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Locally Quasiconvex Spaces and Fixed Point Theorems

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Abstract. We introduced the notions of *locally quasiconvex spaces* and *quasi-seminorms,* and investigate the natural relationships between these two notions. As applications, we obtain generalizations of some well known fixed point theorems and fixed set theorems that require neither metrizability, nor compactness nor the standard notion of convexity.

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1 Introduction

It is widely acknowledged that locally convex spaces (especially Banach spaces) provide the most natural and functional instance in which is done the major part of functional analysis. For thorough and up-to-date treatments of topological vector spaces, we refer the reader to [12]. Nevertheless, there are important examples of vector spaces whose topologies are not determined by norms. The best known examples of non-locally convex spaces are the spaces ℓ_p and $L_p[0,1]$,when $0 < p < 1$. The absence of genuine convexity may appear to be a stumbling block that can

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make simple-looking problems difficult. However, there are very sound reasons, as we shall see in the present note, to want to develop understanding of topological vector spaces beyond the scope of convexity (see for example [9]).

The study of fixed point theory has evolved rapidly over the past fifty years. Systematic studies have been done especially in trying to generalize and strengthen the fundamental ideas of Fixed Point Theorems and several interesting results have been obtained. Researchers have brought their efforts in various separate directions (see for example [1, 2, 3, 4, 6, 10, 11, 13]) but in general, it seems that central keys and unavoidable notions to most of the results in extension of the fixed point theorems are three properties, as a whole or separately: convexity, compactness and at least some form of metrizability.

The central goal of this short note is to obtain useful extensions of some of the classical Fixed Point Theorems to the more general setting of locally quasiconvex topological spaces. Our results not only forgo metrizability, but also weaken the convexity condition and more importantly at the same time relax the compactness requirement. What we want is a treatment of the subject which is not only unified, but also elegant and easily understood. To do so, we begin by giving a glimpse of the theory of quasi-seminormed spaces and then discuss its natural interconnection with the notion of local quasiconvex topological space. Most of the results obtained in this note pertaining to fixed point theorems parallel those results already obtained under different settings by the author in [13], and some parts of the proofs are taken over with trivial notational changes. The major feature of these new results are the fact that compactness is no longer a requirement in most of their statements.

For most of the results obtained in this note, it does not matter whether the field of scalars is real or complex. Therefore, we shall simply use the symbol $\mathbb{F}$ for either of both fields. We have made every effort to use only standard notations throughout the text. New notation are defined as they are introduced in the text.

2 Quasi-seminorm and Local Quasiconvexity

The notions we introduce in this section are certainly of interest of their own.

Definition 2.1. A *quasi-seminorm* on a vector space X is a functional $p : X \to [0, \infty]$ with the properties:

- there exists $\kappa > 0$ such that $p(x + y) \leq \kappa [p(x) + p(y)]$ for $x, y \in X$;
- $p(\alpha x) = |\alpha| p(x)$ for $x \in X$ and $\alpha \in \mathbb{F}$.

κ is known as the *modulus of concavity* of the quasi-seminorm.

Quasi-seminorms arise naturally in many ways in analysis. Clearly, seminorm and quasinorm are quasi-seminorms. If $a_1, \ldots, a_n$ are non-negative scalars and $p_1, \ldots, p_n$ are quasi-seminorms then $\sum_{i=1}^{n} a_i p_i$ and $\max\{a_1 p_1, \ldots, a_n p_n\}$ are quasi-seminorms.

It follows from the above definition that a quasi-seminorm p is symmetric, that is, $p(x) = p(-x)$ for $x \in X$; $p(0) = 0$. The set $\ker p := \{x \in X : p(x) = 0\}$ is a linear subspace of X: indeed if $x, y \in \ker p$, then

$$p(\alpha x + \beta y) \leq \kappa [p(\alpha x) + p(\beta y)] = \kappa [|\alpha| p(x) + |\beta| p(y)] = 0.$$

From $p(x) \leq \kappa \left[p(\frac{y}{\kappa}) + p(x - \frac{y}{\kappa}\right]$ and $p(y) \leq \kappa \left[p(\frac{x}{\kappa}) + p(y - \frac{x}{\kappa}\right]$, it follows that

$$|p(x) - p(y)| \leq \max\{p(\kappa x - y), p(\kappa y - x)\}. \tag{1}$$

If p is a quasi-seminorm on a vector space X, then the set $V_p = \{x \in X : p(x) < 1\}$ is absorbing and balanced and is called the *open unit ball* determined by p. The *closed unit ball* is $\overline{V}_p = \{x \in X : p(x) \leq 1\}$. The following properties only require routine verifications:

- If q is a quasi-seminorm on X then $p \leq q$ if and onlyn if $V_q \subset V_p$;
- For any $\alpha > 0$, $\alpha V_p = \{x \in X : p(x) < \alpha\} = V_{\frac{1}{\alpha}p}$;
- For any $x \in X$, $x + V_p = \{x \in X : p(x - y) < 1\}$.

