Journal of Mathematical Extension
Vol. 6, No. 2, (2012), 41-49

Tuples with Property of Cyclicity Criterions

M. J. Ataei*
Payame Noor University

B. Yousefi
Payame Noor University

Abstract. In this paper we give conditions under which a tuple of operators satisfying the hypercyclicity, supercyclicity and cyclicity criterions.

AMS Subject Classification: 47B37; 47B33
Keywords and Phrases: Tuple of operators, hypercyclic vector, supercyclic vector, cyclic vector

1. Introduction

Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X. We will let

$$\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n} : k_i \in \mathbb{Z}_+, \quad i = 1, ..., n\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set

$$Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}.$$

A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic. Also, a vector x is called a supercyclic vector for $\mathcal{T}$ if $\mathbb{C}Orb(\mathcal{T}, x)$, is dense in X and in

Received: October 2011; Accepted: March 2012
*Corresponding author

this case the tuple $\mathcal{T}$ is called supercyclic. By $\mathcal{T}_d^{(k)}$ we will refer to the set of all k copies of an element of $\mathcal{F}$, i.e.

$$\mathcal{T}_d^{(k)} = \{S_1 \oplus ... \oplus S_k : S_1 = ... = S_k \in \mathcal{F}\}.$$

We say that $\mathcal{T}_d^{(k)}$ is hypercyclic provided there exist $x_1, ..., x_k \in X$ such that

$$\{W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)}\},$$

is dense in the k copies of X, $X \oplus ... \oplus X$, and similarly we say that $\mathcal{T}_d^{(k)}$ is

supercyclic provided there exist $x_1, ..., x_k \in X$ such that

$$\mathbb{C}\{W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)}\},$$

is dense in the k copies of X. Also, we say that x is a cyclic vector for $\mathcal{T}$ if the linear span of the orbit $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called cyclic. By a polynomial p we will mean

$$p(z_1, z_2, ..., z_n) = \sum_{j=1}^{n} \sum_{i_j=1}^{m_j} c_{i_1 ... i_n} z_1^{i_1} z_2^{i_2} ... z_n^{i_n}.$$

Let

$$\mathcal{P} = \{p(T_1, T_2, ..., T_n) : p \textit{ is a polynomial}\},$$

and note that the linear span of the orbit $Orb(\mathcal{T}, x)$ is equal to $\{Sx : S \in \mathcal{P}\}$. Thus the tuple $\mathcal{T}$ is cyclic if $\{Sx : S \in \mathcal{P}\}$ is dense in X. Also, let

$$\mathcal{P}(\mathcal{T})_d = \{S \oplus S : S \in \mathcal{P}\}.$$

We say that $\mathcal{P}(\mathcal{T})_d$ is cyclic if there exist $x_1, x_2 \in X$ such that

$$\{W(x_1 \oplus x_2) : W \in \mathcal{P}(\mathcal{T})_d\},$$

is dense in $X \oplus X$.

In this paper we want to extend some properties of hypercyclicity, supercyclicity and cyclicity criterions from a single operator to a tuple of commuting operators. For some topics we refer to [1–18].

2. Main Results

In the following we will give all three extensions of Supercyclicity, Cyclicity, and Hypercyclicity Criterions for tuples and then we show that each of which implies the supercyclicity, cyclicity and hypercyclicity of $\mathcal{T}_d^{(2)}$, respectively.

Definition 2.1. *Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X. We say that $\mathcal{T}$ satisfies the Supercyclicity Criterion if there exist two dense subsets Y and Z in X, and strictly increasing sequences $\{m_{j_i}\}_j$ for $i = 1, ..., n$, and a sequence of mappings $S_j : Z \to X$ such that:*
1) $T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} S_j z \to z$ for every $z \in Z$,
2) $||T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} y|| \, ||S_j z|| \to 0$ for every $y \in Y$ and every $z \in Z$.

Theorem 2.2. *Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of operators $T_1, T_2, ..., T_n$. Then the followings are equivalent:*
i) $\mathcal{T}$ satisfies the Supercyclicity Criterion.
ii) $\mathcal{T}_d^{(2)}$ is supercyclic on $X \oplus X$.

Proof. $(i) \to (ii)$: Suppose that $\mathcal{T}$ satisfies the Supercyclicity Criterion. Thus there exist two dense subsets Y and Z in H, sequences of positive integers $\{m_{j_i}\}_j$ for $i = 1, ..., n$, and a sequence of mappings $S_k : Z \to X$ such that:
1) $T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} S_j z \to z$ for every $z \in Z$,
2) $||T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} y|| \, ||S_k z|| \to 0$ for every $y \in Y$ and every $z \in Z$.
Now let $\mathcal{Y}$ be the set of all sequences $(y_n)_n \in \oplus_{i=1}^{\infty} Y$ such that $y_n = 0$ for all but finitely many $n \in \mathbb{N}$. Similarly let $\mathcal{Z}$ be the set of all sequences $(z_n)_n \in \oplus_{i=1}^{\infty} Z$ such that $z_n = 0$ for all but finitely many $n \in \mathbb{N}$. Put

$$S'_k = \oplus_{i=1}^{\infty} S_k,$$

and consider it acting on $\mathcal{Z}$. Then both $\mathcal{Y}$ and $\mathcal{Z}$ are dense in $\oplus_{i=1}^{\infty} X$ and clearly the hypotheses of the Supercyclicity Criterion are satisfied. Thus $\mathcal{T}_d^{(\infty)}$ is supercyclic on $\oplus_{i=1}^{\infty} X$ from which we can conclude that clearly $\mathcal{T}_d^{(2)}$ is supercyclic on $X \oplus X$.

$(ii) \to (i)$: Suppose that $\mathcal{T}_d^{(2)}$ is supercylcic and let (x,y) be a supercyclic vector for $\mathcal{T}_d^{(2)}$. In particular x and y are supercyclic vectors for $\mathcal{T}$. For all $k \in \mathbb{N}$, put $U_k = B(0, \frac{1}{k})$. Then there exist $\{m_{j_i}\}_j \subset \mathbb{N}$ for $i = 1, ..., n$ and $\lambda_j \in \mathbb{C}$ such that

$$\lambda_j(T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} \oplus T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}})(x,y) \in U_j \oplus (x + U_j).$$

Thus

$$\lambda_j T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} x \in U_j,$$

and

$$\lambda_g T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} y \in x + U_j,$$

for all $j \in \mathbb{N}$. This implies that

$$\lambda_g T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} x \to 0,$$

and

$$\lambda_g T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} y \to x.$$

Let

$$Y = Z = \mathbb{C}Orb(\mathcal{T}, x),$$

which is dense in X. Also for all $j \in \mathbb{N}, \lambda \in \mathbb{C}$ and $k_{r_i} \in \mathbb{N}\ for\ i = 1, ..., n$, define

$$S_j(\lambda T_1^{k_{r_1}} T_2^{k_{r_2}} ... T_n^{k_{r_n}} x) = \lambda \lambda_j T_1^{k_{r_1}} T_2^{k_{r_2}} ... T_n^{k_{r_n}} y.$$

Note that

$$T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} S_j(\lambda T_1^{k_{r_1}} T_2^{k_{r_2}} ... T_n^{k_{r_n}} x) = \lambda T_1^{k_{r_1}} T_2^{k_{r_2}} ... T_n^{k_{r_n}} (\lambda_j T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} y)$$

which tends to $\lambda T_1^{k_{r_1}} T_2^{k_{r_2}} ... T_n^{k_{r_n}} x$ as $j \to \infty$. So

$$T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} S_j z \to z,$$

for all $z \in Z$. Also for all $\lambda, w \in \mathbb{C}$ and $t_{s_i},\ k_{r_i} \in \mathbb{N}$ for $i = 1, ..., n$, we have

$$||T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} (\lambda T_1^{t_{s_1}} T_2^{t_{s_2}} ... T_n^{t_{s_n}} x)||\ ||S_j(w T_1^{k_{r_1}} T_2^{k_{r_2}} ... T_n^{k_{r_n}} x)||$$

$$= |\lambda|\ |w|\ ||T_1^{t_{s_1}} T_2^{t_{s_2}} ... T_n^{t_{s_n}} (T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} x)||.||\lambda_j T_1^{k_{r_1}} T_2^{k_{r_2}} ... T_n^{k_{r_n}} y||$$

$$\leqslant |\lambda|\ |w|\ |\lambda_j|\ ||T_1^{t_{s_1}} T_2^{t_{s_2}} ... T_n^{t_{s_n}}||\ ||T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} x||\ ||T_1^{k_{r_1}} T_2^{k_{r_2}} ... T_n^{k_{r_n}} y||.$$

Since

$$|\lambda_j|\ ||T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} x|| \to 0,$$

hence we get

$$||T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} (\lambda T_1^{t_{s_1}} T_2^{t_{s_2}} ... T_n^{t_{s_n}} x)||\ ||S_j(w T_1^{k_{r_1}} T_2^{k_{r_2}} ... T_n^{k_{r_n}} x)|| \to 0\ ,$$

as $j \to \infty$. Thus for all $y \in Y$ and $z \in Z$, we get

$$||T_1^{m_{j_1}} T_2^{m_{j_2}} ... T_n^{m_{j_n}} y||\ ||S_j z|| \to 0$$

and so $\mathcal{T}$ satisfies the Supercyclicity Criterion. $\square$

Definition 2.3. *Suppose X is a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of continuous linear mappings on X. Suppose there exist two dense subsets Y and Z in X, a sequence $\{p_k\}$ of polynomials, and a sequence $\{S_k\}$ of maps $S_k : Z \to X$ such that:*

1) for every $y \in Y$, $p_k(T_1, T_2, ..., T_n)y \to 0$,

2) for every $z \in Z$, $S_k z \to 0$,

3) for every $z \in Z$, $p_k(T_1, T_2, ..., T_n)S_k z \to z$.

Then we say that $\mathcal{T}$ satisfies the Cyclicity Criterion.

Theorem 2.4. *Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of operators $T_1, T_2, ..., T_n$. Then the followings are equivalent:*

i) $\mathcal{T}$ satisfies the Cyclicity Criterion.

ii) $\mathcal{P}(\mathcal{T})_d$ is cyclic on $X \oplus X$.

Proof. $(i) \to (ii)$: Suppose that y, Z, p_k *and* S_k are the ones obtained by the property of the Cyclicity Criterion for $\mathcal{T} = (T_1, T_2, ..., T_n)$. Let U_1, U_2, V_1 and V_2 be four nonempty open subsets of X. We want to find a polynomial p such that

$$p(T_1, T_2, ..., T_n)(U_1) \cap V_1 \neq \emptyset$$

and

$$p(T_1, T_2, ..., T_n)(U_2) \cap V_2 \neq \emptyset.$$

For this let u_1, v_1, u_2, v_2 belongs to $U_1, \cap Y, V_1 \cap Z, U_2 \cap Y$ and $V_2 \cap Z$, respectively. Then

$$u_1 + S_k v_1 \to u_1,$$

$$p_k(T_1, T_2, ..., T_n)(u_1 + S_k v_1) \to v_1,$$

$$u_2 + S_k v_2 \to u_2,$$

and

$$p_k(T_1, T_2, ..., T_n)(u_2 + S_k v_2) \to v_2.$$

Hence,

$$p_k(T_1, T_2, ..., T_n)(U_1) \cap V_1 \neq \emptyset$$

and

$$p_k(T_1, T_2, ..., T_n)(U_2) \cap V_2 \neq \emptyset$$

for k large enough. This implies that $\mathcal{T}_d^{(2)}$ is cyclic on $X \oplus X$. The assertion $(ii) \to (i)$ follows by a similar method used in the proof of the previous theorem. $\square$

Theorem 2.5. *(Hypercyclicity Criterion for tuples) Suppose that X is a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be the n-tuple of operators T_1, T_2, ..., T_n acting on X. If there exist two dense subsets Y and Z in X, and strictly increasing sequences $\{m_{j(i)}\}_j$ for $i = 1, ..., n$ such that :*
1. $T_1^{m_{j(1)}} ... T_n^{m_{j(n)}} y \to 0$ for all $y \in Y$
2. There exist a sequence of functions $\{S_j : Z \to X\}$ such that for every $z \in Z$, $S_j z \to 0$, and $T_1^{m_{j(1)}} ... T_n^{m_{j(n)}} S_j z \to z$,
then $\mathcal{T}$ is a hypercyclic tuple.

