics
Department of Mathematics, Faculty of Science
King Abdulaziz University
Jeddah 21589, Saudi Arabia.
E-mail: emmelsayed@yahoo.com