We denote by $\widehat{A}$ the convex hull of a given subset A of a vector space, i.e. the set of all convex combinations of elements of A. It is easy to see that if the set A is balanced and absorbing then so is its convex hull.

Definition 2.2. We say that a subset A of a topological vector space is *κ-convex* if $A \subset \widehat{A} \subset \kappa A$ for some $\kappa \geq 1$.

Plainly, a subset A of X is 1-convex if and only if it is convex. If A is κ_0-convex then it is κ-convex for all $\kappa \geq \kappa_0$. If p is a quasi-seminorm, and if $p(x), p(y) < 1$, then for all $t \in [0,1]$,

$$p\left(tx + (1-t)y\right) \leq \kappa\left[tp(x) + (1-t)p(y)\right] < \kappa.$$

That is, the ball V_p is κ-convex. On the other hand, since the ball V_p is balanced and absorbing, it is quickly seen that so is $\widehat{V_p}$.

The converse is the object of the following important proposition. First, we define the *gauge (Minkowski functional)* of an absorbing and balanced (not necessarily convex) subset K of a vector space X as

$$p_K(x) = \inf\left\{t > 0 : x/t \in K\right\}.$$

Proposition 2.3. *Let K be a κ-convex balanced absorbing set in a vector space X. Then the functional $x \mapsto p_K(x)$ defines a quasi-seminorm on X with modulus of concavity κ.*

Proof. Since K is aborbing, the set $\{t > 0 : x/t \in K\}$ is nonempty. For $z \in aK + bK$, then $z = ax + by$ for some $x, y \in K$. Since K is κ-convex,

$$\frac{z}{s+t} = \frac{s}{s+t}x + \frac{t}{s+t}y \in \kappa K$$

and thus $z \in (a+b)\kappa K$. Hence, $aK + bK \subset (a+b)\kappa K$. Thus if $x \in aK$ and $y \in bK$ then $x + y \in (a+b)\kappa K$. Thus $p_K(x+y) \leq \kappa(a+b)$. Since a and b are arbitrary, it follows that $p_K(x+y) \leq \kappa(p_k(x) + p_K(y))$.

Let $a \in \mathbb{F} \setminus \{0\}$. For $t > 0$, $ax \in tK$ if and only if $x \in \frac{t}{a}K = \left|\frac{t}{a}\right| K = \frac{t}{|a|}K$. Thus

$$p_K(ax) = |a| \inf\left\{\frac{t}{|a|} > 0 : x \in \frac{t}{|a|}K\right\} = |a|\, p_K(x).$$

The proof is complete. $\square$

Up to now we were in the realm of vector space with no topology. We now introduce the following definitions:

Definition 2.4. A topological vector space X is said to be *locally quasiconvex* if X has a neighborhood base of 0 consisting of κ-convex sets for some $\kappa \geq 1$. The smallest such κ is called the *index of quasiconvexity* of X.

For example, for $X = \mathbb{R}^2$ endowed with the quasinorm $p((x,y)) = \left(\sqrt{|x|} + \sqrt{|y|}\right)^2$, the topological space $\left(\mathbb{R}^2, p\right)$ is locally quasiconvex with index of quasiconvexity equal to 2.

Let $\mathcal{P}$ be a family of quasi-seminorms on a vector space X all with modulus of concavity κ. Since $\mathcal{S} = \{V_p : p \in \mathcal{P}\}$ consists of balanced absorbing κ-convex sets, the collection of positive multiples of finite intersections of sets from S is a base at 0 for a locally quasiconvex topology $T_{\mathcal{P}}$ for X with quasiconvexity index κ. It is called the topology determined by $\mathcal{P}$. The topology $T_{\mathcal{P}}$ is Hausdorff if and only if $\mathcal{P}$ separates points in X, that is, if and only if for each nonzero $x \in X$, there is a $p \in \mathcal{P}$ such that $p(x) \neq 0$. Thus we have:

Proposition 2.5. *Every family of quasi-seminorms on a vector space generates a locally quasiconvex vector topology.*

Our next result shows that this is the only way, that is, any locally quasiconvex topological vector space is determined by a family of quasi-seminorms.

