A Note on Lyapunov-type Inequalities for Fractional Boundary Value Problems with Sturm-Liouville Boundary Conditions

Anil Chavada
The Maharaja Sayajirao University of Baroda

Nimisha Pathak*
The Maharaja Sayajirao University of Baroda

Abstract. In this note, we study Cauchy-Schwarz-type inequality for fractional Strum-Liouville boundary value problem containing Caputo derivative of order $\alpha$, $1 < \alpha \leq 2$. A lower bound for the smallest eigenvalue is determined using this inequality. We give a comparison between the smallest eigenvalue and its lower bound obtained from the Lyapunov-type and Cauchy-Schwarz-type inequalities which indicate the properties of eigenvalues.

AMS Subject Classification: 34A08; 34A40; 26D10; 34C10; 33E12
Keywords and Phrases: Lyapunov inequality; Caputo fractional derivative; Cauchy-Schwarz inequality; Mittag-Leffler function.

1 Introduction

The Lyapunov inequality [10] has proved to be very useful in the study of spectral properties and oscillation theory of ordinary differential equations. This inequality can be stated as follows [1]:

Received: April 2020; Accepted: November 2020

*Corresponding Author
The nontrivial solution to the boundary value problem
\[ u''(t) + q(t)u(t) = 0, \quad a < t < b, \quad u(a) = u(b) = 0, \]
exists, where \( q: [a, b] \to \mathbb{R} \) is a continuous function, then
\[
\int_a^b |q(s)| \, ds > \frac{4}{b - a}.
\]

The research on Lyapunov-Type Inequalities (LTIs) for Fractional Boundary Value Problems (FBVPs) has begun since 2013. In [3], [4], [5], [6], [7], [8], [11], [12], [13] and [15], the authors have established LTIs for FBVPs of order \( \alpha \) with different boundary conditions. In [12], Pathak obtained LTI for fractional boundary value problem with Hilfer derivative of order \( \alpha, \quad 1 < \alpha \leq 2 \). Furthermore, the author applied LTI to obtain the lower bound for the smallest eigenvalue of corresponding eigenvalue problem. In addition, the Cauchy-Schwarz type inequality (CSI) is established to improve the lower bound estimation of the smallest eigenvalue and applied it to obtain intervals where certain Mittag-Leffler (M-L) function has no real zeros. The CSI provides better results than that of LTI.

Motivated by the above work, we consider the following problem with Sturm-Liouville boundary conditions [8]:

\[
(C_a D^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b, \quad 1 < \alpha < 2 \quad (1)
\]

\[
pu(a) - ru'(a) = u(b) = 0, \quad (2)
\]

where \( p > 0, \quad r \geq 0 \) and \( q: [a, b] \to \mathbb{R} \) is a continuous function. In [8], Jleli and Samet established a Lyapunov-type inequality for FBVP (1)-(2) as follows:

For \( \frac{r}{p} > \frac{b-a}{\alpha-1} \)

\[
\int_a^b |q(s)| \, ds \geq \left( 1 + \frac{p}{r}(b-a) \right) \frac{\Gamma\alpha}{(b-a)^{\alpha-1}} \quad (3)
\]

and for \( 0 \leq \frac{r}{p} \leq \frac{b-a}{\alpha-1} \)

\[
\int_a^b |q(s)| \, ds \geq \frac{\Gamma\alpha}{max\{A(\alpha, \frac{p}{r}), B(\alpha, \frac{p}{r})\}}. \quad (4)
\]
We establish CSI for FBVP (1)-(2). The outline of the paper is as follows: first, we provide some preliminaries in Section 2 which we will use in this paper. In section 3, we establish CSI for fractional Strum-Liouville boundary value problem containing Caputo derivative of order $\alpha$, $1 < \alpha \leq 2$. We also give a comparison between the lower bound estimates of the smallest eigenvalue obtained from the LTI and CSI. In section 4, we use these inequalities to obtain an interval where a linear combination of certain Mittag-Leffler functions have no real zeros. Finally, a conclusion is given in Section 5.

2 Preliminaries

In this section, we recall some basic definitions which are further used in this paper.

Definition 2.1. The Caputo derivative of fractional order $\alpha > 0$ is defined by

$$(^{C}aD^{\alpha}f)(t) = \frac{1}{\Gamma(m-\alpha)} \int_{a}^{t} (t-s)^{m-\alpha-1} f^{m}(s)ds, \ t \in [a,b],$$

where $m$ is the smallest integer greater of equal to $\alpha$.

