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Original Research Paper

Quasi-Duality Result and Linearization in Multiobjective Quasiconvex Programming

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Abstract. In this paper, multiobjective optimization problems with nondifferentiable quasiconvex functions are considered. We obtain some duality results and a linear representation for the considered problems. Since the well-known strong duality result is not valid for the problems, we present a weaker form of it, named quasi-strong duality result.

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Keywords and Phrases: Quasiconvex optimization, Duality results, Linearization, subdifferential, Multiobjective programming.

1 Introduction

We consider the following multiobjective problem

$$
\begin{aligned}
(\mathrm{P}): \qquad & \inf\ \big(\varphi_1(x), \ldots, \varphi_p(x)\big) \\
& \text{s.t.} \quad \psi_t(x) \leq 0, \qquad t = 1, \ldots, q, \\
& x \in \Omega,
\end{aligned}
$$

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where Ω is a convex set in $\mathbb{R}^n$, and $\psi_1, \ldots, \psi_q, \varphi_1, \ldots, \varphi_p$, are quasiconvex functions from $\mathbb{R}^n$ to $\mathbb{R}$.

Since the feasible set of (P) is not necessarily convex, applying common methods of convex analysis is not applicable here. Therefore, we take the quasiconvex analysis approach. Because of we do not assume that the emerging functions are differentiable, we replace the derivatives appearing in the classical results by some subdifferentials, named star subdifferential and Penot subdifferential [14]. We use these subdifferentials as they have links with variational subdifferentials and they are more natural for quasiconvexity properties [14, 15].

If $p = 1$, (P) was studied by Penot [13] and the results are extended for semi-infinite case (i.e., the number of ψ_ts is infinite) in [6, 11]. Very recently, Kanzi *et al* [7] presented Karush-Kuhn-Tucker (KKT) types necessary and sufficient optimality conditions for (P) with $p > 1$. Since the presentation of duality results and also the linearization of nonlinear programming problems are two important applications of optimality conditions, in the present paper we focus on Mond-Weir [12] type dual problem and linear approximation of problem (P).

The structure of the subsequent sections of this paper is as follows: In Sect. 2, we define required definitions, theorems, and relations of quasiconvex analysis. The Mond-Weir type dual problem and linearization of (P) are studied in Sections 2 and 3, respectively.

2 Preliminaries

In this section we briefly overview some notions of quasiconvex analysis widely used in formulations and proofs of main results of the paper. For more details and discussion see [1, 5, 14, 15].

Given $x, y \in \mathbb{R}^n$, we write $x < y$ (resp. $x \leq y$) when $x_i < y_i$ (resp. $x_i \leq y_i$ and $x \neq y$) for $i = 1, \ldots, n$. Also, $x \leqq y$ means $x_i \leq y_i$ for $i = 1, \ldots, n$. The zero vector of $\mathbb{R}^n$ is denoted by 0_n.

Given a convex set $H \subseteq \mathbb{R}^n$, we denote by $N_H(x_0)$, the normal cone of H at $x_0 \in H$, i.e.,

$$N_H(x_0) := \big\{ d \in \mathbb{R}^n \mid \big\langle d, h - x_0 \big\rangle \leq 0, \qquad \forall h \in H \big\}.$$

The topological interior of H is denoted by $int(H)$.

Let ϕ be a function from $\mathbb{R}^n$ to $\mathbb{R}$, and let $x_0 \in \mathbb{R}^n$. The sublevel set and the strictly sublevel set of ϕ at x_0 are, respectively, defined by

$$\mathcal{S}(\phi, x_0) := \big\{x \in \mathbb{R}^n \mid \phi(x) \leq \phi(x_0)\big\},$$

$$\mathcal{S}^s(\phi, x_0) := \big\{x \in \mathbb{R}^n \mid \phi(x) < \phi(x_0)\big\}.$$