Proof. Let U and V be two nonempty open sets in X and choose $y \in Y \cap U$ and $z \in Z \cap V$. Define $x_j = y + S_j z$. Then $x_j \to y$ and we have

$$T_1^{m_{j(1)}} ... T_n^{m_{j(n)}} x_j = T_1^{m_{j(1)}} ... T_n^{m_{j(n)}} y + T_1^{m_{j(1)}} ... T_n^{m_{j(n)}} S_j z,$$

which tends to z as $j \to \infty$. Thus for large j, we have $x_j \in U$ and

$$T_1^{m_{j(1)}}...T_n^{m_{j(n)}} x_j \in V.$$

Hence we get

$$T_1^{m_{j(1)}}...T_n^{m_{j(n)}}(U) \cap V \neq \emptyset,$$

and so $\mathcal{T}$ is topologically transitive. Thus by Lemma 2.1, $\mathcal{T}$ is a hypercyclic tuple. This completes the proof. □

The method of the proof of the following theorem is similar to the proof of Theorem 2.4 and so we omit it.

Theorem 2.6. *Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of operators acting on a separable infinite dimensional Banach space X. Then the followings are equivalent:*
(i) $\mathcal{T}$ satisfies the Hypercyclicity Criterion.
(ii) $\mathcal{T}_d^{(2)}$ is hypercyclic.

References

[1] J. Bes and A. Peris, Hereditarily hypercyclic operators, *J. Func. Anal.*, 167 (1) (1999), 94-112.

[2] P. S. Bourdon, orbits of hyponormal operators, *Mich. Math. Journal*, 44 (1997), 345-353.

[3] P. S. Bourdon and N. Feldman, Somewhere dense orbits are everywhere dense, *Indiana Univ. Math. Journal*, 52 (2003), 811-819.

[4] P. S. Bourdon and J. H. Shapiro, Cyclic phenomena for composition operators, *Memoirs of the Amer. Math. Soc., 125, Amer. Math. Soc. Providence*, RI., 1997.

[5] P. S. Bourdon and J. H. Shapiro, Hypercyclic operators that commute with the Bergman backward shift, *Trans. Amer. Math. Soc.*, 352 (2000), 5293-5316.

[6] K. C. Chan, Hypercyclicity of the operator algebra for a separable Hilbert space, *J. Operator Theory*, 42 (1999), 231-244.

[7] G. Costakis and M. Sambarino, Topologically mixing hypercyclic operators, *Proc. Amer. Math. Soc.*, 132 (2003), 385-389.

[8] N. S. Feldman, Hypercyclic pairs of coanalytic Toeplitz operators, *Integral Equations Operator Theory*, 58 (20) (2007), 153-173.

[9] N. S. Feldman, Hypercyclic tuples of operators and somewhere dense orbits, *J. Math. Appl.*, 346 (2008), 82-98.

[10] G. Godefroy and J. H. Shapiro, operators with dense invariant cyclic manifolds, *J. Func. Anal.*, 98 (1991), 229-269.

[11] S. Grivaux, Hypercyclic operators, mixing operators and the bounded steps problem, *J. Operator Theory*, 54 (2005), 147-168.

[12] H. N. Salas, Hypercyclic weighted shifts, *Trans. Amer. Math. Soc.*, 347 (1995), 993-1004.

[13] B. Yousefi and H. Rezaei, Hypercyclicity on the algebra of Hilbert-Schmidt operators, *Results in Mathematics*, 46 (2004), 174-180.

[14] B. Yousefi and A. Farrokhinia, On the hereditarily hypercyclic vectors, *Journal of the Korean Mathematical Society*, 43 (6) (2006), 1219-1229.

[15] B. Yousefi and H. Rezaei, Hypercyclic property of weighted composition operators, *Proc. Amer. Math. Soc.*, 135 (10) (2007), 3263-3271.

[16] B. Yousefi and H. Rezaei, On the supercyclicity and hypercyclicity of the operator algebra, *Acta Mathematica Sinica*, 24 (7) (2008), 1221-1232.

[17] B. Yousefi and J. Izadi, Weighted composition operators and supercyclicity criterion, *International Journal of Mathematics and Mathematical Sciences*, Volume 2011, DOI 10.1155/2011/514370 (2011).

[18] B. Yousefi, Hereditarily transitive tuples, *Rend. Circ. Mat. Palermo*, Volume 2011, DOI 10.1007/S12215-011-0066-y (2011).

Mohammad Javad Ataei
Department of Mathematics
Assistant Professor of Mathematics
Payame Noor University

P.O. Box 19395-3697
Tehran, Iran
E-mail: ataeymj@pnu.ac.ir

Bahmann Yousefi
Department of Mathematics
Professor of Mathematics
Payame Noor University
P.O. Box 19395-3697
Tehran, Iran
E-mail: b_yousefi@pnu.ac.ir

Journal of Mathematical Extension
Vol. 6, No. 2, (2012), 51-64

General Decay of Energy for a Class of Integro-Differential Equation with Nonlinear Damping

E. Hesameddini*
Shiraz University of Technology

Y. Khalili
Shiraz University of Technology

Abstract. In this paper, a class of non-linear Integro-differential equations is considered in a bounded domain Ω with a smooth boundary $\partial\Omega$ as follows:

$$u_{tt} + M(\|D^m u\|_2^2)(-\Delta)^m u(t) + \int_0^t g(t-s)(-\Delta)^m u(s)ds + |u_t|^{\alpha-1}u_t = |u|^{p-1}u.$$

The asymptotic behavior of solutions is discussed by some conditions on g. Decay estimates of the energy function of solutions are also given.

AMS Subject Classification: 34A34; 65L05
Keywords and Phrases: Integro-differential, energy decay, relaxation function

1. Introduction

Consider the initial boundary value problem for a higher-order integro-differential equation:

$$\begin{cases} u_{tt} + M(\|D^m u\|_2^2)(-\Delta)^m u(t) + \int_0^t g(t-s)(-\Delta)^m u(s)ds \\ +|u_t|^{\alpha-1}u_t = |u|^{p-1}u, & t > 0 \\ \frac{\partial^i u}{\partial \nu^i} = 0, \quad i = 0,1,2,...,m-1, & x \in \partial\Omega, t \geqslant 0 \\ u(x,0) = u_0, u_t(x,0) = u_1 & x \in \Omega, \end{cases} \tag{1}$$

Received: July 2011; Accepted: April 2012
*Corresponding author

where $p > 1, m \geqslant 1, \alpha \geqslant 1$, Ω is a bounded domain of $\mathbb{R}^n, n \geqslant 1$, with the smooth boundary $\partial\Omega$, so that, divergence theorem can be applied, ν is unit outward normal on $\partial\Omega$, and $\frac{\partial^i u}{\partial \nu^i}$ denotes the $i-$order normal derivation of u, and D denotes the gradient operator, that is $Du = (u_{x_1}, u_{x_2}, ..., u_{x_n})$, and:

$$D^m = \overbrace{\nabla.\nabla.\cdots.\nabla}^{m}.$$

Before further progress, without the viscoelastic term, that is $g = 0$, for the case that m=1 and M being not a constant function, equation(1) is Kirchhoff equation which has been introduced in order to describe the nonlinear vibrations of an elastic string. Kirchhoff ([6]) was the first one to study the oscillations a stretched string and plates. In this case, the existence and nonexistence of solutions were discussed by many authors ([3,9,13]).

With $g \neq 0$, in the case of $M = 1$, equation (1) becomes a semilinear viscoelastic equation. Cavalcanti el.al ([2]) treated equation (1) with damping term $a(x)u_t$; here $a(x)$ may be null on apart of boundary. By assuming the kernel g in the memory term decays exponentially, they obtained an exponentially decay rate. On the other hand, Jiang and Rivera ([4]) proved, in the framework of nonlinear viscoelasticity, the exponential decay of the energy provided that the kernel g decays exponentially without imposing damping term. In the case M is not a constant function, equation (1) is a model in which describe the motion of deformable solids as hereditary effect is incorporated. This equation was first studied by Torrejon and Young ([12]) who proved the existence of weakly asymptotic stable solution for large analytical datum. Later, Rivera ([6]) showed the existence of global solutions for small datum and the total energy decays to zero exponentially under some restrictions. Recently, Wu and Tsai ([11]) discussed the global as well as energy decay of equation(1). In that paper, the following assumption on the negative kernel $g'(t) \leqslant 0$, for all $t \geqslant 0$ for some $r > o$, which motivated the present researcher to consider the problem of how to obtain the energy decay of the solutions when the above assumption is replaced by $g'(t) \leqslant 0$, for all $t \geqslant 0$.

In this paper, the global solution and the energy decays exponentially and polynomially under some conditions on g were established. The content of this paper is organized as follows: In Section 2, some important Lemmas and assumptions which will be used later and the state the local existence Theorem 2.1. are given. In Section 3, the results of global existence and decay property of the solutions of equation(1) are given by Theorem 3.1.

2. Preliminary Notes

In this section, the material needed for proving the main result is introduced. The standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H^m(\Omega)$ are used with their usual scalar products and norms. Meanwhile,

$$H_0^m(\Omega) = \{u \in H^m(\Omega) : \frac{\partial^i u}{\partial \nu^i} = 0, \quad i = 0, 1, 2, ..., m-1\}$$

is defined and the following abbreviations are introduced:
$\|.\|_{H^m} = \|.\|_{H^m(\Omega)}$, $\|.\|_{H_0^m} = \|.\|_{H_0^m(\Omega)}$, $\|.\|_2 = \|.\|_{L^2(\Omega)}$, $\|.\|_p = \|.\|_{L^p(\Omega)}$ for any real number $p > 1$.
It is assumed that:

(A1) $M(s)$ is positive C^1-function for $s \geqslant 0$ and $M(s) = m_0 + s^q$ for $m_0 > 0, q \geqslant 1$ and $s \geqslant 0$.

(A2) $g \in C^1([0, \infty))$ is a bounded function satisfying:

$$m_0 - \int_0^t g(s) = l > 0, \quad \forall t > 0,$$

and there exist positive constants ξ_1 and ξ_2 such that:

$$-\xi_1 g(t) \leqslant g'(t) \leqslant -\xi_2 g(t). \tag{2}$$

(A3) $1 < p < \infty$ for $n \leqslant 2m$, $1 < p < \frac{2m}{n-2m}$ for $n > 2m$.

It is necessary to state that the local existence theorem for equation(1) will be established by combining the arguments of [3] and [12].