Definition 2.2. The two-parameter M-L function is defined by

$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k + \beta)}, \ (\alpha,\beta,z \in \mathbb{R}; \alpha,\beta > 0).$

Definition 2.3. The Pfaff Transformation is defined as

$$2F_{1}(a,b;c;t) = (1-t)^{-a}2F_{1}\left(a,c-b;c;\frac{t}{t-1}\right); |t| < \frac{1}{2},$$

where $2F_{1}(a,b;c;t)$ is a hypergeometric function.

For more details, refer [9] and [14].
3 Main Result

The main result of this note is given in Theorem 3.3.

Lemma 3.1. The FBVP (1)-(2) can be written in its equivalent integral form as [8]
\[
  u(t) = \int_{a}^{b} G(t, s)q(s)u(s)ds, \quad t \in [a, b],
\]
where \(G\) is the Green’s function given by
\[
  G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} 
  \frac{(\frac{r}{p} + t - a)(b - s)^{\alpha - 1}}{(\frac{r}{p} + b - a)} - (t - s)^{\alpha - 1}, & a \leq s \leq t \leq b, \\
  \frac{(\frac{r}{p} + t - a)(b - s)^{\alpha - 1}}{(\frac{r}{p} + b - a)}, & a \leq t \leq s \leq b.
\end{cases}
\]

Lemma 3.2. [12] Let \(u \in L^2[a, b]\), then the Cauchy-Schwarz-type inequality of FBVP (1)-(2) is given by
\[
1 \leq \left\{ \int_{a}^{b} \int_{a}^{b} |G(t, s)q(s)|^2dsdt \right\}.
\]

Proof. Taking the Cauchy-Scharz inequality in (5) we get,
\[
|u(t)| \leq \left[ \int_{a}^{b} |G(t, s)q(s)|^2ds \right]^{\frac{1}{2}} \left[ \int_{a}^{b} |u(s)|^2ds \right]^{\frac{1}{2}}.
\]

Squaring and integrating from \(a\) to \(b\) w.r.to. \(t\) gives
\[
\int_{a}^{b} |u(t)|^2dt \leq \int_{a}^{b} \left\{ \left[ \int_{a}^{b} |G(t, s)q(s)|^2ds \right] \left[ \int_{a}^{b} |u(s)|^2ds \right] \right\} dt
\]
\[
||u||_2 \leq \int_{a}^{b} \int_{a}^{b} |G(t, s)q(s)|^2dsdt||u||_2,
\]
which proves the Lemma. □

Now, we consider Fractional Sturm-Liouville eigen value problem (FEP):

\[
\begin{cases}
(C_0^\alpha D^\alpha u)(t) + \lambda u(t) = 0, & a < t < b, 1 < \alpha < 2 \\
pu(a) - ru'(a) = u(b) = 0.
\end{cases}
\tag{8}
\]

We are ready to state and prove our main results.

**Theorem 3.3.** If a nontrivial continuous solution of the problem (8) exists, then for FEP (8) the CSI is

\[
\lambda \geq \frac{1}{(\Gamma(\alpha))} \left\{ \frac{1}{(2\alpha - 1)(\frac{r}{p} + b - a)^2} \left[ \frac{r - a}{p} \right] (b - a) \\
+ \left( \frac{r - a}{p} \right) (b^2 - a^2) + \frac{(b^3 - a^3)}{3} \right] + \frac{(b - a)^{2\alpha}}{2\alpha(2\alpha - 1)} \\
- \frac{2\beta(1, \alpha)(b - a)^\alpha}{\left( \frac{r}{p} + b - a \right)} \int_a^b \left( \frac{r - a + t}{p} \right) (t - a)^\alpha \frac{2F_1\left( 1, 2\alpha, 1 + \alpha, \frac{a - t}{b - t} \right)}{(b - t)} dt \right\}^{-\frac{1}{2}},
\tag{9}
\]

where $\beta(m, n)$ is a Beta function.

**Proof.** Taking $q(t) = \lambda$ in (7) gives the inequality

\[
\lambda \geq \left[ \int_a^b \int_a^b G(t, s)^2 ds dt \right]^{-\frac{1}{2}}. \tag{10}
\]

By substituting equation (6) in (10), after some simplifications we obtain (9), which concludes the proof. □

We consider following two cases.

Case 1: Taking $a = 0, b = 1, p = 1$ and $r = 2$ in (8), we get the following FEP:

\[
\begin{cases}
(C_0^\alpha D^\alpha u)(t) + \lambda u(t) = 0, & 0 < t < 1, 1 < \alpha < 2 \\
u(0) - 2u'(0) = u(1) = 0.
\end{cases}
\tag{11}
\]

Case 2: Taking $a = 0, b = 1, p = 2$ and $r = 1$ in (8), gives the eigenvalue problem:

\[
\begin{cases}
(C_0^\alpha D^\alpha u)(t) + \lambda u(t) = 0, & 0 < t < 1, 1 < \alpha < 2 \\
u(0) - u'(0) = u(1) = 0.
\end{cases}
\tag{13}
\]
Next, we give three methods to estimate the lower bound for the smallest eigenvalue of problems (11)-(12) and (13)-(14) by using the following definitions given in [12].