ϕ is said to be a quasiconvex and stictly quasiconvex function if for each $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, respectively, one has

$$\phi(\lambda x + (1-\lambda)y) \leq \max\ \{\phi(x), \phi(y)\},$$

$$\phi(\lambda x + (1-\lambda)y) < \max\ \{\phi(x), \phi(y)\}.$$

we can see if ϕ is quasiconvex function, its corresponding sublevel and strictly sublevel sets $\mathcal{S}(\phi, x_0)$ and $\mathcal{S}^s(\phi, x_0)$ are convex for all $x_0 \in \mathbb{R}^n$. Therefore, the following subdifferentials of ϕ at x_0 are well-defined:

$$\partial^{\S}\phi(x_0) := N_{\mathcal{S}^s(\phi,x_0)}(x_0) \setminus \{0\}, \qquad \partial^{\ddagger}\phi(x_0) := N_{\mathcal{S}(\phi,x_0)}(x_0).$$

Equivalently,

$$\partial^{\S}\phi(x_0) = \{d \in \mathbb{R}^n \setminus \{0\} \mid \phi(x) < \phi(x_0) \Rightarrow \langle d, x - x_0\rangle \leq 0\},$$

$$\partial^{\ddagger}\phi(x_0) = \{d \in \mathbb{R}^n \mid \phi(x) \leq \phi(x_0) \Rightarrow \langle d, x - x_0\rangle \leq 0\}.$$

$\partial^{\S}\phi(x_0)$ and $\partial^{\ddagger}\phi(x_0)$ are called, respectively, the star subdifferential and the Penot subdifferential of ϕ at x_0 in [6, 7, 14, 15]. Observe that, if ϕ is upper-semicontinuous (u.s.c.) on $\mathcal{S}^s(\phi, x_0)$ and there is no local minimizer of ϕ in $\phi^{-1}\big(\phi(x_0)\big)$, then ([14])

$$\partial^{\S}\phi(x_0) = \partial^{\ddagger}\phi(x_0) \setminus \{0\}. \tag{1}$$

3 Quasi-Duality Results

Assume that $\varphi_r : \mathbb{R}^n \to \mathbb{R}$, for $r \in I := \{1, \ldots, p\}$, are quasiconvex functions, and $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$\Phi(x) := \big(\varphi_1(x), \ldots, \varphi_p(x)\big).$$

A point $\tilde{x} \in \mathbb{R}^n$ is called the weak minimizer of Φ if there is no $x \in \mathbb{R}^n$ satisfying $\Phi(x) < \Phi(\tilde{x})$. The set of all weak minimizer of Φ is denoted by W^Φ.

Note that the problem (P) can be rewritten as

$$
\begin{aligned}
(\mathrm{P}): \qquad\qquad & \inf\ \Phi(x) \\
& \text{s.t.} \quad \psi_t(x) \leq 0, \qquad t \in T, \\
& x \in \Omega,
\end{aligned}
$$

in which $T := \{1, \dots, q\}$. The feasible set of (P) is denoted by M, i.e.,

$$M := \{x \in \Omega \mid \psi_t(x) \leq 0, \quad t \in T\}.$$

A point $\tilde{x} \in M$ is said to be a efficient (resp. an weakly efficient) solution to (P) if there is no $x \in M$ satisfying $\Phi(x) \leq \Phi(\tilde{x})$ (resp. $\Phi(x) < \Phi(\tilde{x})$). The set of all efficient (resp. weakly efficient) solutions of (P) is denoted by $E^{(P)}$ (resp. $W^{(P)}$).

For each feasible point $\tilde{x} \in M$, let

$$T(\tilde{x}) := \{t \in T \mid \psi_t(\tilde{x}) = 0\}.$$

Very recently, Kanzi and Soleimani-damaneh prove the following important theorem in [7, Theorem 3.1].

Theorem 3.1. (KKT necessary condition) *Suppose that $\tilde{x} \in W^{(P)}$ while $\tilde{x} \notin W^F$. If the functions φ_r, for $r \in I$, are strictly quasicinvex and u.s.c., the functions ψ_t, for $t \in T$, are u.s.c., and there exists a vector $z_* \in int(M)$ with $\psi_t(z_*) < 0$ for all $t \in T$ (Slater condition), then one has*

$$0 \in \sum_{r \in I} \partial^\S \varphi_r(\tilde{x}) + \sum_{t \in T(\tilde{x})} \partial^\ddagger \psi_t(\tilde{x}) + N_\Omega(\tilde{x}).$$

Notice, the above theorem is in the line of papers [2, 9, 10]. In fact, these results present the necessary KKT conditions for nondifferentiable multiobjective optimization problems under various assumptions.