Theorem 2.1. *Assume that* $M(s)$,$g(x)$ *and* p *satisfy* $(A1)$, $(A2)$ *and* $(A3)$ *respectively. Then for any given* $(u_0, u_1) \in (H_0^m(\Omega) \cap H^{2m}(\Omega)) \times H_0^m(\Omega)$, *the problem (1) has a unique local solution satisfying:*

$$u \in C([0,T]; H_0^m(\Omega)), \quad u_t \in C([0,T]; L^2(\Omega)) \cap L^2(\Omega \times (0,T)),$$
$$u_{tt} \in L^\infty((0,T); L^2(\Omega)).$$

Lemma 2.2.(Sobolev-Poincare inequality [1]). *If* p *satisfies* (A_3) *for all* $u \in H_0^m(\Omega)$, *then* $H_0^m(\Omega) \longrightarrow L^p(\Omega)$, $\|u\|_{p+1} \leqslant B\|D^m u\|_2$, *where* B *is the optimal constant of the Sobolev embeding.*

Lemma 2.3. ([7]) *Let* $\phi(t)$ *be a nonincreasing and nonnegative function defined on* $[0,T]$, $T > 1$, *satisfying:*

$$\phi^{1+r}(t) \leqslant k_0(\phi(t) - \phi(t+1)),$$

for $t \in [0,T]$, $k_0 > 1$ *and* $r \geqslant 0$. *Then we have for each* $t \in [0,T]$,

$$\begin{cases} \phi(t) \leqslant \phi(0)e^{k(t-1)^+}, & r = 0 \\ \phi(t) \leqslant (\phi^{-r}(0) + k_0 r^{-1}(t-1)^+)^{\frac{-1}{r}}, & r > 0 \end{cases}$$

where $(t-1)^+ = \max\{t-1, 0\}$ *and* $k = \ln(\frac{k_0}{k_0-1})$. *Furthermore, the energy function* $E(t)$ *of the problem (1) is defined by:*

$$\begin{aligned} E(t) \quad &= \tfrac{1}{2}(m_0 - \textstyle\int_0^t g(s)ds)\|D^m u(t)\|_2^2 + \tfrac{1}{2}(goD^m u)(t) \\ &+ \tfrac{1}{2(q+1)}\|D^m u(t)\|_2^{2(q+1)} + \tfrac{1}{2}\|u_t\|_2^2 - \tfrac{1}{p+1}\|u\|_{p+1}^{p+1}, \end{aligned} \tag{3}$$

where $(goD^m u)(t) = \int_0^t g(t-s)\|D^m u(s) - D^m u(t)\|_2^2 ds$.

Lemma 2.4. *Assume that* $(A1)$,$(A2)$ *and* $(A3)$ *hold and let* u *be the solution of problem (1). Then* $E(t)$ *decreases, in other words:*

$$E'(t) = \frac{1}{2}(g' oD^m u)(t) - \frac{1}{2}g(t)\|D^m u(t)\|_2^2 - \|u_t\|_{\alpha+1}^{\alpha+1} \leqslant 0,$$

furthermore, for all $t \geqslant 0$,

$$E(t) \leqslant E(0). \tag{4}$$

Proof. By multiplying equation(1) by u_t and integrating the result over Ω, the following result is obtained:

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\|u_t\|_2^2 + M(\|D^m u(t))\|_2^2 \int_\Omega (-\Delta)^m u(t) u_t dx + \|u_t\|_{\alpha+1}^{\alpha+1} \\ &+ \int_0^t g(t-s) \int_\Omega (-\Delta)^m u(s) u_t dx ds = \tfrac{d}{dt}\tfrac{1}{p+1}\|u\|_{p+1}^{p+1}, \end{aligned} \tag{5}$$

for any regular solution, this result remains valid for weak solutions by a simple density argument. After being integrated m times by parts for the second term on the left side of (4) and noting $\frac{\partial^i u}{\partial \nu^i} = 0$, the following identity will be obtained:

$$\begin{aligned} \int_\Omega [(-\Delta)^m u(t)] u_t dx = (-1)^m \int_\Omega D^{2m} u u_t dx &= \int_\Omega D^m u(t).D^m u_t(t) dx \\ &= \tfrac{1}{2}\tfrac{d}{dt}\|D^m u(t)\|_2^2. \end{aligned} \tag{6}$$

Inserting (6) in (5) and applying $(A1)$, result in:

$$\begin{aligned} &\tfrac{d}{dt}\Big\{ \tfrac{1}{2(q+1)}\|D^m u(t))\|_2^{2(q+1)} + \tfrac{m_0}{2}\|D^m u(t))\|_2^2 + \tfrac{1}{2}\|u_t\|_2^2 - \tfrac{1}{p+1}\|u\|_{p+1}^{p+1} \Big\} \\ &= \int_0^t g(t-s) \int_\Omega D^m u_t . D^m u(s) dx ds - \|u_t\|_{\alpha+1}^{\alpha+1}, \end{aligned} \tag{7}$$

Also:

$$\begin{aligned} &\int_0^t g(t-s) \int_\Omega D^m u_t . D^m u(s) dx ds \\ &= \int_0^t g(t-s) \int_\Omega D^m u_t . [D^m u(s) - D^m u(t)] dx ds \\ &+ \int_0^t g(t-s) ds \int_\Omega D^m u_t . D^m u(t) dx \\ &= -\tfrac{1}{2}\int_0^t g(t-s) \tfrac{d}{dt} \int_\Omega |D^m u(s) - D^m u(t)|^2 dx ds \\ &+ \int_0^t g(s) ds \tfrac{d}{dt}\tfrac{1}{2}(\int_\Omega |D^m u(t)|^2 dx) ds. \end{aligned}$$

But,

$$= -\frac{1}{2}\frac{d}{dt}\Big[\int_0^t g(t-s)\int_\Omega |D^m u(s) - D^m u(t)|^2 dxds\Big]$$

$$+\frac{d}{dt}\frac{1}{2}\Big[\int_0^t g(s)\int_\Omega |D^m u(t)|^2 dxds\Big] \tag{8}$$

$$+\frac{1}{2}\int_0^t g'(t-s)\int_\Omega |D^m u(s) - D^m u(t)|^2 dxds - \frac{1}{2}g(t)\int_\Omega |D^m u(t)|^2 dx.$$

Then, (8) is inserted in (7) to get:

$$\frac{d}{dt}\Big\{\frac{1}{2}(m_0 - \int_0^t g(s)ds)\|D^m u(t))\|_2^2 + \frac{1}{2(q+1)}\|D^m u(t))\|_2^{2(q+1)}$$

$$+\frac{1}{2}(goD^m)u(t) + \frac{1}{2}\|u_t\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}\Big\} \tag{9}$$

$$= -\|u_t\|_{\alpha+1}^{\alpha+1} + \frac{1}{2}\int_0^t g'(t-s)\|D^m u(s) - D^m u(t)\|_2^2 ds - \frac{1}{2}\|D^m u(t)\|_2^2.$$

Using the definition of $E(t)$, the proof is completed. □

3. The Main Result

In this section, the main result is proved.

Theorem 3.1. (Global existence and energy decay) *Let the assumptions of Theorem 2.1. hold and* $1 \leqslant \alpha \leqslant \frac{n+2}{n-2}$. *If the initial datum satisfies,*

$$\|u_0\|_{p+1} < \lambda_0 = l^{\frac{1}{p-1}} B^{\frac{-2}{p-1}}, \qquad E(0) < E_0 = \frac{p-1}{2(p+1)}\lambda_0^{p+1}, \tag{10}$$

where B *is the optimal constant of the Sobolev embedding(Sobolev-Poincare inequality). Then the cauchy problem (1) has a unique global solution. Moreovere,*

$$E(t) \leqslant E(0)e^{-k(t-1)^+}, \qquad t \geqslant 0, \qquad \alpha = 1, \tag{11}$$

$$E(t) \leqslant (E^{\frac{\alpha-1}{2}}(0) + \frac{\alpha-1}{2}c_{12}^{-1}(t-1)^+)^{\frac{-2}{\alpha-1}} \qquad t \geqslant 0, \qquad \alpha > 1, \tag{12}$$

where $k = \ln(\frac{3c_{10}}{3c_{10}-1})$ *and* c_{10} *and* c_{12} *are given in (41)and (44) respectively.*

Proof. By decreasing of energy $E(t)$, one obtains:

$$E(t) \leqslant E(0) < E_0 = \frac{p-1}{2(p+1)}\lambda_0^{p+1}. \tag{13}$$

Therefore the following inequality is claimed:

$$\|u(.,t)\|_{p+1} < \lambda_0, \qquad \forall t \geqslant 0. \tag{14}$$

Suppose (14) is not true, by continuity of $\|u(.,t)\|_{p+1}$-norm; then there exist a t_0 such that $\|u(.,t_0)\|_{p+1} = \lambda_0$. Using Sobolev-Poincare inequality the following relation can be presented:

$$E(t) \geqslant \frac{1}{2}lB^{-2}\|u(t)\|_{p+1}^2 - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1} \qquad \forall t \geqslant 0. \tag{15}$$

Then,

$$\begin{aligned} E(t_0) &\geqslant \tfrac{1}{2}lB^{-2}\|u(t_0)\|_{p+1}^2 - \tfrac{1}{p+1}\|u(t_0)\|_{p+1}^{p+1} \\ &= \tfrac{p-1}{2(p+1)}\lambda_0^{p+1} = E_0, \end{aligned} \tag{16}$$

in which (16) contradicts with (13).
On the other hand for all $t \geqslant 0$,

$$\begin{aligned} l\|D^m u(t)\|_2^2 &= 2E(t) - \|u_t\|_2^2 - \tfrac{1}{q+1}\|D^m u(t)\|_2^{2(q+1)} \\ &+ \tfrac{2}{p+1}\|u(t)\|_{p+1}^{p+1} + (goD^m u)(t) \\ &\leqslant \tfrac{p-1}{p+1}l^{\frac{p+1}{p-1}}B^{\frac{-2(p+1)}{p-1}} + \tfrac{2}{p+1}l^{\frac{p+1}{p-1}}B^{\frac{-2(p+1)}{p-1}} \\ &= \lambda_0^{p+1}. \end{aligned} \tag{17}$$

By continuation argument and (17), the local solution constructed by Theorem 2.1. will be exist globally. Furthermore, the large time behavior of equation(1) is considered.
According to (17), the initial condition and Sobolev-Poincare inequality, the following relation can be concluded:

$$\begin{aligned} l\|D^m u(t)\|_2^2 &< 2E(t) + \tfrac{2}{p+1}B^{p+1}\|D^m u(t)\|_2^{p+1} \\ &< 2E(t) + \tfrac{2}{p+1}l\|D^m u(t)\|_2^2, \end{aligned} \tag{18}$$

and consequently:

$$\|D^m u(t)\|_2^2 < (\frac{2(p+1)}{l(p-1)}E(0))^{\frac{1}{2}}. \tag{19}$$

The parameter β is defined as follows:

$$0 \leqslant \beta = \frac{B^{p+1}}{l}(\frac{2(p+1)}{l(p-1)}E(0))^{\frac{p-1}{2}} < 1. \tag{20}$$

From (19), (20) and Sobolev-Poincar inequality, the following can be received:

$$\begin{aligned} \|u(t)\|_{p+1}^{p+1} &< \beta l \|D^m u(t)\|_2^2 \\ &< l\|D^m u(t)\|_2^2. \end{aligned} \tag{21}$$

Therefore, if $I(t)$ is defined as follows:

$$I(t) = l\|D^m u(t)\|_2^2 + \|D^m u(t)\|_2^{2(q+1)} + (goD^m u)(t) - \|u(t)\|_{p+1}^{p+1}, \tag{22}$$

then, by considering (21), the following can be presented:

$$I(t) > l(1-\beta)\|D^m u(t)\|_2^2 > 0. \tag{23}$$

Now, $F(t)$ is set as follows:

$$\begin{aligned} F^{\alpha+1}(t) &= -\tfrac{1}{2}\int_t^{t+1}\int_0^t g'(t-s)\|D^m u(s) - D^m u(t)\|_2^2 ds dt \\ &+ \int_t^{t+1}\|u_t(t)\|_{\alpha+1}^{\alpha+1} dt + \tfrac{1}{2}\int_t^{t+1} g(t)\|D^m u(t)\|_2^2 dt. \end{aligned} \tag{24}$$