Proposition 2.6. *Let X be a locally quasiconvex topological vector space with quasiconvexity index $\kappa > 0$. Let $\mathcal{N}$ be a local base of neighborhood consisting of κ-convex sets. Then we have:*

1. *$V \subset \{x \in X : p_V(x) < 1\} \subset \kappa V$ for every $V \in \mathcal{N}$.*
2. *$\{p_V : V \in \mathcal{N}\}$ is a separating family of continuous quasi-seminorms.*

Proof. Let $x \in V$. Since V is open, there exists $t > 1$ such that $x/t \in V$, i.e. $p_V(x) < 1$. If $p_V(x) < 1$, then there exists $\alpha > 1$ such that $\alpha x \in V$. Since V is κ-convex, $x \in \kappa V$. Hence, $V \subset \{x \in X : p_V(x) < 1\} \subset \kappa V$.

It is a consequence of the continuity at 0 of the application $\alpha \mapsto \alpha x$ that p_V takes values in $[0, \infty]$. Clearly, $p_V(\alpha x) = |\alpha| \, p_V(x)$ for $x \in X$ and $\alpha \in \mathbb{F}$. Since K is a neighborhood of 0 it is absorbing. It follows from the κ-convexity of V and the relation

$$\frac{x+y}{s+t} = \frac{s}{s+t}\frac{x}{s} + \frac{t}{s+t}\frac{y}{t},$$

that if $\frac{x}{s}, \frac{y}{t} \in V$, then $\frac{x+y}{s+t} \in \kappa V$. Thus $p_V(x+y) \leq \kappa \, [p_V(x) + p_V(y)]$ for $x, y \in X$. That is, p_V is a quasi-seminorm on X.

Let $x \neq 0$. There exists $V \in \mathcal{N}$such that $x \notin V$. Then $p_V(x) \geq 1 > 0$. Thus $\{p_V : V \in \mathcal{N}\}$ is separating.

The continuity of the p_V's follows from the inequality

$$|p_V(x) - p_V(0)| \leq \max\{p_V(\kappa x - 0), p_V(k0 - x)\} = \kappa p_V(x).$$

Since $\left\{\frac{1}{n}V : n \in \mathbb{N}\right\}$ is a local base, if $x \in \frac{1}{n}V$ then

$$|p_V(x) - p_V(0)| \leq \kappa p_V(x) = \frac{\kappa}{n} p_V(nx) < \frac{\kappa}{n}.$$

Thus for every $\epsilon > 0$, there exists $U \in \mathcal{N}_0$ such that $p_V(U) \subset [0, \kappa\epsilon)$. □

We shall next see that it is quite easy and natural to give generalizations of the all important notions of continuity, Cauchy-ness, and boundedness relative to the seminorm settings to the more general case of quasi-seminormed spaces.

Let $\mathcal{P}$ be a base of continuous quasi-seminorms for a locally quasi-convex topological vector space X with quasiconvexity index $\kappa > 0$.

- A subset A of X is said to converge to $x_0 \in X$ if for every $\epsilon > 0$, there exists $N \in 2^{|A|}$ such that $\sup_{p \in \mathcal{P}} p(x - x_0) < \epsilon$ for every $x \in A \setminus N$.

- A subset A of X is said to be Cauchy if for every $\epsilon > 0$, there exists $N \in 2^{|A|}$ such that $\sup_{p \in \mathcal{P}} p\,(x - y) < \epsilon$ for $x, y \in A \setminus N$.

- A subset A of X is said to be bounded if there exists $M > 0$ such that $\sup_{p \in \mathcal{P}} p(x) < M$ for all $x \in A$.

We shall also use the following definition (see page 172 of [12]) to help us weaken the compactness requirement in fixed point results.

Definition 2.7. A topological vector space X is said to be *quasicomplete* if each closed bounded subset of X is complete.