The following technical lemma will be used in sequel.

Lemma 3.2. *Suppose that the quasiconvex function $f : \mathbb{R}^n \to \mathbb{R}$ is u.s.c. at $z_0 \in \mathbb{R}^n$. Then, for each $z \in \mathbb{R}^n$ with $f(z) < f(z_0)$ we have*

$$\left\langle \vartheta_0^\S, z - z_0 \right\rangle < 0, \qquad \forall \vartheta_0^\S \in \partial^\S f(z_0). \tag{2}$$

Proof. For given $\vartheta_0^{\S} \in \partial^{\S} f(z_0)$, we have $\vartheta_0^{\S} \neq 0$ by (1). Owing to $f(z) < f(z_0)$, we can write $\langle \vartheta_0^{\S}, z - z_0 \rangle \leq 0$. By indirect proof assume the inequality (2) does not hold, and hence $\langle \vartheta_0^{\S}, z - z_0 \rangle = 0$. Thus, there exists a sequence $\{p_\nu\}$ in $\mathbb{R}_+ := (0, +\infty)$ converging to $\langle \vartheta_0^{\S}, z - z_0 \rangle$.

From well-known Riez Representation Theorem [1, 5], we can find some $z_\nu \in \mathbb{R}^n$, for $\nu \in \mathbb{N}$, such that $p_\nu = \langle \vartheta_0^{\S}, z_\nu \rangle$. Therefore, we conclude that $\{z_\nu\}$ converges to $z - z_0$ and $\langle \vartheta_0^{\S}, z_\nu \rangle > 0$ for all $\nu \in \mathbb{N}$. This means

$$\langle \vartheta_0^{\S}, (z_\nu + z_0) - z_0 \rangle > 0.$$

So, the definition of $\partial^{\S} f(x_0)$ implies that $f(z_\nu + z_0) \geq f(z_0)$. From this inequality and upper-semicontinuity of f we conclude that

$$\lim_{\nu \to \infty} f(z_\nu + z_0) \geq f(z_0) \quad \Longrightarrow \quad f(z) = f(z - z_0 + z_0) \geq f(z_0),$$

which contradicts the assumption of lemma. Hence, (2) holds. $\square$

We now consider the following Mond-Weir [12] type dual problem to (P):

$$\begin{aligned} &(\text{MWD}) && \max\ \Phi(y) \\ &\qquad\text{s.t.} && 0_n \in \sum_{r \in I} \partial^{\S} \varphi_r(y) + \sum_{t \in T(y)} \partial^{\ddagger} \psi_t(y) + N_\Omega(y). \end{aligned}$$

Let Y denotes the feasible solution of (MWD), i.e.,

$$Y := \Big\{ y \in \mathbb{R}^n \mid 0_n \in \sum_{r \in I} \partial^{\S} \varphi_r(y) + \sum_{t \in T(y)} \partial^{\ddagger} \psi_t(y) + N_\Omega(y) \Big\}. \tag{3}$$

From now on, an weakly efficient solution of a "max" problem like the dual problem (MWD) is similarly defined as "min" problem by replacing "$<$" by "$>$".

The following theorem is quasiconvex version of results that presented in [4] by Clarke subdifferential for the problems with locally Lipschitz data.