Thanks to mean value Theorem and Holder inequality,

$$\begin{aligned} \tfrac{1}{4}\|u_t(t_1)\|_2^2 + \tfrac{1}{4}\|u_t(t_2)\|_2^2 &\leqslant \int_t^{t+1}\|u_t(t)\|_2^2 dt \\ &|\Omega|^{\frac{\alpha-1}{\alpha+1}}(\int_t^{t+1}\|u_t(t)\|_{\alpha+1}^{\alpha+1}dt)^{\frac{2}{\alpha+1}}, \end{aligned} \tag{25}$$

holds for some $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$.
Hence, by (24), the following is presented:

$$\|u_t(t_i)\|_2^2 \leqslant cF^2(t), \qquad i = 1, 2, \tag{26}$$

where $c = 4|\Omega|^{\frac{2(\alpha-1)}{(\alpha+1)^2}}$.
Afterwards, multiplying equation(1) by u and integrating it over $\Omega \times [t_1, t_2]$ the following identity is presented:

$$\begin{aligned} &\int_{t_1}^{t_2} [l\|D^m u(t)\|_2^2 + \|D^m u(t)\|_2^{2(q+1)} - \|u(t)\|_{p+1}^{p+1}]dt \\ &= -\int_{t_1}^{t_2} \int_\Omega u(t)u_{tt}(t)dxdt - \int_{t_1}^{t_2} \int_\Omega u(t)|u_t(t)|^{\alpha-1}u_t(t)dxdt \\ &+ \int_{t_1}^{t_2} \int_\Omega \int_0^t g(t-s)D^m u(t).[D^m u(s) - D^m u(t)]dsdxdt. \end{aligned} \tag{27}$$

Then, using (22), the following is obtained:

$$\begin{aligned} &\int_{t_1}^{t_2} I(t)dt \\ &= -\int_{t_1}^{t_2} \int_\Omega u(t)u_{tt}(t)dxdt - \int_{t_1}^{t_2} \int_\Omega u(t)|u_t(t)|^{\alpha-1}u_t(t)dxdt \\ &+ \int_{t_1}^{t_2} (goD^m)u(t)dt \\ &+ \int_{t_1}^{t_2} \int_\Omega \int_0^t g(t-s)D^m u(t).[D^m u(s) - D^m u(t)]dsdxdt. \end{aligned} \tag{28}$$

Note that by integrating by parts and Holder inequality, the following inequality is achieved:

$$-\int_{t_1}^{t_2} \int_\Omega u(t)u_{tt}(t)dxdt \leqslant \sum_{i=1}^{2} \|u_t(t_i)\|_2^2 + \int_{t_1}^{t_2} \int_\Omega u_t^2(t)dxdt. \tag{29}$$

Also the following relation is obtained by considering Young inequality:

$$\begin{aligned} &\int_{t_1}^{t_2} \int_\Omega \int_0^t g(t-s)D^m u(t).[D^m u(s) - D^m u(t)]dsdxdt \\ &\leqslant \delta \int_{t_1}^{t_2} \int_0^t g(t-s)\|D^m u(t)\|_2^2 dsdt + \frac{1}{4\delta} \int_{t_1}^{t_2} (goD^m)u(t)dt, \end{aligned} \tag{30}$$

where δ is some positive constant to be chosen later.
By using (18) and Sobolev-Poincare inequality, the following result is concluded:

$$\|u(t_i)\|_{p+1} \leqslant c_1 \sup_{t_1 \leqslant s \leqslant t_2} E^{\frac{1}{2}}(s), \tag{31}$$

where $c_1 = (\frac{2(p+1)}{l(p-1)})^{\frac{1}{2}}$.
Then, by (26) and (29)-(31), the following relation is deduced:

$$\begin{aligned}\int_{t_1}^{t_2} I(t)dt &\leqslant c_2 F(t)\sup_{t_1\leqslant s\leqslant t_2} E^{\frac{1}{2}}(s) + cF^2(t)\\ &+ \int_{t_1}^{t_2}\int_\Omega |u(t)||u_t(t)|^\alpha dxdt + (\tfrac{1}{4\delta}+1)\int_{t_1}^{t_2}(goD^m u)(t)dt\\ &+\delta\int_{t_1}^{t_2}\int_0^t g(t-s)\|D^m u(t)\|_2^2 dsdt,\end{aligned} \tag{32}$$

where $c_2 = \sqrt{2}c_1 c$.
On the other hand, the following inequality is obtained from (2) and (24):

$$\int_{t_1}^{t_2}(goD^m u)(t)dt \leqslant -\frac{1}{\xi_2}\int_{t_1}^{t_2}(g'oD^m u)(t)dt \leqslant \frac{2}{\xi_2}F^{\alpha+1}(t). \tag{33}$$

The following relation is achieved by considering (2) and (23):

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^t g(t-s)\|D^m u(t)\|_2^2 dsdt &\leqslant \tfrac{1}{\xi_1}\int_{t_1}^{t_2}\int_0^t g'(t-s)\|D^m u(t)\|_2^2 dsdt\\ &\leqslant \tfrac{1}{\xi_1}\int_{t_1}^{t_2} g(0)\|D^m u(t)\|_2^2 dt\\ &\leqslant \tfrac{g(0)}{(1-\beta)l\xi_1}\int_{t_1}^{t_2} I(t)dt.\end{aligned} \tag{34}$$

Hence, by choosing δ such that $\frac{\delta g(0)}{(1-\beta)l\xi_1} = \frac{1}{2}$ and by using (32)-(34), the following is obtained:

$$\begin{aligned}\int_{t_1}^{t_2} I(t)dt &\leqslant 2c_2 F(t)\sup_{t_1\leqslant s\leqslant t_2} E^{\frac{1}{2}}(s) + 2cF^2(t)\\ & c_3 F^{\alpha+1}(t) + \int_{t_1}^{t_2}\int_\Omega |u(t)||u_t(t)|^\alpha dxdt,\end{aligned} \tag{35}$$

where $c_3 = (1+\frac{g(0)}{(1-\beta)l\xi_1})\frac{1}{\xi_2}$.
By using Holder inequality and Sobolev-Poincare inequality, the following is resulted:

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega |u(t)||u_t(t)|^\alpha dxdt &\leqslant B\int_{t_1}^{t_2}\|u_t(t)\|_{\alpha+1}^\alpha \|D^m u(t)\|_2 dt\\ &\leqslant c_1 \sup_{t_1\leqslant s\leqslant t_2} E^{\frac{1}{2}}(s)F^\alpha(t).\end{aligned} \tag{36}$$

By putting (36) into (35) and due to decreasing of energy $E(t)$, it can be concluded that:

$$\int_{t_1}^{t_2} I(t)dt \leqslant c_4(F(t)E^{\frac{1}{2}}(t) + F^{\alpha}(t)E^{\frac{1}{2}}(t) + F^2(t) + F^{\alpha+1}(t)), \quad (37)$$

where $c_4 = \max\{c, c_1, c_2, c_3\}$.
Moreover, from (3), (22) and (23), it is seen that:

$$E(t) \leqslant \frac{1}{2}\|u_t\|_2^2 + c_5 l\|D^m u(t)\|_2^2 + c_5(goD^m u)(t) + c_6 I(t), \quad (38)$$

where $c_5 = \frac{1}{2} - \frac{1}{p+1}$ and $c_6 = (\frac{1}{p+1} + \frac{1}{2(q+1)})$.
By integrating (38) over (t_1, t_2) also using (23), (26) and (33), the following is achieved:

$$\int_{t_1}^{t_2} E(t)dt \leqslant \frac{c}{2}F^2(t) + c_7\int_{t_1}^{t_2} I(t)dt + c_8 F^{\alpha+1}(t), \quad (39)$$

where $c_7 = c_6 + \frac{c_5}{1-\beta}$ and $c_8 = \frac{2c_5}{\xi_2}$.
On the other hand, from the nonincreasing of $E(t)$ one obtain:

$$\int_{t_1}^{t_2} E(t)dt \geqslant \frac{1}{2}E(t_2).$$

Therefore, from (39),

$$\begin{aligned} E(t) &= E(t_2) - \tfrac{1}{2}\textstyle\int_t^{t_2}\int_0^t g'(t-s)\|D^m u(s) - D^m u(t)\|_2^2 ds dt \\ &\textstyle\int_t^{t_2}\|u_t(t)\|_{\alpha+1}^{\alpha+1}dt + \tfrac{1}{2}\int_t^{t_2} g(t)\|D^m u(t)\|_2^2 dt \\ &\leqslant 2\textstyle\int_{t_1}^{t_2} E(t)dt + F^{\alpha+1}(t) \\ &\leqslant c_9(F(t)E^{\frac{1}{2}}(t) + F^{\alpha}(t)E^{\frac{1}{2}}(t) + F^2(t) + F^{\alpha+1}(t)), \end{aligned} \quad (40)$$

where $c_9 = \max\{c + 2c_7c_4, 2c_7c_4, 1 + 2c_8 + c_7c_4\}$.
After that, by Young inequality, the following inequality is achieved:

$$E(t) \leqslant c_{10}(F^2(t) + F^{2\alpha}(t) + F^{\alpha+1}(t)), \quad (41)$$

where $c_{10} = |\frac{2c_9}{1-c_9}|$.
If $\alpha = 1$ in (41), then

$$E(t) \leqslant 3c_{10}(F^2(t)) = 3c_{10}(E(t) - E(t+1)), \tag{42}$$

therefore (11) follows from (42) and Lemma 2.3.
If $\alpha > 1$, since:

$$F^{\alpha+1}(t) = E(t) - E(t+1) \leqslant E(0),$$

then, the following relation is obtained:

$$\begin{aligned} E(t) &\leqslant c_{10}(1 + F^{2(\alpha-1)}(t) + F^{\alpha-1}(t))F^2(t) \\ &\leqslant c_{10}(1 + (E(0))^{\frac{2(\alpha-1)}{\alpha+1}}(t) + (E(0))^{\frac{\alpha-1}{\alpha+1}}(t))F^2(t) \\ &\leqslant c_{11}F^2(t), \end{aligned} \tag{43}$$

where $c_{11} = (1 + (E_0)^{\frac{2(\alpha-1)}{\alpha+1}}(t) + (E_0)^{\frac{\alpha-1}{\alpha+1}}(t))$.
Thus, (43) implies (44) as follows:

$$E^{\frac{\alpha+1}{2}}(t) \leqslant c_{11}^{\frac{\alpha+1}{2}} F^{\alpha+1}(t) = c_{12}(E(t) - E(t+1)), \tag{44}$$

where $c_{12} = c_{11}^{\frac{\alpha+1}{2}}$.
Hence,(12)follows from (44) and Lemma 2.3. This finishes the proof. □

4. Conclusion

In this paper, a difference inequality [Lemma 2.3.] for a class of Integro-Diferential equations with nonlinear damping has been applied. The main goal of this work is estimating general decay energy of these equations. The mentioned target is satisfied by the propose method. Also, the asymptotic behavior of solutions is discussed.

References

[1] R. A. Adams , *Sobolev Spaces, Pure Appl. Math.,* 65, Academic Press, New York-London, 1975.

[2] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and J. A. Soriano, Exponential decay for the solution of semilinear viscoleastic wave equation with localized damping, *Electronic J. Diff. Eqns.*, 44 (2002), 1-14.

[3] Q. Gao, F. Li, and Y. Wang, Blow up of the solution for higher-order Kirchhoff-type equations with nonlinear dissipation, *Cent. Eur. J. Math.*, 9 (2011), 686-698.

[4] S. Jiang and J. E. Munoz Rivera, A global existence theorem for the Dirichlet problem in nonlinear n-dimensional viscoelastic, *Differential and Integral Equations*, 9 (1996), 791-810.