Theorem 2.8. *Let p be a quasi-seminorm on a topological vector space X and $\mathcal{N}_0$ local base of neighborhoods of 0. Then the following are equivalent:*

1. *p is continuous at 0.*
2. *p is uniformly continuous.*
3. *V_p is open.*

Proof. The chain of implications $2 \Rightarrow 3 \Rightarrow 1$ is clear. To see $1 \Rightarrow 2$, we note that continuity at 0 means that for every $\epsilon > 0$, there exists a balanced $V \in \mathcal{N}_0$ such that $p(V) \subset [0, \epsilon)$. Take a neighborhood $U \in \mathcal{N}_0$ such that $\kappa U - U \subset V$. For $x, y \in U$, $\kappa x - y, \kappa y - x \in \kappa U - U \subset V$, so $\max\{p(\kappa x - y), p(\kappa y - x)\} < \epsilon$. The inequality (1) yields the uniform continuity. $\square$

Recall that a nest is a set of subsets that is linearly ordered by inclusion. The *Cantor Intersection Principle* states that a nest of nonempty compact subsets of a topological space has nonempty intersection. The following extension of such a result to the setting of locally quasiconvex topological vector space X will be useful later.

Theorem 2.9. *Let $\mathcal{P}$ be a base of continuous quasi-seminorms for a locally quasiconvex topological vector space X. Assume that X is quasicomplete. Let $\mathcal{A} = \{A_\alpha : \alpha \in \Omega\}$ be a nested net of nonempty closed bounded subsets of X. If $\lim_{\mathcal{A}} \sup_{p\in\mathcal{P}} \sup_{x,y\in A_\alpha} p(x-y) = 0$, then $\bigcap_\alpha A_\alpha$ contains exactly one point.*

The limit in the statement $\lim_{\mathcal{A}} \sup_{p\in\mathcal{P}} \sup_{x,y\in A_\alpha} p(x-y) = 0$ is to be understood in the sens that for every $\epsilon > 0$, there exists $A_{\alpha_\epsilon} \in \mathcal{A}$ such that for every $A_\beta \in \mathcal{A}$, $A_\beta \subset A_{\alpha_\epsilon}$ implies $\sup_{p\in\mathcal{P}} \sup_{x,y\in A_\beta} p(x-y) < \epsilon$.
Proof. Let β such that $A_\beta \subset A_\alpha$. Fix $A_\alpha \in \mathcal{A}$. For each $A_\beta \subset A_\alpha$ pick $x_\beta \in A_\beta$. Let E_α be the collection of such x_β. The condition $\lim_{\mathcal{A}} \sup_{p\in\mathcal{P}} \sup_{x,y\in A_\alpha} p(x-y) = 0$ implies that the set E_α is Cauchy.

Since A_α is closed and bounded and X is quasicomplete, E_α converges to some point $x \in A_\alpha$. This holds for all $\alpha \in \Omega$, thus $x \in \bigcap_\alpha A_\alpha$. Now suppose to the contrary that the intersection $\bigcap_\alpha A_\alpha$ contains another point $y \neq x$. Then there exists $\epsilon > 0$ and $p \in \mathcal{P}$ such that $p(y - x) > 0$. This contradicts the fact that $\lim_{\mathcal{A}} \sup_{p\in\mathcal{P}} \sup_{x,y\in A_\alpha} p(x - y) = 0$. The proof is complete. $\square$

3 Fixed Point Theorems in Locally Quasiconvex Spaces

Let X be a topological vector space. Let $T : X \to X$ be a mapping and A a nonempty subset of X satisfying $T(A) \subset A$. A point $x^* \in A$ is said to be a fixed point of T if $T(x^*) = x^*$. We shall use the common standard notation for the n-th iteration of a mapping $f : E \to E$ as follows

$$f^n(x) = f\left(f\left(\cdots\left(f(x)\right)\right)\right).$$

Let us agree to say that a function $\varphi : [0, \infty) \to [0, \infty)$ is *contractant* if it is increasing and $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. An example of contractant function is $t \mapsto qt$ where $q \in (0, 1)$. Note that every contractant function $\varphi : [0, \infty) \to [0, \infty)$ has the property that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.