Theorem 3.3. (Weak duality) *Suppose that $\tilde{x} \in M$ and $\tilde{y} \in Y$. If φ_i is u.s.c. for each $r \in I$, then $\Phi(\tilde{x}) \nless \Phi(\tilde{y})$.*

Proof. We present our proof in three steps.
Step 1: By the assumption of $\tilde{y} \in Y$ and (3), we find some $\vartheta_r^{\S} \in \partial^{\S}\varphi_r(\tilde{y})$, for $r \in I$, and $\vartheta_t^{\ddagger} \in \partial^{\ddagger}\psi_t(\tilde{y})$, for $t \in T(\tilde{y})$, and $\omega \in N_\Omega(\tilde{y})$ such that

$$\sum_{r\in I} \vartheta_r^{\S} + \sum_{t\in T(\tilde{y})} \vartheta_t^{\ddagger} + \omega = 0_n. \tag{4}$$

The assumption of $\tilde{x} \in M$ implies that

$$\psi_t(\tilde{x}) \leq 0 = \psi_t(\tilde{y}), \qquad \forall t \in T(\tilde{y}).$$

So, $\tilde{x} \in \mathcal{S}(\psi_t, \tilde{y})$ for each $t \in T(\tilde{y})$, and hence

$$\left\langle \vartheta_t^{\ddagger}, \tilde{x} - \tilde{y} \right\rangle \leq 0, \qquad \forall t \in T(\tilde{y}). \tag{5}$$

Step 2: By indirect proof we assume that $\Phi(\tilde{x}) < \Phi(\tilde{y})$, i.e., $\varphi_r(\tilde{x}) < \varphi_r(\tilde{y})$, for all $r \in I$. In view of Lemma 3.2 we deduce that

$$\left\langle \vartheta_r^{\S}, \tilde{x} - \tilde{y} \right\rangle < 0 \qquad \forall r \in I. \tag{6}$$

Step 3: Since $\omega \in N_\Omega(\tilde{y})$, the definition of normal cone implies that $\left\langle \omega, \tilde{x} - \tilde{y} \right\rangle \leq 0$. Adding this inequality with (5) and (6), we conclude that

$$\Big\langle \sum_{r\in I} \vartheta_r^{\S} + \sum_{t\in T(\tilde{y})} \vartheta_t^{\ddagger} + \omega \ , \ \tilde{x} - \tilde{y} \Big\rangle < 0.$$

which contradicts (4). This contradiction completes the proof. □

The following example shows that, Theorem 3.3 may not be valid if one replaces $\partial^{\ddagger}$ with $\partial^{\S}$ in definition of Y in (3).

Example 1. Consider the problem (P) by following data:

$$p = 2, \quad n = 1, \quad q = 1, \quad \varphi_1(x) = \varphi_2(x) = x, \quad \Omega = \mathbb{R},$$

$$\psi_1(x) = \begin{cases} -1 & x \in [2,4] \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $M = \mathbb{R}$. Considering $\tilde{y} = -1$, we get $T(-1) = \{1\}$, and

$$\mathcal{S}^s(\psi_1, -1) = [2,4] \implies \partial^{\S}\psi_1(-1) = (-\infty, 0),$$
$$\mathcal{S}^s(\varphi_1, -1) = \mathcal{S}^s(\varphi_2, -1) = (-\infty, -1)$$
$$\implies \partial^{\S}\varphi_1(-1) = \partial^{\S}\varphi_2(-1) = (0, +\infty).$$

Hence, $0 \in \partial^{\S}\varphi_1(-1) + \partial^{\S}\varphi_2(-1) + \partial^{\S}\psi_1(-1)$, while $\Phi(\tilde{x}) < \Phi(\tilde{y})$ for $\tilde{x} = -2$.

Notice, since $\partial^{\ddagger}\phi(\tilde{y}) \subseteq \partial^{\S}\phi(\tilde{y})$ for each quasiconvex function $\phi : \mathbb{R}^n \to \mathbb{R}$, Theorem 3.3 is still true if we replace $\partial^{\S}$ with $\partial^{\ddagger}$ in definition of Y in (3).

The forthcoming theorem presents quasi-strong duality relation between the prime problem (P) and the dual problem (MWD).