**Definition 3.4.** A Lyapunov-Type Inequality Lower Bound (LTILB) is defined as a lower bound estimate for the smallest eigenvalue obtained from Lyapunov-type inequalities given by (3) and (4).

We obtain a lower bound for the smallest eigenvalue of problem (11) with boundary conditions (12) is:

\[
\lambda \geq \frac{3}{2} \Gamma(\alpha). \quad (15)
\]

and for the problem (13)-(14) it is:

\[
\lambda \geq \frac{\Gamma\alpha}{\max\{A(\alpha, \frac{1}{2}), B(\alpha, \frac{1}{2})\}}. \quad (16)
\]

**Definition 3.5.** A Cauchy-Schwarz Inequality Lower Bound (CSILB) is defined as an estimate of the lower bound for the smallest eigenvalue obtained from the Cauchy-Schwarz inequality of type given in equation (9).

We obtain the CSIs of problems (11)-(12) and (13)-(14), after some simplifications and using Pfaff transformation in (9) respectively as follows:

\[
\lambda \geq \frac{1}{\Gamma(\alpha)} \left\{ \frac{1}{(2\alpha - 1)} \left[ \frac{19}{27} + \frac{1}{2\alpha} \right] - \frac{2}{3} \int_0^1 (2 + t) t^\alpha \beta(1, \alpha) \, _2F_1(1 - \alpha, 1; \alpha + 1, t) dt \right\}^{-\frac{1}{2}}; \alpha > \frac{1}{2}, \quad (17)
\]

\[
\lambda \geq \frac{1}{\Gamma(\alpha)} \left\{ \frac{1}{(2\alpha - 1)} \left[ \frac{13}{27} + \frac{1}{2\alpha} \right] - \frac{4}{3} \int_0^1 \left( \frac{1}{2} + t \right) t^\alpha \beta(1, \alpha) \, _2F_1(1 - \alpha, 1; \alpha + 1, t) dt \right\}^{-\frac{1}{2}}; \alpha > \frac{1}{2}. \quad (18)
\]
In [2], eigenvalues $\lambda \in \mathbb{R}$ of problems (11)-(12) and (13)-(14) are the solutions of the linear combination of certain M-L functions are respectively as follows:

$$2E_{\alpha,1}(-\lambda) + E_{\alpha,2}(-\lambda) = 0,$$

$$E_{\alpha,1}(-\lambda) + 2E_{\alpha,2}(-\lambda) = 0.$$  \hspace{1cm} (19) \hspace{1cm} (20)

Now, comparing the non-zero solutions of equations (19)-(20) for $1.5 < \alpha \leq 2$ with CSILB given by equations (17)-(18) and LTILB given by the equations (15)-(16) respectively, we get the following comparison figures.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{comparison_figure.png}
\caption{Comparison of the lower bounds for $\lambda$ obtained from Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. (– o –: LTILB; – * –: CSILB; – x –: LE - the Lowest Eigenvalue $\lambda$)}
\end{figure}
Figure 2: Comparison of the lower bounds for $\lambda$ obtained from Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LTILB; $-\ast-$: CSILB; $-\times-$: LE - the Lowest Eigenvalue $\lambda$)

These figures clearly demonstrates that between the two estimates considered here, the LTILB provides the worse estimate and the CSILB provide better estimate for the smallest eigenvalues of (11)-(12) (Figure 1) and (13)-(14) (Figure 2). We use MATHEMATICA and MATLAB codes to find the smallest eigenvalue of the M-L functions.

We consider the integer order case, i.e. $\alpha = 2$. For this case, the LTILB and CSILB for the smallest $\lambda$ of (11)-(12) are given as 1.5 and 3.3310 and for (13)-(14), 2.6667 and 5.1117 respectively. (See equations (15), (16), (17) and (18)). For $\alpha = 2$, the problems (11)-(12) and (13)-(14) can be solved in closed form using the tools from integer order calculus. Results show, the smallest eigenvalues of (11)-(12) and (13)-(14) are the roots of equations (19) and (20) respectively, which give the smallest eigenvalues as 3.3731 and 5.2392. Comparing these $\lambda$ with its
estimate above, it is clear that between LTILB and CSILB for the integer \( \alpha \) the CSILB provides the best estimate for the smallest eigenvalue.