[5] G. Kirchhoff, *Vorlesungen Uber Mechanik*, 3rd ed., Teuber, Leipzig, 1883.

[6] J. E. Munoz Rivera, Global solution on a quasilinear wave equation with memory, *Bolletino U. M. I.*, 7 (1994), 289-303.

[7] M. Nako, A difference inequality and its application to nonlinear evolution equations, *J. Math. Soc. Japan*, 30 (1978), 747-762.

[8] K. Ono, On global solutions and blow-up solutions of nonlinear Kirchhoff strings with nonlinear dissipation, *J. Math. Anal. and Appli.*, 216 (1997), 321-342.

[9] R. M. Torrejon and J. Young, On a quasilinear wave equation with memory, *Nonlinear Anal., Theory, Methods and Applications*, 16 (1991), 61-78.

[10] S. T. Wu and L. Y. Tsai, Blow-up of solutions for some nonlinear wave equations of Kirchhoff type with some dissipation, *Nonlinear Anal., Theory, Methods and Applications*, 65 (2006), 243-264.

[11] S. T. Wu and L. Y. Tsai, Blow up of positive-initial-energy solutions for an integro-differential equation with nonlinear nonlinear damping, *Taiwanese journal of mathematics*, 14 (2010), 2043-2058.

[12] S. T. Wu and L. Y. Tsai, On global solutions and blow-up solutions for a nonlinear viscoelastic wave equation with nonlinear damping, *National Chengchi University*, 9 (2004), 479-500.

Esmaeil Hesameddini
Department of Mathematics

Faculty of Basic Sciences
Associate Professor of Mathematics
Shiraz University of Technology
P. O. Box 71555-313
Shiraz, Iran
E-mail: Hesameddini@sutech.ac.ir

Yasser Khalili
Department of Mathematics
Faculty of Basic Sciences
M.Sc. student of Mathematics
Shiraz University of Technology
P. O. Box 71555-313
Shiraz, Iran
E-mail: yasser.khalili@yahoo.com

Journal of Mathematical Extension
Vol. 6, No. 2, (2012), 65-73

On the Irreducibility of Some Composite Polynomials

M. Alizadeh
Islamic Azad University-Ahvaz Branch

Abstract. In this paper we study the irreducibility of some composite polynomials, constructed by a polynomial composition method over finite fields. Finally, a recurrent method for constructing families of irreducible polynomials of higher degree from given irreducible polynomials over finite fields is given.

AMS Subject Classification: 12A20
Keywords and Phrases: Composition polynomial, finite fields, irreducible polynomial

1. Introduction

The problem of irreducibility of polynomials over Galois fields is a case of spacial interest and plays an important role in modern engineering. One of the methods to construct irreducible polynomials is the polynomial composition method that allows constructions of irreducible polynomials of higher degree from given irreducible polynomials over finite fields.

Let $\mathbb{F}_q$ be the Galois field of order $q = p^s$, where p is a prime and s is a natural number. For a finite field $\mathbb{F}_q$ we denote by $\mathbb{F}_q^*$ the multiplicative group of nonzero elements of $\mathbb{F}_q$. Recall that the trace function of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is defined by

$$Tr_{\mathbb{F}_{q^n}|\mathbb{F}_q}(\alpha) = \sum_{i=0}^{n-1} \alpha^{q^i},\ \alpha \in \mathbb{F}_{q^n},$$

Received: June 2011; Accepted: January 2012

where $\mathbb{F}_{q^n}$ is an extension field of the finite field $\mathbb{F}_q$. For convince we denote $Tr_{\mathbb{F}_{q^n}|\mathbb{F}_q}$ by $Tr_{q^n|q}$. Notice also the transitivity of the trace in the sense that

$$Tr_{\mathbb{F}_{q^n}|\mathbb{F}_p}(\alpha) = Tr_{\mathbb{F}_q|\mathbb{F}_p}(Tr_{\mathbb{F}_{q^n}|\mathbb{F}_q}(\alpha)), \qquad \alpha \in \mathbb{F}_{q^n}. \tag{1}$$

Suppose that $P(x) = \sum_{i=0}^{n} c_i x^i$ be an irreducible polynomial over $\mathbb{F}_q$ of degree n. Its reciprocal polynomial is defined as

$$P^*(x) = x^n P(1/x).$$

Some authors have been studied the irreducibility of the polynomial

$$F(x) = (dx^p - rx + h)^n P(\frac{ax^p - bx + c}{dx^p - rx + h}), \tag{2}$$

for some particular cases. Varshamov studied one case from (2) and gave the following proposition:

Proposition 1.1. ([10, Theorem 3.13]) *Let $P(x) = \sum_{i=0}^{n} c_i x^i$ be an irreducible polynomial over $\mathbb{F}_q$ and p be the characteristic of $\mathbb{F}_q$. Then the polynomial $P(x^p - x - \delta_0)$ is an irreducible polynomial over $\mathbb{F}_q$ if and only if*

$$Tr_{q|p}(n\delta_0 - c_{n-1}) \neq 0.$$

Also, for this case, Kyuregyan gave a recurrent method for constructing irreducible polynomials in the following proposition:

Proposition 1.2. (Kyuregyan [8, Theorem 2]) *Let $F(x) = \sum_{u=0}^{n} c_u x^u$ be an irreducible polynomial over $\mathbb{F}_q$ and suppose that there exist an element $\delta_0 \in \mathbb{F}_p$ such that $F(\delta_0) = a$, with $a \in \mathbb{F}_p^*$ and*

$$Tr_{q|p}(n\delta_0 - c_{n-1}) Tr_{q|p}(F'(\delta_0)) \neq 0.$$

Let $g_0(x) = x^p - x + \delta_0$ and $g_k(x) = x^p - x + \delta_k$, where $\delta_k \in \mathbb{F}_p^*$, $k \geqslant 1$. Define $F_0(x) = F(g_0(x))$, and $F_k(x) = F_{k-1}^*(g_k(x))$ for $k \geqslant 1$, where $F_{k-1}^*(x)$ is the reciprocal polynomial of $F_{k-1}(x)$. Then for each $k \geqslant 0$, the polynomial $F_k(x)$ is an irreducible polynomial of degree $n_k = np^{k+1}$ over $\mathbb{F}_q$.

We note that the above proposition is the generalization of Varshamov's theorem, that the reader can find it in [10]. He also gave another recurrent method for constructing irreducible polynomials in the following proposition:

Proposition 1.3. (Kyuregyan [7], Corollary 2) *Let s be odd integer, δ be any element of $\mathbb{F}_{2^s}^*$, and the sequence of functions $\varphi_m(x)$ be defined by*

$$\varphi_m(x) = a_m(x) + \delta b_m(x)$$

under the initial condition

$$\varphi_0(x) = x + \delta.$$

Then the polynomial $\varphi_m(x)$ of degree $2m$ defined by the recurrent relation

$$\varphi_m(x) = x^{2^{m-1}} \varphi_{m-1}(x + \frac{\delta^2}{x})$$

is an irreducible polynomial over $\mathbb{F}_{2^s}$, where

$$a_1(x) = x^2 + \delta^2, \quad b_1(x) = x$$

and

$$a_m(x) = a_{m-1}^2(x) + b_{m-1}^2(x)$$

and also

$$b_m(x) = a_{m-1}(x) b_{m-1}(x).$$

The aim of this paper is to determine under what conditions

$$F(x) = x^{2n} P(\frac{x^2 - \delta_0 x + \delta_1}{x^2}), \qquad \delta_0, \delta_1 \in \mathbb{F}_{2^s}^*$$

is an irreducible polynomial over $\mathbb{F}_{2^s}$, where $P(x)$ is an irreducible polynomial of degree n over $\mathbb{F}_{2^s}$, and also giving a recurrent method for constructing families of irreducible polynomials $F_k(x)$, for $k \geqslant 0$ over finite fields, when $F_0(x) = P(x)$. Such polynomials are used to implement arithmetic in extension fields and are found in many applications, including coding theory [1] and [6], cryptography [2], [4] and [5], computer algebra system [3].

In [9] Melsik K. Kyuregyan and Gohar M. Kyuregyan presented a new method for constructing irreducible polynomials over finite fields. They proved the following results which will be used in the proof of our results.

Proposition 1.4. (M. K. Kyuregyan and G. M. Kyureghyan [9], Lemma 1) *A monic polynomial $f(x) \in \mathbb{F}_q[x]$ of degree $n = dk$ is irreducible over $\mathbb{F}_q$ if and only if there is a monic irreducible polynomial $h(x) = \sum_{i=0}^{k} h_i x^i$ over $\mathbb{F}_{q^d}$ of degree k such that $\mathbb{F}_q(h_0, h_1, ..., h_k) = \mathbb{F}_{q^d}$ and $f(x) = \prod_{v=0}^{d-1} h^{(v)}(x)$ on $\mathbb{F}_{q^d}[x]$, where*

$$h^{(v)}(x) = \sum_{i=0}^{k} h_i^{q^v} x^i.$$

2. Irreducibility of Composition Polynomials

In this section we examine the irreducibility of composite polynomial $x^{2n} P(\frac{x^2-\delta_0 x+\delta_1}{x^2})$ over $\mathbb{F}_{2^s}$. We prove some results that will be helpful to construct sequences of high degree irreducible polynomials over a finite fields. The following proposition will be helpful to derive our results.

Proposition 2.1. *([10], Corollary 3.6) For $a, b \in \mathbb{F}_q^*$ the trinomial $x^p - ax - b$ is irreducible over $\mathbb{F}_q$ if and only if $a = A^{p-1}$, for some $A \in \mathbb{F}_q$ and $Tr_{q|p}(\frac{b}{A^p}) \neq 0$.*

Theorem 2.2. *Let $P(x) = \sum_{i=0}^{n} c_i x^i$ be an irreducible polynomial over $\mathbb{F}_{2^s}$ of degree n. Then*

$$F(x) = x^{2n} P(\frac{x^2 - \delta_0 x + \delta_1}{x^2}), \qquad \delta_0, \delta_1 \in \mathbb{F}_{2^s}^*$$

is an irreducible polynomial of degree $2n$ over F_{2^s} if and only if

$$Tr_{2^s|2}(\frac{\delta_1}{\delta_0^2}(\frac{P^{*\prime}(0)}{P^*(0)} + n)) \neq 0.$$

Proof. Let $\alpha \in \mathbb{F}_{2^{sn}}$ be a root of $P(x)$. Irreducibility of $P(x)$ over $\mathbb{F}_{2^s}$ implies that it can represented over $\mathbb{F}_{2^{sn}}$ as

$$P(x) = c_n \prod_{u=0}^{n-1} (x - \alpha^{2^{su}}).$$

By substituting $\frac{x^2-\delta_0 x+\delta_1}{x^2}$ for x and multiplying its both sides by x^{2n}, we get

$$\begin{aligned} F(x) &= x^{2n} P(\frac{x^2 - \delta_0 x + \delta_1}{x^2}) \\ &= c_n x^{2n} \prod_{u=0}^{n-1} (\frac{x^2 - \delta_0 x + \delta_1}{x^2} - \alpha^{2^{su})} \\ &= c_n \prod_{u=0}^{n-1} (1 - \alpha^{2^{su}})(x^2 - (\frac{\delta_0}{1-\alpha})^{2^{su}} x - (\frac{\delta_1}{\alpha - 1})^{2^{su}}) \\ &= c_n (1-\alpha)^{\frac{2^{sn}-1}{2^s-1}} \prod_{u=0}^{n-1} (x^2 - (\frac{\delta_0}{1-\alpha})^{2^{su}} x - (\frac{\delta_1}{\alpha - 1})^{2^{su}}). \end{aligned}$$