Theorem 3.4. (Quasi-strong duality) *Assume that $\tilde{x} \in W^{(P)}$ and $\tilde{y} \in W^{(MWD)}$. Under the hypothesis of Theorem 3.1, one has*

$$\Phi(\tilde{y}) \geqq \Phi(\tilde{x}) \not< \Phi(\tilde{y}).$$

Proof. Theorem 3.1 concludes that

$$0 \in \sum_{r \in I} \partial^{\S}\varphi_i(\tilde{x}) + \sum_{t \in T(\tilde{x})} \partial^{\ddagger}\psi_t(\tilde{x}) + N_{\Omega}(\tilde{x}).$$

This implies $\tilde{x} \in Y$, and hence $\Phi(\tilde{x}) \leqq \Phi(\tilde{y})$ by $\tilde{y} \in W^{(MWD)}$. Now, owing to weak duality Theorem 3.3, we obtain the result. □

Observe that, unlike when $p > 1$, if $p = 1$ (i.e., (P) is single-objective optimization problem), Theorem 3.4 guarantees that $\Phi(\tilde{y}) \geqq \Phi(\tilde{x}) \geqq \Phi(\tilde{y})$ which implies $\Phi(\tilde{x}) = \Phi(\tilde{y})$, named "strong duality result" in [6, Theotrem 4.2]. Thus, the quasi-strong Theorem 3.4 is an extension of strong duality theorem from single-objective to multi-objective quasiconvex optimization.

4 Linearization

Let $x_0 \in M$ be a feasible point for problem (P). For given $\vartheta^{\S} := (\vartheta_1^{\S}, \ldots, \vartheta_p^{\S}) \in \prod_{r \in I} \partial^{\S}\varphi_r(x_0)$, we consider the following linear multi-objective problem:

$$
\begin{aligned}
(\mathrm{LP}_{x_0}^{\vartheta^{\S}}) \qquad & \inf \Phi_{x_0}^{\vartheta^{\S}}(x) := \Big(\varphi_1(x_0) + \langle \vartheta_1^{\S}, x - x_0 \rangle, \ldots, \varphi_p(x_0) + \langle \vartheta_p^{\S}, x - x_0 \rangle\Big) \\
\text{s.t.} \qquad & \langle \vartheta_t^{\ddagger}, x - x_0 \rangle \leq 0, \qquad \forall t \in T(x_0),\ \forall \vartheta_t^{\ddagger} \in \partial^{\ddagger}\psi_t(x_0) \backslash \{0\}, \\
& x \in \Omega.
\end{aligned}
$$

Observe that (LP$_{x_0}^{\vartheta^\S}$) has infinite number of constraints in general, and so, it is a linear semi-infinite multiobjective programming problem (LSIP). The LSIPs have widely used in various theoretical and practical field; see, e.g., [3, 8].

The feasible set of (LP$_{x_0}^{\vartheta^\S}$) is denoted by $M_{x_0}^{\vartheta^\S}$, i.e.,

$$M_{x_0}^{\vartheta^\S} := \big\{x \in \Omega \mid \langle \vartheta_t^\ddagger, x - x_0\rangle \leq 0, \qquad t \in T(x_0),\ \vartheta_t^\ddagger \in \partial^\ddagger \psi_t(x_0)\backslash\{0\}\big\}.$$

The following theorems establish the relationship between the optimality of x_0 for (P) and (LP$_{x_0}^{\vartheta^\S}$). At first, we establish the efficient solutions of these problems coincide.

Theorem 4.1. *Let $\tilde{x} \in E^{(P_{\tilde{x}}^{\vartheta^\S})}$ for some $\vartheta^\S = (\vartheta_1^\S, \dots, \vartheta_p^\S) \in \prod_{r\in I} \partial^\S \varphi_r(\tilde{x})$. If all the functions $\varphi_r,\ r \in I,$ and $\psi_t,\ t \in T,$ are u.s.c., then $\tilde{x} \in E^{(P)}$.*

Proof. By contradiction assume that there exists some $x^* \in M$ such that $\Phi(x^*) \leq \Phi(\tilde{x})$, i.e., there exists $\ell \in I$ such that

$$\varphi_r(x^*) \leq \varphi_r(\tilde{x}), \quad \forall r \in I, \qquad \text{and} \qquad \varphi_\ell(x^*) < \varphi_\ell(\tilde{x}). \tag{7}$$