4 Applications

We now consider an application of the lower bounds for the smallest eigenvalues of FEPs (11)-(12) and (13)-(14) found in equations (15)-(20).

**Theorem 4.1.** Let \( 1.5 < \alpha \leq 2 \). The linear combination of certain Mittag-Leffler functions \( 2E_{\alpha,1}(-z) + E_{\alpha,2}(-z) \) have no real zeros in the following domains:

**LTILB:**
\[
z \in \left( -\frac{3}{2} \Gamma(\alpha), 0 \right]
\]  

**CSILB:**
\[
z \in \left( -\frac{1}{\Gamma(\alpha)} \left\{ \frac{1}{(2\alpha - 1) \left( \frac{19}{27} + \frac{1}{2\alpha} \right)} - \frac{2}{3} C_1(\alpha) \right\}^{-\frac{1}{2}}, 0 \right],
\]

where \( C_1(\alpha) = \int_0^1 (2 + t)^{\alpha} \beta(1, \alpha) \frac{1}{2} \) for \( \alpha = 1 + \alpha, 1 + \alpha, t \) dt.

**Proof.** Let \( \lambda \) be the smallest eigenvalue of the equation (12), then \( z = \lambda \) is the smallest value for which \( 2E_{\alpha,1}(-z) + E_{\alpha,2}(-z) = 0 \). If there is another \( z \) smaller than \( \lambda \) for which \( 2E_{\alpha,1}(-z) + E_{\alpha,2}(-z) = 0 \), then it will contradict that \( \lambda \) is the smallest eigenvalue. Therefore, \( 2E_{\alpha,1}(-z) + E_{\alpha,2}(-z) \) have no real zeros for \( z \in (-\lambda, 0] \). Thus, \( 2E_{\alpha,1}(-z) + E_{\alpha,2}(-z) \) have no real zeros for

\[
z \in \left( -\frac{1}{\Gamma(\alpha)} \left\{ \frac{1}{(2\alpha - 1) \left( \frac{19}{27} + \frac{1}{2\alpha} \right)} - \frac{2}{3} C_1(\alpha) \right\}^{-\frac{1}{2}}, 0 \right].
\]

This proves the equation (22). Proof of equation (21) is given in [8]. \( \square \)

**Theorem 4.2.** Let \( 1.5 < \alpha \leq 2 \). The linear combination of certain Mittag-Leffler functions \( E_{\alpha,1}(-z) + 2E_{\alpha,2}(-z) \) have no real zeros in the
following domains:
\textbf{LTILB:} 
\[ z \in \left( -\frac{\Gamma \alpha}{\max\{A(\alpha, \frac{1}{2}), B(\alpha, \frac{1}{2})\}}, 0 \right], \quad (23) \]
\textbf{CSILB:} 
\[ z \in \left( -\frac{1}{\Gamma(\alpha)} \left\{ \frac{1}{2(2\alpha-1)} \left[ \frac{13}{27} + \frac{1}{2\alpha} \right] - \frac{4}{3} C_2(\alpha) \right\}^{-\frac{1}{2}}, 0 \right], \quad (24) \]

where 
\[ C_2(\alpha) = \int_0^1 \left( \frac{1}{2} + t \right) t^\alpha \beta(1, \alpha) ~_2F_1(1 - \alpha, 1; \alpha + 1, t) dt. \]

\textbf{Proof.} The proof is similar to the proof of Theorem 4.1. \qed

5 Conclusion

In this note, we established Cauchy-Schwarz-type inequality for fractional Strum-Liouville boundary value problem containing Caputo derivative of order \( \alpha \), \( 1 < \alpha \leq 2 \) to determine a lower bound for the smallest eigenvalues. We give a comparison between the smallest eigenvalues and its lower bounds obtained from the Lyapunov-type and Cauchy-Schwarz-type inequalities. The results indicate that the Cauchy-Schwarz-type inequality gives better lower bound estimates for the smallest eigenvalues than the Lyapunov-type inequality. We then used these inequalities to obtain an interval where a linear combination of certain Mittag-Leffler functions have no real zeros.
References


**Anil Chavada**  
Department of Applied Mathematics  
Assistant Professor(Temporary)  
The Maharaja Sayajirao University of Baroda  
Vadodara, India  
E-mail: anile-appmath@msubaroda.ac.in

**Nimisha Pathak**  
Department of Applied Mathematics  
Assistant Professor  
The Maharaja Sayajirao University of Baroda  
Vadodara, India  
E-mail: nimisha.pathak-appmath@msubaroda.ac.in