Let X be a locally quasiconvex topological vector space with quasiconvexity index $\kappa > 0$. Let $\mathcal{P}$ be a base of continuous quasi-seminorms for the topology of X. We denote by δ the set function $\delta : 2^X \to [0, \infty]$ defined by

$$\delta(E) = \sup_{p\in\mathcal{P}} \sup_{x,y\in E} p(x - y).$$

We say that a mapping $T : X \to X$ is *a quasicontraction* on a nonempty subset A of X if $T(A) \subset A$ and if there exists a contractant function $\varphi : [0, \infty) \to [0, \infty)$ such that for every $E \subset A$ such that $TE \subset E$

$$\delta\left(TE\right) < \varphi\left(\kappa\delta\left(E\right)\right). \tag{2}$$

We note that if X is a locally quasiconvex topological vector space, a mapping $T : X \to X$ is continous at a point $a \in X$ if for every $\epsilon > 0$, there exists $r > 0$ such that $\sup_{p\in\mathcal{P}} p\left(Tx - Ta\right) \leq \epsilon$ whenever

$\sup_{p\in\mathcal{P}} p(x-a) < r$. It is then clear that if $T : X \to X$ is a quasicontraction mapping then it is necessary continuous.

We are now ready to state and prove a quasi-seminormed version of the *Matkowski's Fixed Point Theorem.*

Theorem 3.1. *Let X be a quasicomplete locally quasiconvex topological vector space with quasiconvexity index $\kappa > 0$. Let $T : X \to X$ be a quasicontraction mapping on a closed bounded subset A of X. Then T admits a unique fixed point $a \in A$. Furthermore, if $x_0 \in A$, the sequence $x_n = T(x_{n-1})$, $n = 1, 2, \ldots$ of elements of A converges to a.*

Proof. Let $\varphi : [0,\infty) \to [0,\infty)$ be a contractant mapping for T and let $\psi : [0,\infty) \to [0,\infty)$ be defined by $\psi(t) = \varphi(\kappa t)$. Choose an arbitrary $x \in A$ and consider the set $C = \{T^n x : n \in \mathbb{N}\}$. Then clearly, $C \subset A$. Thus $\delta(TC) < \varphi(\kappa\delta(C)) = \psi(\delta(C))$. Iteratively, $\delta(T^n C) < \psi^n(\delta(C))$. Since $\lim_{n\to\infty} \psi^n(\delta(A)) = 0$, given $\epsilon > 0$, we can choose N large enough so that for $n > N$, we have $\delta(T^n C) < \epsilon$. Since for every $k, n \in \mathbb{N}$, $T^n x$ and $T^{n+k}x$ are both in $T^n C$, it follows that for every $k \in \mathbb{N}$ and $n > N$

$$\sup_{p\in\mathcal{P}} p\left(T^n x - T^{n+k}x\right) \leq \delta(T^n C) < \epsilon.$$

This shows that the set C is Cauchy. Since X is quasicomplete, $C \subset A$, A is closed and bounded, C converges to some $a = \lim_{n\to\infty} C \in A$. The continuity of T implies that $a = Ta$.

For the uniqueness, assume that $b \in A$ such that $b = Tb$ and $b \neq a$. Then there exists $p \in \mathcal{P}$ such that $p(b-a) > 0$. It follows that

$$p(b-a) = p(Tb - Ta) = \cdots = p(T^n b - T^n a) \leq \delta(T^n A)$$

for all $n \in \mathbb{N}$. Since $\delta(T^n A) \to 0$ as $n \to 0$, it follows that $p(b-a) = 0$. Contradiction! The proof is complete. $\square$

Remark 3.2. It is worth noticing that the strength of the result of Theorem 3.1 lies on the facts that its statement weakens the convexity requirement and at the same time relaxes the compactness condition.

We say that a mapping $T : X \to X$ is *quasi-Lipschitz* on a subset A of X such that $TA \subset A$, if there exists a constant $q \in (0, \kappa^{-1})$ such that for every $E \subset A$ such that $TE \subset E$, $\delta(TE) < q\kappa\delta(E)$. As an immediate corollary of the above extension of the Matkowski's Fixed Point Theorem, we have the following extension of the Banach Fixed Point Theorem.