Proposition 4 implies that $F(x)$ is an irreducible polynomial over $\mathbb{F}_{2^s}$ if and only if

$$x^2 - \frac{\delta_0}{\alpha - 1} x - \frac{\delta_1}{\alpha - 1}$$

is an irreducible polynomial over $\mathbb{F}_{2^{sn}}$. Then by Proposition 5, $F(x)$ is an irreducible polynomial over $\mathbb{F}_{2^s}$ if and only if

$$Tr_{2^{sn}|2}(\frac{\frac{\delta_1}{\alpha-1}}{(\frac{\delta_0}{1-\alpha})^2}) = Tr_{2^{sn}|2}(\frac{\delta_1}{\delta_0^2}(\alpha - 1)) \neq 0.$$

On the other side by (1),

$$Tr_{2^{sn}|2}(\frac{\delta_1}{\delta_0^2}(\alpha - 1)) = Tr_{2^s|2}(Tr_{2^{sn}|2^s}(\frac{\delta_1}{\delta_0^2}(\alpha - 1))). \tag{3}$$

Recall that for an irreducible polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ of degree n over $\mathbb{F}_q$, we have

$$Tr_{q^n|q}(\beta) = -\frac{a_{n-1}}{a_n},$$

where $\beta \in \mathbb{F}_{q^n}$ is a root of $f(x)$(see [6], page 51). So

$$\begin{aligned} Tr_{2^{sn}|2^s}(\frac{\delta_1}{\delta_0^2}(\alpha-1)) &= \frac{\delta_1}{\delta_0^2} Tr_{2^{sn}|2^s}(\alpha - 1) \\ &= \frac{\delta_1}{\delta_0^2}(Tr_{2^{sn}|2^s}(\alpha) - Tr_{2^{sn}|2^s}(1)) \\ &= \frac{\delta_1}{\delta_0^2}(\frac{c_{n-1}}{c_n} + n). \end{aligned} \tag{4}$$

Hence (3) and (4) imply that

$$Tr_{2^{sn}|2}(\frac{\delta_1}{\delta_0^2}(\alpha-1)) = Tr_{2^s|2}(\frac{\delta_1}{\delta_0^2}(\frac{c_{n-1}}{c_n} + n)).$$

By the given condition $Tr_{2^s|2}(\frac{\delta_1}{\delta_0^2}(\frac{P^{*\prime}(0)}{P^*(0)} + n)) \neq 0$, $F(x)$ is an irreducible polynomial over $\mathbb{F}_{2^s}$. □

Example 2.3. Consider the irreducible polynomial $P(x) = x^2 + x + (\alpha + 1)$ over the Galois field $\mathbb{F}_4 = \{0, 1, \alpha, \alpha + 1\}$, where α is a root of the irreducible polynomial $x^2 + x + 1$ over $\mathbb{F}_2$. According to Theorem 2.

$$\begin{aligned} F(x) &= x^4 P(\frac{x^2 - (\alpha+1)x + \alpha}{x^2}) \\ &= (x^2 - (\alpha+1)x + \alpha)^2 + x^2(x^2 - (\alpha+1)x + \alpha) + (\alpha+1)x^4 \\ &= (\alpha+1)x^4 + (\alpha+1)x^3 + (\alpha+1) \end{aligned}$$

is an irreducible polynomial over $\mathbb{F}_4$.

Corollary 2.4. *Let $P(x) = \sum_{i=0}^{n} c_i x^i$ be an irreducible polynomial over $\mathbb{F}_2$ of degree n. Then*

$$F(x) = x^{2n} P(\frac{x^2 - x + 1}{x^2})$$

is an irreducible polynomial of degree $2n$ over $\mathbb{F}_2$ if and only if

$$\frac{c_{n-1}}{c_n} + n \neq 0.$$

3. Recurrent Method

In this section we shall describe a computationally simple and explicit recurrent method for constructing higher degree irreducible polynomials over finite field $\mathbb{F}_{2^s}$ starting from an irreducible polynomial.

Theorem 3.1. *Let $P(x)$ be an irreducible polynomial of degree n over $\mathbb{F}_{2^s}$. Define*

$$F_0(x) = P(x),$$

$$F_k(x) = x^{n2^k} F_{k-1}(\frac{x^2 - x + 1}{x^2}) \qquad k \geqslant 1. \tag{5}$$

Suppose that

$$Tr_{2^s|2}(\frac{P'(1)}{P(1)}) \cdot Tr_{2^s|2}(\frac{P^{*\prime}(0)}{P^*(0)} + n) \neq 0.$$

Then $(F_k(x))_{k \geqslant 1}$ is a sequence of irreducible polynomials over $\mathbb{F}_{2^s}$ of degree $n2^k$.

Proof. We start our proof by setting $\delta_0, \delta_1 = 1$ in Theorem 2.2. According to Theorem 2.2. and hypothesis of theorem , $F_1(x)$ is an irreducible polynomial over $\mathbb{F}_{2^s}$ of degree $2n$. Also by Theorem 2.2. for every $k \geqslant 2$, $F_k(x)$ is an irreducible polynomial over $\mathbb{F}_{2^s}$ if and only if $Tr_{2^s|2}(\frac{{F^*_{k-1}}'(0)}{F^*_{k-1}(0)}) \neq 0$. On the other hand, from (5), we have

$$\begin{aligned} F^*_k(x) &= x^{n2^k} F_k(\frac{1}{x}) \\ &= x^{n2^k} ((\frac{1}{x})^{n2^k} F_{k-1}(\frac{(\frac{1}{x})^2 - (\frac{1}{x}) + 1}{(\frac{1}{x})^2}) \\ &= F_{k-1}(x^2 - x + 1), \end{aligned} \tag{6}$$

for every $k \geqslant 1$. So

$$F^*_k(0) = F_{k-1}(1), \tag{7}$$

and

$$F_k^{*\prime}(0) = F'_{k-1}(1), \tag{8}$$

for every $k \geqslant 1$. On the other side

$$F'_k(x) = x^{n2^k-2} F'_{k-1}(\frac{x^2-x+1}{x^2}). \tag{9}$$

So

$$F'_k(1) = F'_{k-1}(1), \tag{10}$$

for every $k \geqslant 1$. Using (8) and (9), we get

$$F_k^{*\prime}(0) = P'(1). \tag{11}$$

Obviously by (5)

$$F_k(1) = F_{k-1}(1), \tag{12}$$

for every $k \geqslant 1$. So (7) and (12) imply that $F_k^*(0) = P(1)$, for every $k \geqslant 1$. Thus by hypothesis of theorem $F_k(x)$ is an irreducible polynomial over $\mathbb{F}_{2^s}$, for every $k \geqslant 2$, and the proof is completed. □

Corollary 3.2. *Consider the irreducible polynomial $F_0(x) = x^2 + x + 1$ over the Galois field $\mathbb{F}_2$. According to Theorem 3., for each $k \geqslant 1$,*

$$F_k(x) = x^{2^{k+1}} F_{k-1}(\frac{x^2-x+1}{x^2}),$$

is a sequence of irreducible polynomials over $\mathbb{F}_2$ of degree 2^{k+1}.

References

[1] E. R. Berlekamp, *Algebraic coding theory*, Mc Graw-Hill, New York, 1968.

[2] I. F. Blake, G. Seroussi, and N. P. Smart, *Elliptic curves in cryptography*, Cambridge University Press, Cambridge, reprinted, 2000.

[3] J. Calment, Algebraic algorithms in GF(q), *Discrete Math.,* 56 (1985), 101-109.

[4] B. Chor and R. Rivest, *Aknapsack-type public key cryptosystem based on arithmetic in finite fields*, IEEE Trans. Inform. Theory, 34 (1988), 901-909.

[5] N. Koblitz, *Algebraic a spects of cryptography*, Springer, Berlin, 1998.

[6] R. Lidl and H. Niederreiter, *Finite Fields*, Cambridge University, Press Cambridge, 1987.

[7] M. K. Kyuregyan, Recurrent methods for constructing irreducible polynomials over $GF(2^s)$, *Finite Fields and Their Applications,* 8 (2002), 52-68.

[8] M. K. Kyuregyan, Iterated constructions of irreducible polynomials over finite fields with linearly independent roots, *Finite Fields and Their Applications,* 10 (2004), 323-341.

[9] M. K. Kyuregyan and G. M. Kyureghyan, *Irreducible compositions of polynomials over finite fields*, Designs Codes and Cryptography, DOI: 10.1007/s 10623-010-9476-7.

[10] A. J. Menezes, I. F. Blake, X. Gao, R. C. Mullin, S. A. Vanstone, and T. Yaghoobian, *Applications of finite fields*, Kluwer Academic publishers, Boston, Dordrecht, Lancaster, 1993.

Mahmood Alizadeh
Department of Mathematics
Assistant Professor of Mathematics
Islamic Azad University-Ahvaz Branch
Ahvaz, Iran
E-mail: alizadeh@iauahvaz.ac.ir

Journal of Mathematical Extension
Vol. 6, No. 2, (2012), 75-90

Solving Integro-Differential Equations by a Semi-Analytic Method

M. Fardi*
Islamic Azad University, Najafabad Branch

M. Ghasemi
Shahrekord University

F. Hemati Boroujeni
Islamic Azad University, Boroujen Branch

Abstract. In this paper, we propose a method to obtain approximate solutions to Fredholm integral-differential equations by employing the homotopy analysis method (HAM). The HAM gives the possibility to increase convergence region and rate of series solution. we show that the adomian decomposition method (ADM) cannot give better results than the present method. Five examples are presented to illustrate convergence and accuracy of the method to the solution. Also, we compute the absolute error to show that obtained results have reasonable accuracy.

AMS Subject Classification: 45J05; 65T60
Keywords and Phrases: Integro-differential equations, Homotopy analysis method

1. Introduction

The homotopy analysis method have been used for many years for solving mathematical problems. This new method has been presented by Liao ([1]) and applied to nonlinear oscillators with discontinuities ([2-4]), heat transfer ([5,6]), boundary layer flows ([7-9]), chaotic dynamical systems ([10]), systems of ODEs ([11]), delay differential equation ([12]),

Received: August 2011; Accepted: December 2012
*Corresponding author

ordinary differential equations ([13]), Glauert-jet problem ([14]), chaotic dynamical systems ([15]) and strongly nonlinear oscillatory system ([16]). Consider the following Feredholm integro-Differential equation

$$\sum_{j=0}^{m} P_j(t)u^{(j)}(t) = F(t) + \int_a^b K(t,s)G(u(s),...,u^{(m)}(s))ds, \quad (1)$$
$$\frac{\partial^i u(t)}{\partial t^i}|_{t=a} = \lambda_i, i = 0,1,...,m_1,$$
$$\frac{\partial^i u(t)}{\partial t^i}|_{t=b} = \lambda_i, i = 0,1,...,m_2.$$

where $u(t) : [a,b] \to \Re$ is the unknown function. where $K(t,s)$ and $P_j(t)$, $j = 0,1,2,...,m$ are known functions.
In this paper, we propose an analytical method to solve the Feredholm's Integro-Differential Equations. Comparisons are made between ADM and the proposed method. It is demonstrated that the solutions obtained by the ADM are special cases of the present method. For the purpose, we first give the following definition and theorems.

Definition 1.1. *Let ϕ be a function of the homotopy parameter q, then*

$$D_n(\phi) = \tfrac{1}{n!}\tfrac{\partial^n \phi}{\partial q^n}\Big|_{q=0},$$

is called the nth-order homotopy-derivative of ϕ, where $n \geqslant 0$ is an integer.