Thus, by definition of star subdifferential and Lemma 3.2, we deduce that

$$\langle \vartheta_r^\S, x^* - \tilde{x}\rangle \leq 0, \quad \forall r \in I, \qquad \text{and} \qquad \langle \vartheta_\ell^\S, x^* - \tilde{x}\rangle < 0.$$

The last inequalities and (7) imply that for each $r \in I$ we have

$$\varphi_r(x^*) + \langle \vartheta_r^\S, x^* - \tilde{x}\rangle \leq \varphi_r(\tilde{x}) + 0 = \varphi_r(\tilde{x}) + \langle \vartheta_r^\S, \tilde{x} - \tilde{x}\rangle,$$

and $\varphi_\ell(x^*) + \langle \vartheta_\ell^\S, x^* - \tilde{x}\rangle < \varphi_\ell(\tilde{x}) + \langle \vartheta_\ell^\S, \tilde{x} - \tilde{x}\rangle$. Thus,

$$\Phi_{\tilde{x}}^{\vartheta^\S}(x^*) \leq \Phi_{\tilde{x}}^{\vartheta^\S}(\tilde{x}). \tag{8}$$

On the other hand, since $\psi_t(x^*) \leq 0 = \psi_t(\tilde{x})$ for all $t \in T(\tilde{x})$, we have $\langle \vartheta_t^\ddagger, x^* - \tilde{x}\rangle \leq 0$ for $t \in T(\tilde{x})$. This implies that x^* is a feasible point for (LP$_{\tilde{x}}^{\vartheta^\S}$), and thus (8) gives a contradiction. The proof is complete. □

The following theorem shows that the weak efficient solutions of (P) and (LP$_{x_0}^{\vartheta^\S}$) are equal.

Theorem 4.2. *Let* $\tilde{x} \in W^{(P_{\tilde{x}}^{\vartheta^{\S}})}$ *for some* $\vartheta^{\S} = (\vartheta_1^{\S}, \ldots, \vartheta_p^{\S}) \in \prod_{r \in I} \partial^{\S}\varphi_r(\tilde{x})$. *If all the functions* φ_r, $r \in I$, *and* ψ_t, $t \in T$, *are u.s.c., then* $\tilde{x} \in W^{(P)}$.

Proof. In direct proof assume $\Phi(x^*) < \Phi(\tilde{x})$ for some $x^* \in M$. Similar to the proof of (8), we get

$$\Phi_{\tilde{x}}^{\vartheta^{\S}}(x^*) < \Phi_{\tilde{x}}^{\vartheta^{\S}}(\tilde{x}). \tag{9}$$

Repeating the proof of Theorem 4.1 shows that x^* is a feasible point for $(\mathrm{LP}_{\tilde{x}}^{\vartheta^{\S}})$. Thus, (9) contradicts $\tilde{x} \in W^{(P_{\tilde{x}}^{\vartheta^{\S}})}$, as required. $\square$

The converse of Theorem 4.1 is not true in general, and the following theorem presents a "weak version" of this converse. Of course, the following theorem implies that the converse of Theorem 4.2 is true.

Theorem 4.3. *For a given* $\tilde{x} \in W^{(P)}$, *suppose that the hypothesis of Theorem 3.1 are fulfilled. Then,* $\tilde{x} \in E^{(LP_{\tilde{x}}^{\vartheta^{\S}})}$ *for some* $\vartheta^{\S} \in \prod_{r \in I} \partial^{\S}\varphi_i(\tilde{x})$.

Proof. Take $T(\tilde{x}) = \{t_1, \ldots, t_k\}$. Applying Theorem 3.1, there exist some $\vartheta_r^{\S} \in \partial^{\S}\varphi_r(\tilde{x})$, $r \in I$, some $\vartheta_{t_v}^{\ddagger} \in \partial^{\ddagger}\psi_{t_v}(\tilde{x})$, $v = 1, \ldots, k$, and a $\eta \in N_{\Omega}(\tilde{x})$ satisfying

$$\sum_{r \in I} \vartheta_r^{\S} = -\sum_{v=1}^{k} \vartheta_{t_v}^{\ddagger} - \eta.$$