Theorem 3.3. *Let X be a quasicomplete locally quasiconvex topological vector space with quasiconvexity index $\kappa > 0$. Assume that $T : X \to X$ is quasi-Lipschitz with constant $q \in (0, \kappa^{-1})$ on a closed bounded subset A of X. Then T admits a unique fixed point $a \in A$. Furthermore, if $x_0 \in A$, the sequence $x_n = T(x_{n-1})$, $n = 1, 2, \ldots$ of elements of A converges to a.*

Proof. It suffices to notice that for $q \in (0, \kappa^{-1})$ the mapping $\varphi : [0, \infty) \to [0, \infty)$ given by $t \mapsto qt$ is contractant, and the quasi-Lipschitz proposition of T implies that T is a quasicontraction. It suffices then to apply Theorem 3.1. □

An immediate corollary is as follows:

Theorem 3.4. *Let X be a quasicomplete locally quasiconvex topological vector space with quasiconvexity index $\kappa > 0$. Assume that $T : X \to X$ is a mapping such that for some natural number m, T^m is quasi-Lipschitz on a closed bounded subset A of X. Then T admits a unique fixed point.*

Proof. The case $m = 1$ is exactly that of Theorem 3.3. Assume that $m > 1$. The mapping $S = T^m$ satisfies the hypotheses of Theorem 3.3, hence it admits a unique fixed point, say a in A. Then

$$STa = T^{m+1}a = TSa = Ta.$$

In other words, Ta is also a fixed point of S. By uniqueness of fixed point, $T(a) = a$. That is, a is a fixed point for T. To see that a is unique, assume that $b = Tb$. Then $Sb = T^m b = b$. That is, b is a fixed point for S and hence $b = a$. The proof is complete. □

Our next result is a consequence of our version of the Cantor Intersection Principle in Theorem 2.9. Let X be a quasicomplete locally quasiconvex topological vector space with quasiconvexity index $\kappa > 0$. Fix a sequence $\{a_n\}$ of positive numbers converging to 0. Given a subset A of X, let us agree to say that a mapping $T : X \to X$ is *nearly* *quasi-Lipschitz* with respect to $\{a_n\}$ on A if for each $n \in \mathbb{N}$ there exists $q_n \geq 0$ such that for every $E \subset A$ such that $TE \subset E$, we have $\delta(T^nE) < q_n\left(\kappa\delta(E) + a_n\right)$. The smallest such constant q_n will be denoted by $q(T^n)$.

Theorem 3.5. *Let X be a quasicomplete locally quasiconvex topological vector space with quasiconvexity index $\kappa > 0$. Assume that $T : X \to X$ is nearly quasi-Lipschitz with respect to $\{a_n\}$ on a closed bounded $A \subset X$. Suppose that* $\limsup_{n\to\infty}[q(T^n)]^{1/n} < 1$. *Then T admits a unique fixed point $a \in A$. Furthermore, if $x_0 \in A$, the sequence $x_n = T(x_{n-1})$, $n = 1, 2, \ldots$ of elements of A converges to a.*

Proof. Let $M = \sup\{a_n : n \in \mathbb{N}\}$. Let $x_0 \in X$ and consider the sequence defined by $x_n = T^n x_0$. Fix an open and bounded set U containing both x and Tx_0. Then for each $n \in \mathbb{N}$, both $T^n x_0$ and $T^{n+1}x_0$ are in T^nU, and we observe that

$$\delta\left(T^nU\right) < q_n\left(\kappa\delta(U) + a_n\right) \leq q_n\left(\kappa\delta(U) + M\right).$$

It follows that $\sup_{p\in\mathcal{P}} p\left(T^nx_0 - T^{n+1}x_0\right) < q_n\left(\kappa\delta(U) + M\right)$. Iteratively, for each $k \in \mathbb{N}$we have

$$\sup_{p\in\mathcal{P}} p\left(T^nx_0 - T^{n+k}x_0\right) < \sum_{i=1}^{k} q_{n+i}\left(\kappa\delta(U) + M\right).$$

Now $\limsup_{n\to\infty}[q(T^n)]^{1/n} < 1$ implies that the series $\sum q_{n+i}\left(\kappa\delta(U) + M\right)$ converges and thus the sequence $\epsilon_{n,k} = \sum_{i=1}^{k} q_{n+i}\left(\kappa\delta(U) + M\right) \to 0$ as $n \to \infty$.