Theorem 1.2. *For homotopy-series*

$$\phi = \sum_{k=0}^{\infty} u_k q^k,$$

it holds the recurrence formulas

$$D_0(e^{\phi}) = e^{u_0},$$

$$D_m(e^{\phi}) = \sum_{k=0}^{m-1} (1 - \frac{k}{m}) D_k(e^{\phi}) D_{m-k}(\phi),$$

where $m \geqslant 1$ is integer.

Proof. See [17]. □

Theorem 1.3. *For homotopy-series*

$$\phi = \sum_{k=0}^{\infty} u_k q^k,$$

it holds the recurrence formulas

$$D_0(\sin(\phi)) = \sin(u_0),\ \ D_0(\cos(\phi)) = \cos(u_0),$$

$$D_m(\sin(\phi)) = \sum_{k=0}^{m-1}(1-\frac{k}{m})D_k(\cos(\phi))D_{m-k}(\phi),$$
$$D_m(\cos(\phi)) = \sum_{k=0}^{m-1}(1-\frac{k}{m})D_k(\sin(\phi))D_{m-k}(\phi),$$

where $m \geqslant 1$ is integer.

Proof. See [17]. □

2. Main Results

2.1 Analysis of the Method for the Feredholm Integro-Differential Equations

From (4) we define the nonlinear operator

$$\begin{aligned} N(S(t;q),q) &= \sum_{j=0}^{m} P_j(t)\frac{\partial^j S(t;q)}{\partial t^j} - F(t) \\ &- \int_a^b K(t,s)G(S(s;q),...,\frac{\partial^m S(s;q)}{\partial s^m})ds, \end{aligned} \tag{2}$$

and we choose the auxiliary linear operator as follows

$$L(S(t;q)) = \frac{\partial^m S(t;q)}{\partial t^m},$$

where $q \in [0,1]$ is an embedding parameter; $S(t;q)$, is real function of t and q, respectively. Let $\hbar$ denote a nonzero auxiliary parameter. Also, assume u_0 denote the initial guess of the exact solution $u(t)$.
We construct the zero-order deformation equation

$$(1-q)L[S(t;q)-u_0] = q\hbar N(S(t;q),q), \tag{3}$$

subject to the boundary conditions

$$\frac{\partial^i S(t;q)}{\partial t^i}|_{t=a} = \lambda_i, i = 0,1,...,m_1,$$
$$\frac{\partial^i S(t;q)}{\partial t^i}|_{t=b} = \lambda_i, i = 0,1,...,m_2.$$

Using Taylor's theorem, we expand $S(t;q)$ in the power series of q as follows

$$S(t;q) = u_0 + \sum_{j=1}^{\infty} u_j(t)q^j, \tag{4}$$

where

$$u_j(t) = D_j(S(t;q)).$$

Note that (2.) contains an auxiliary parameter $\hbar$. Assuming that is correctly chosen so that (2.) is convergent at $q = 1$, we have the series solution

$$u(t) = S(t;1) = u_0 + \sum_{j=1}^{\infty} u_j(t),$$

Operating on both sides of (2), we have the so-called nth-order deformation equation

$$L[u_n(t) - \chi_n u_{n-1}(t)] = \hbar R_n(u_0, u_1, ..., u_{n-1}, t),$$

$$\frac{\partial^i u_{n-1}(t)}{\partial t^i}|_{t=a} = 0, i = 0,1,...,m_1,$$
$$\frac{\partial^i u_{n-1}(t)}{\partial t^i}|_{t=b} = 0, i = 0,1,...,m_2.$$

where

$$\chi_n = \begin{cases} 0, & n \leqslant 1, \\ 1, & otherwise. \end{cases}$$

and

$$R_n(u_0, ..., u_{n-1}, t) = D_{n-1}(N(u_0 + \sum_{j=1}^{\infty} u_j(t)q^j)).$$

Then, we have

$$\begin{aligned} R_n &= \sum_{j=0}^{m} P_j(t)u_{n-1}^{(j)} - F(t)(1-\chi_n) \\ &- \int_a^b K(t,s)D_{n-1}(G(S(s;q), ..., \frac{\partial^m S(s;q)}{\partial s^m}))ds. \end{aligned} \tag{5}$$

We gain u_n $(n = 1, 2, 3, ...)$, successively. At the Mth-order approximation we have

$$u(t) \approx U_M(t, \hbar) = u_0 + \sum_{j=1}^{M} u_j(t).$$

2.2 Convergence of Method and Comparison to ADM

Theorem 2.2.1. *If the series solution*

$$u_0(t) + \sum_{j=1}^{\infty} u_j(t),$$

converges then it is an exact solution of of (4).

Proof. If the series solution:

$$u_0(t) + \sum_{j=1}^{\infty} u_j(t),$$

is convergent, then:

$$\lim_{j\to\infty} u_j(t) = 0. \tag{6}$$

Using (6), we obtain:

$$\lim_{m\to\infty} \sum_{n=1}^{m} L\big[u_n(t) - \chi_n u_{n-1}(t)\big] = \lim_{m\to\infty} u_m(t) = 0.$$

Since $\hbar \neq 0$, we deduce:

$$\sum_{n=1}^{\infty} R_n(u_0, u_1, ..., u_{n-1}, t) = 0.$$

Now, from (2), it conclude:

$$\begin{aligned}\sum_{n=1}^{\infty} R_n &= \sum_{n=1}^{\infty}\sum_{j=0}^{m} P_j(t) u_{n-1}^{(j)} - \sum_{n=1}^{\infty} F(t)(1-\chi_n) \\ &- \int_a^b K(t,s) \sum_{n=1}^{\infty} D_{n-1}(G(S(s;q), ..., \frac{\partial^m S(s;q)}{\partial s^m}))ds] = 0. \end{aligned} \tag{7}$$

If the series solution

$$u(t) = u_0(t) + \sum_{j=1}^{\infty} u_j(t),$$

is convergent, then the series

$$\sum_{n=1}^{\infty} D_{n-1}(G(S(s;q), ..., \frac{\partial^m S(s;q)}{\partial s^m})),$$

will converge to $G(u(s), ..., u^{(m-1)}(s), u^{(m)}(s))$ (see [18]).
Now, by using (7) we have:

$$\sum_{j=0}^{m} P_j(t) u^{(j)}(t) = F(t) + \int_a^b K(t,s) G(u(s), ..., u^{(m-1)}(s), u^{(m)}(s)) ds. \tag{8}$$

This completes the proof. □

Remark 2.2.2. *The valid region of $\hbar$ for convergence of series solution $u_0(t) + \sum_{j=1}^{\infty} u_j(t)$, can be found although approximately by plotting the curves of unknown quantities versus $\hbar$. Let $t_0 \in [a, b]$, then $U_M(t_0, \hbar)$, is function of $\hbar$. In accordance with $\hbar$-curve of $U_M(t_0, \hbar)$, we can find the valid region of $\hbar$ **[1]**.*

Theorem 2.2.3. *(Comparison to ADM) If $\hbar = -1$ and $L[u_0(t)] = F(t)$, the present method will be converted to ADM.*

Proof. See [1]. □

3. Test Examples

In this section, we solve five test problems to demonstrate the accurate nature of the proposed method. The validity of the method is based on assumption that the series (2.) converges at $q = 1$.
There is the convergence-control parameter $\hbar$ which guarantees that this assumption can be satisfied. We need to concentrate on the convergence of the obtained results by properly choosing $\hbar$.

Example 3.1. Consider the following nonlinear integro differential equation

$$\begin{cases} u''(t) = 2 - \frac{sin(1)}{2}(t^2 + 1) - \int_0^1 z(t^2 + 1)\cos(u(z))dz, \\ u(0) = 0, \quad u'(0) = 0. \end{cases}$$

The exact solution of this problem is $u(t) = t^2$ ([18]).
We choose $u_0(t) = 0$ as initial approximation guess. We study the influence of $\hbar$ on the convergence of $U_6'(0.5, \hbar)$. We can investigate the influence of $\hbar$ on the convergence region of $U_6'(0.5, \hbar)$ by means of $\hbar$-curve as shown in Fig. 1. From Fig. 1, the convergence region of $U_6'(0.5, \hbar)$ is $[-1.2, -0.5]$. The Error function $|U_M(t, \hbar) - u(t)|$ with M $= 6$ has been plotted for different $\hbar$ in Fig. 2.

Example 3.2. Consider the following linear integro differential equation

$$\begin{cases} u''(t) = t - \sin(t) - \int_0^{\frac{\pi}{2}} tzu(z)dz, \\ u(0) = 0, \quad u'(0) = 1. \end{cases}$$

The exact solution of this problem is $u(t) = \sin(t)$ ([18]).
We choose $u_0(t) = t$ as initial approximation guess. The curve of $U_6(\frac{\pi}{2}, \hbar)$ is plotted in Fig. 3 to determine the valid region of $\hbar$. As shown in Fig. 3, the series solutions of $U_6(\frac{\pi}{2}, \hbar)$ converge at $[-1.05, -0.55]$. The results presented in Table 1 clearly show the good accuracy of present method.

Example 3.3. Consider the following nonlinear integro differential equation

$$\begin{cases} u'(t) = 1 - e^{-1} + \int_0^1 e^{-u'(t)} dt, \\ u(0) = 0. \end{cases}$$

The exact solution of this problem is $u(t) = t$ ([19]).
Let us choose $u_0(t) = 0$ as initial approximation guess. Fig. 4 shows the $\hbar$-curve obtained from the $\frac{\partial U_6}{\partial t}$ at $t = 0.5$. From this figure, the valid values of $\hbar$ fall in the range $[-0.45, -0.35]$. The Error function $|U_M(t, \hbar) - u(t)|$ with $M = 6$ has been plotted for different value of $\hbar$ in Fig. 5.

Example 3.4. Consider the following nonlinear integro differential equation

$$\begin{cases} u'(t) = \frac{5}{4} - \frac{x^2}{3} + \int_0^1 (x^2 - t) u^2(t) dt, \\ u(0) = 0. \end{cases}$$

The exact solution of this problem is $u(t) = t$ ([20]).
Let us choose $u_0(t) = 0$ as initial approximation guess. In Fig. 6, $\hbar$-curve of $U_8(0.4, \hbar)$ has been plotted, as we see the valid region of $\hbar$ is $[-1.2, -0.3]$. The numerical solution obtained from the present method is much more accurate than the numerical solution given by the ADM, as shown in Table 2.

Example 3.5. Consider the following linear integro differential equation

$$\begin{cases} u''(t) + xu'(t) - xu(x) = e^t - 2\sin(t) - \int_{-1}^1 \sin(t) e^{-z} u(z) dz, \\ u(0) = 1, \quad u'(0) = 1. \end{cases}$$

The exact solution of this problem is $u(t) = e^t$ ([21]).
We choose $u_0(t) = 1 + t$ as initial approximation guess. To find the valid

region of $\hbar$, the $\hbar$-curve given by the $\frac{\partial U_5}{\partial t}$ at $t = -0.5$ is drawn in Fig. 7, which indicates that the valid region of $\hbar$ is about $[-1.2, -0.4]$. The convergence region of the solution given by ADM is $t \in [-0.5, 0.5]$, as shown in Fig. 8. When $\hbar = -1, -0.9, -0.7$, we obtain an approximate solution which is much more accurate than the solution given by the ADM as shown in Table 3.

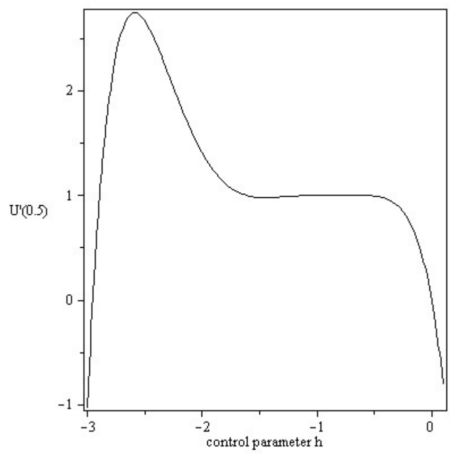

Figure 1: The $\hbar$-curve of $U_6'(0.5, \hbar)$ (Example 3.1).