Assume that $x_* \in M_{\tilde{x}}^{\vartheta^{\S}}$ is arbitrarily given. Since

$$x_* \in \Omega, \qquad \text{and} \qquad \psi_{t_v}(\tilde{x}) \leq -\langle \vartheta_{t_v}^{\ddagger}, x_* - \tilde{x} \rangle, \quad \forall v = 1, \ldots, k,$$

we deduce that

$$\langle \sum_{r \in I} \vartheta_r^{\S}\ ,\ x_* - \tilde{x} \rangle = -\langle \sum_{v=1}^{k} \vartheta_{t_v}^{\ddagger}\ ,\ x_* - \tilde{x} \rangle - \underbrace{\langle \eta, x_* - \tilde{x} \rangle}_{\leq 0} \geq \sum_{v=1}^{k} \psi_{t_v}(\tilde{x}) = 0.$$

This implies that

$$\sum_{r \in I} \varphi_r(\tilde{x}) + \sum_{r \in I} \langle \vartheta_r^{\S}, x_* - \tilde{x} \rangle \geq \sum_{r \in I} \varphi_r(\tilde{x}), \qquad \forall x_* \in M_{\tilde{x}}^{\vartheta^{\S}}. \tag{10}$$

Now, if $\tilde{x}$ is not an efficient solution for $(\mathrm{LP}_{\tilde{x}}^{\vartheta^\S})$ with $\vartheta^\S := (\vartheta_1^\S, \ldots, \vartheta_p^\S) \in \prod_{r\in I} \partial^\S \varphi_r(\tilde{x})$, there exist $z \in M_{\tilde{x}}^{\vartheta^\S}$ and $\ell \in I$ such that

$$\varphi_r(\tilde{x}) + \langle \vartheta_r^\S, z - \tilde{x}\rangle \leq \varphi_r(\tilde{x}) + \langle \vartheta_r^\S, \tilde{x} - \tilde{x}\rangle = \varphi_r(\tilde{x}), \quad \forall r \in I,$$

$$\varphi_\ell(\tilde{x}) + \langle \vartheta_\ell^\S, z - \tilde{x}\rangle < \varphi_\ell(\tilde{x}) + \langle \vartheta_\ell^\S, \tilde{x} - \tilde{x}\rangle = \varphi_\ell(\tilde{x}).$$

Adding these inequalities for $r \in I$, we deduce that

$$\sum_{r\in I}\Big[\varphi_r(\tilde{x}) + \langle \vartheta_r^\S, z - \tilde{x}\rangle\Big] < \sum_{r\in I}\varphi_r(\tilde{x}),$$

which contradicts (10). □

An important point about the problem $(\mathrm{LP}_{\tilde{x}}^{\vartheta^\S})$ is the index set of its constraints, i.e., $T(\tilde{x})$. The following example shows that if we replace $T(\tilde{x})$ with T there, Theorems 4.1 and 4.2 will not be valid.

Example 2. Consider the problem (P) by following data:

$$p = 1, \quad n = 1, \quad T = \{-1, 1\}, \quad \varphi_1(x) = x^3, \quad \Omega = \mathbb{R},$$

$$\psi_1(x) = x^3, \qquad \psi_{-1}(x) = -x - 1.$$

Clearly, $M = [-1, 0]$. Considering $\tilde{x} = 0$, we get $T(0) = \{1\}$, and

$$\partial^\S\varphi_1(0) = (0, +\infty), \quad \partial^\ddagger\psi_1(0) = (0, +\infty), \quad \partial^\ddagger\psi_{-1}(0) = (-\infty, 0).$$

Hence, if we replace $T(\tilde{x})$ by T in $(\mathrm{LP}_{\tilde{x}}^{\vartheta^\S})$, we receive to the following problem:

$$\begin{array}{lll} (Q): \qquad & \max & \alpha x \\ & \text{s.t.} & \beta x \leq 0, \qquad \forall \beta > 0, \\ & & \gamma x - 1 \leq 0, \quad \forall \gamma < 0, \end{array}$$

where $\alpha > 0$. Since the feasible set of (Q) is $\{0\}$, then $\tilde{x}$ is the solution of (Q) while it is not solution of (P).

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