Now let $\mathbb{N}\times\mathbb{N}$ be dierected as follows: $(n, k) \succ (n', k')$ if $n > n'$ or $n = n'$ and $k > k'$. Consider

$$C_{n,k} = \left\{x \in A : \sup_{p\in\mathcal{P}} p\left(T^nx - x\right) \leq \sum_{i=1}^{k} q_{n+i}\left(\kappa\delta(U) + M\right)\right\}.$$

We observe that $\{C_{n,k} : (n, k) \in \mathbb{N}\times\mathbb{N}\}$ is nested net of subsets of X such that

$$\lim_{(n,k)} \sup_{p\in\mathcal{P}} \sup_{x,y\in C_{n,k}} p(x - y) = 0.$$

The extension Theorem 2.9 now finishes the proof. □

4 Fix Set Theorems in Quasiconvex Spaces

We denote by $\mathcal{K}(X)$ the space of non-empty compact subsets of a given metric space X. It is a well-known fact that $\mathcal{K}(X)$ is a complete metric space when endowed with the Hausdorff metric. The Hutchinson's Theorem (see for example [5, 7]) states that if $\{K_1, ..., K_n\}$ is a family of contractions on X with respective Lipschitz constants $\{k_1, ..., k_n\}$, then the operator K defined on $\mathcal{K}(X)$ by $\mathcal{K}(A) = \bigcup_{i=1}^{n} \mathcal{K}(A_i)$ is a contraction with Lipschitz constant equal to $k = \max\{k_1, ..., k_n\}$. The Banach Contraction Principle then implies the existence of a compact set E such that $K(E) = E$.

In this section, we seek for a version of such a result in the setting of locally quasiconvex vector spaces. First, we notice that if X is a vector space, then the set $\mathring{2}^X$ of all nonempty subsets of X has a structure of a vector space with the operations:

1. $A + B = \{a + b : a \in A, b \in B\}$ for $A, B \in \mathring{2}^X$.
2. $\lambda A = \{\lambda a : a \in A\}$ for $A \in \mathring{2}^X$ and for $\lambda \in \mathbb{F}$.

Assume that X has a topology that makes it a vector space, and let $\mathcal{B}$ be a local base for the such a topology. Then the space $\mathring{2}^X$ can naturally be topologized by defining a neighborhood of $A \in \mathring{2}^X$, a set of the form $A + V$ where $V \in \mathcal{B}$. If X is quasicomplete locally quasiconvex topological vector space with quasiconvexity index $\kappa > 0$ then so is $\mathring{2}^X$. If $\mathcal{P}$ is a base of continuous quasi-seminorms for a locally quasiconvex topological vector space X with quasiconvexity index $\kappa > 0$, then for every $p \in \mathcal{P}$, the functionals defined by $A \mapsto p(A) = \sup_{x \in A} p(x)$ is a base of continuous quasi-seminorms for $\mathring{2}^X$.

Our next result is an extension of the *Hutchinson's Fixed Point Theorem.*

Theorem 4.1. *Let X be a quasicomplete locally quasiconvex topological vector space with quasiconvexity index $\kappa > 0$. Let $\mathcal{B}(X)$ be the space of nonempty closed and bounded subsets of X. Let $T : \mathcal{B}(X) \to \mathcal{B}(X)$ be a monotone mapping, that is, $TA \subset TA'$ whenever $A \subset A'$. If there exists $A \in \mathcal{B}(X)$ such that $TA \subset A$, then there exists $B \subset A$ such that $TB = B$.*

Proof. The quasicompletenss quickly implies that $\mathcal{B}(X)$ is a subspace of $\mathring{2}^X$. Let $\mathcal{H}(X)$ be the subsets of $\mathcal{B}(X)$ consisting of sets A satisfying $TB \subset B$. By hypothesis, $\mathcal{H}(X)$ is not empty. We order $\mathcal{H}(X)$ by inclusion. The Hausdorff Maximality Principle implies the existence of maximal nest $\Gamma = \{A_i : i \in I\}$ of $\mathcal{H}(X)$. Let $B = \bigcap_{i\in I} A_i = \lim_\Gamma A_i$. Clearly,

$$\lim_\Gamma \sup_{p\in\mathcal{P}} p(A_i - B) = \lim_\Gamma \sup_{p\in\mathcal{P}} \sup_{A_j, A_k\in\Gamma} p(A_j - A_k) = 0.$$

By the Cantor Intersection Theorem 2.9, B is nonempty closed and bounded of X. On the other hand, since $TA_i \subset A_i$ for all $i \in I$, we also have $TB \subset B$ and hence by monotonicity $T^2B \subset TB$. By maximality of Γ, we have $TB = B$. □

We note that no continuity properties is required in the above Theorem 2.9. A special case is as follows:

Theorem 4.2. *Let X be a quasicomplete locally quasiconvex topological vector space with quasiconvexity index $\kappa > 0$. Let $T_i : X \to X$, $i = 1, \dots, n$ be a finite collection of continuous mappings. If there exists a closed bounded subset A of X such that $T_iA \subset A$ for $i = 1, \dots, n$, then there exists a closed bounded subset B of A such that $TB = B$.*

Proof. It suffice to notice that the mapping $T : \mathcal{B}(X) \to \mathcal{B}(X)$ defined by $TA = \bigcup_{i=1}^n T_iA$ satisfies the hypothesis of Theorem 4.1. □

We note that compactness was not required for the result of Theorem 4.1. We finish this note with another variant of the above Theorem 4.2:

Theorem 4.3. *Let X be a locally quasiconvex topological vector space with quasiconvexity index $\kappa > 0$. Let $\mathcal{C}(X)$ be the space of nonempty complete subsets of X. Let $T : \mathcal{C}(X) \to \mathcal{C}(X)$ be a monotone mapping, that is, $TA \subset TA'$ whenever $A \subset A'$. If there exists $A \in \mathcal{C}(X)$, A bounded such that $TA \subset A$ then there exists $B \subset A$ such that $TB = B$.*

Proof. It suffices to notice that the space $\mathcal{B}(X)$ of nonempty closed and bounded subsets of X is a subspace of the complete locally quasiconvex topological vector space $\mathcal{C}(X)$ and that if A is bounded then $A \in \mathcal{K}(X)$. Theorem 4.1 then applies and finishes the proof. □

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References

[1] M. Abtahi, Z. Kadelburg, S. Radenovic, *Fixed points of Ciric-Matkowski-type contractions in ν-generalized metric spaces*, RACSAM, (2016).

[2] R. P. Agarwal, D. O. Regan, D. R. Sahu, *Fixed Point Theory for Lipschitziantype Mappings with Applications*, Springer, New York, (2009).

[3] I. A. Bakhtin, The contraction mapping principle in quasimetric spaces, *Funct. Anal., Unianowsk Gos. Ped. Inst*, 30 (1989) 26-37.

[4] M. Bota, V. Ilea, E. Karapinar, O. Mlesnite, On φ–contractive multi-valued operators in b-metric spaces and applications, *Appl. Math. Inf. Sci*, 9 (2015) 2611-2620.

[5] R. Brooks, K. Schmitt, B. Warner, Fixed set theorems for discrete dynamics and nonlinear boundaryvalue problems, *Electronic Journal of Differential Equations*, 2011 (56) (2011) 1-15.

[6] C. Chifu, G. Petrusel, Fixed points for multivalued contractions in b-metric spaces with applications to fractals, *Taiwanese J. Math*, 18 (2014) 1365-1375. https://doi.org/10.11650/tjm.18.2014.4137

[7] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, *Atti Sem. Mat. Fis. Univ. Modena*, 46 (1998) 263-276.

[8] J. Hutchinson, Fractals and self similarity, *Indiana Univ. Math. J.*, 30 (1981) 713-747. https://doi.org/10.1512/iumj.1981.30.30055

[9] N. Kalton, *Quasi-Banach Spaces*, Chap. 25 of *Handbook of Geometry of Banach spaces*, Vol.2, Elsevier Science, 2003, ISBN 0-444-51305-1

[10] W. A. Kirk, N. Shazhad, Fixed points and Cauchy sequences in semimetric spaces, *J. Fixed Point Theory Appl.*, 17 (2015) 541-555. https://doi.org/10.1007/s11784-015-0233-4

[11] B. Mohammadi, F. Golkarmanesh, V. Parvaneh, Some Fixed Point Theorems for Nonexpansive Self-Mappings and Multi-Valued Mappings in b-Metric Spaces, *Journal of Mathematical Extension* Vol. 14, No. 1, (2020), 1-18

[12] L. Narici, E. Beckenstein, *Topological Vector Spaces*, 2nd ed., CRC Press, Taylor & Francis Group, Boca Raton, London, New York, (2011) ISBN 978-1-58488-866-6

[13] M. A. Robdera, On size function topology and fixed point theorems, *Journal of Nonlinear Analysis and Application* No. 1 (2017) 31-42 42 http://www.ispacs.com/journals/jnaa/2017/jnaa-00336

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