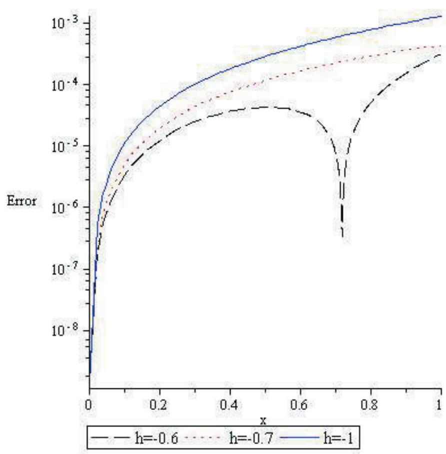

Figure 2: The error with $\hbar = -0.6$, $\hbar = -0.7$ and $\hbar = -1$ (Example 3.1).

Table 1: Absolute error (Example 3.2).

t	u_{exact}	$\hbar=-1$	$\hbar=-0.8$	$\hbar=-0.7$	ADM
$\pi.10^{-1}$	0.3090169944	4.9E-6	$2.6E-8$	$2.9E-7$	$5.4E-6$
$\pi.9^{-1}$	0.3420201433	6.8E-6	$3.6E-8$	$4.0E-7$	$7.4E-6$
$\pi.8^{-1}$	0.3826834325	9.6E-6	$4.9E-8$	$5.6E-7$	$1.1E-5$
$\pi.7^{-1}$	0.4338837393	1.4E-5	$7.1E-8$	$8.1E-7$	$1.5E-5$
$\pi.6^{-1}$	0.5000000000	2.2E-5	$1.0E-7$	$1.2E-6$	$2.5E-5$
$\pi.5^{-1}$	0.5877852524	3.9E-5	$1.7E-7$	$1.9E-6$	$4.3E-5$
$\pi.4^{-1}$	0.7071067810	7.7E-5	$2.7E-7$	$3.1E-6$	$8.4E-5$
$\pi.3^{-1}$	0.8660254040	1.8E-4	$3.8E-7$	$4.3E-6$	$2.0E-4$
$\pi.2^{-1}$	1.0000000000	6.2E-4	$1.3E-6$	$1.5E-5$	$6.8E-4$

Table 2: Absolute error (Example 3.4).

t	u_{exact}	$\hbar=-1.0$	$\hbar=-0.8$	$\hbar=-1.1$	ADM
0	0.0	0.0	0.0	0.0	0.0
0.2	0.2	5.1E-3	$5.0E-4$	$5.7E-4$	$5.9E-3$
0.4	0.4	9.6E-3	$9.5E-4$	$1.1E-3$	$1.1E-3$
0.6	0.6	1.3E-2	$1.2E-3$	$1.4E-3$	$1.5E-3$
0.8	0.8	1.5E-2	$1.3E-3$	$1.5E-3$	$1.6E-3$
1.0	1.0	1.4E-2	$1.4E-3$	$1.6E-3$	$1.5E-3$

Table 3: Absolute error (Example 3.5).

t	u_{exact}	$\hbar=-1$	$\hbar=-0.9$	$\hbar=-0.7$	ADM
-1	0.3678794412	2.7E-4	$4.9E-6$	$6.0E-6$	*divergent*
-0.8	0.4493289641	7.7E-5	$6.4E-6$	$1.5E-5$	*divergent*
-0.6	0.5488116361	1.7E-5	$2.2E-6$	$4.9E-6$	*divergent*
-0.4	0.6703200460	5.0E-6	$5.2E-7$	$5.2E-6$	$3.7E-2$
-0.2	0.8187307531	3.6E-7	$1.9E-6$	$4.5E-6$	$4.5E-3$
0	1	0	$1.7E-7$	$1.6E-7$	0
0.2	1.221402758	5.6e-6	$2.0E-6$	$6.7E-6$	$4.0e-3$
0.4	1.491824698	1.9E-6	$6.4E-6$	$2.6E-5$	$2.9E-2$
0.6	1.822118800	2.2E-5	$3.0E-6$	$3.3E-5$	$8.8E-2$
0.8	2.225540928	7.3E-5	$6.5E-7$	$3.1E-5$	*divergent*
1	2.718281828	1.7E-4	$7.4E-7$	$2.6E-5$	*divergent*

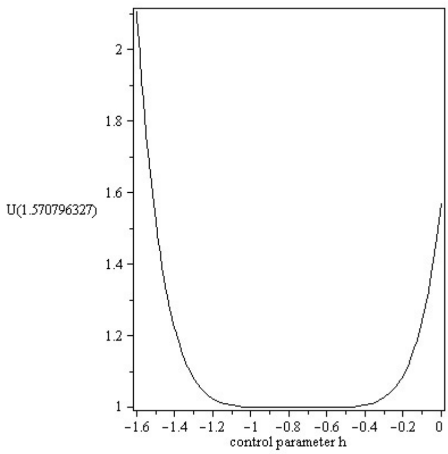

Figure 3: The $\hbar$-curve of $U_6(\frac{\pi}{2}, \hbar)$ (Example 3.2).

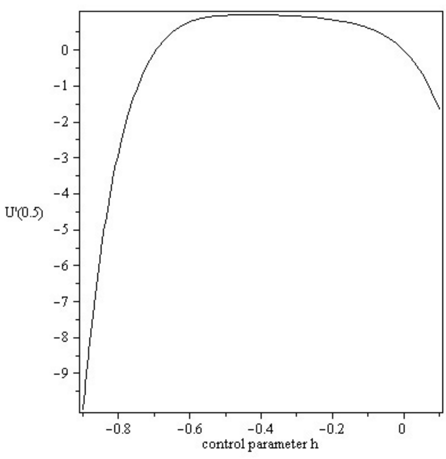

Figure 4: The $\hbar$-curve of $U'_6(0.5, \hbar)$ (Example 3.3).

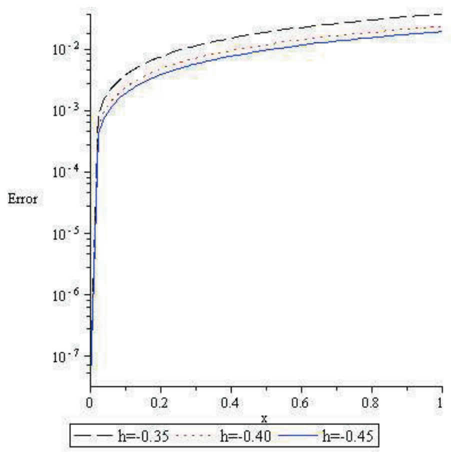

Figure 5: The error with $\hbar = -0.35$, $\hbar = -0.4$ and $\hbar = -0.45$ (Example 3.3).

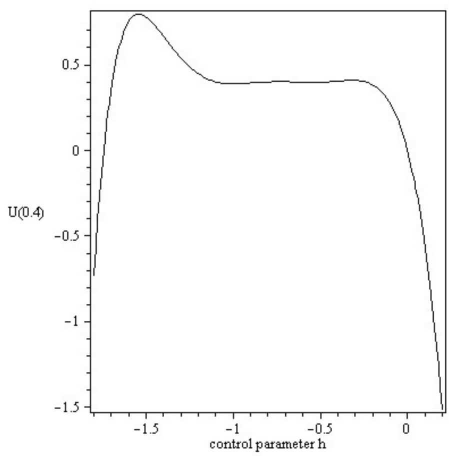

Figure 6: The $\hbar$-curve of $U_8(0.4, \hbar)$ (Example 3.4).

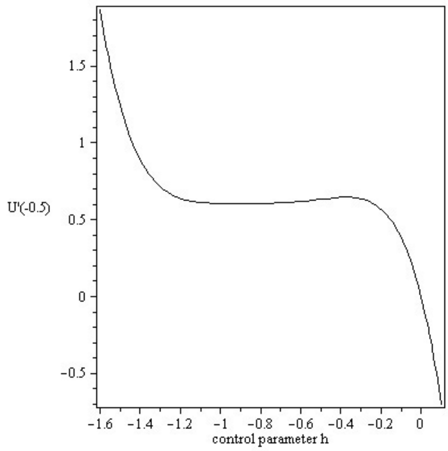

Figure 7: The $\hbar$-curve of $U_5^{'}(-0.5,\hbar)$ (Example 3.5).

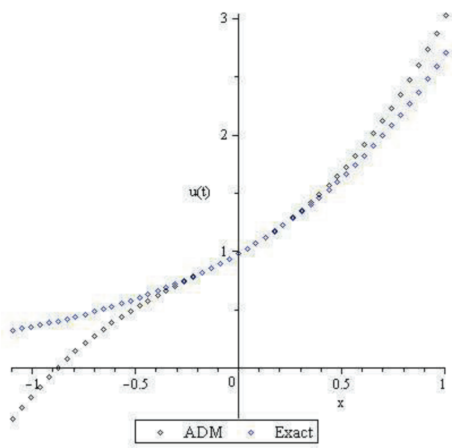

Figure 8: Comparison of the exact result with the 5th-order approximation given by ADM (Example 3.5).

4. Conclusion

In this paper, an semi-analytical method was proposed for solving Feredholm Integro-Differential Equations. The efficiency of this method is demonstrated by solving five examples. We have illustrated that the ADM cannot give better results than the present method. In fact, the ADM are only the especial case of the present method.

References

[1] S. J. Liao, Beyond Perturbation: *Introduction to Homotopy Analysis Method,* Chapman & Hall/ CRC Press, Boca Raton 2003.

[2] S. J. Liao and A. T. Chwang, Application of homotopy analysis method in nonlinear oscillations, *ASME J. of Appl. Mech.*, 65 (1998), 914-922.

[3] S. J. Liao, An analytic approximate technique for free oscillations of positively damped systems with algebraically decaying amplitude, *Int. J. Non-Linear Mech.*, 38 (2003), 1173-1183.

[4] S. J. Liao, An analytic approximate approach for free oscillations of self-excited systems, *Int. J. Non-Linear Mech.*, 39 (2004), 271-280.

[5] S. J. Liao and A. Campo, Analytic solutions of the temperature distri bution in Blasius viscous flow problems, *J. of Fluid Mech.*, 453 (2002), 411-425.

[6] C. Wang et al, On the explicit analytic solution of cheng-chang equation, *Int. J. Heat and Mass Transfer*, 46 (2003), 1855-1860.

[7] S. J. Liao, A kind of approximate solution technique which does not depend upon small parameters (II): an application in fluid mechanics, *Int. J. of Non-Linear Mech.*, 32 (1997), 815-822.

[8] S. J. Liao, An explicit, totally analytic approximation of Blasius viscous flow problems, *Int. J. of Non-Linear Mech.*, 34 (1999), 759-778.

[9] S. J. Liao, An analytic approximation of the drag coe?cient for the viscous flow past a sphere, *Int. J. of Non-Linear Mech.*, 37 (2002), 1-18.

[10] F. M. Allan, Construction of analytic solution to chaotic dynamical systems using the Homotopy analysis method, *Chaos Solitons Fractals,* 39 (2009), 1744-1752.

[11] A. S. Bataineh, M. S. M. Noorani, and I. Hashim, Solving systems of ODEs by homotopy analysis method, *Commun. Nonlinear Sci. Numer. Simul*, 13 (2008), 2060-2070.

[12] A. K. Alomari, M. S. M. Noorani, and R. Nazar, Solution of delay differential equation by means of homotopy analysis method, *Acta Applicandae Mathematicae,* 108 (2009), 395-412.

[13] F. T. Akyildiz, K. Vajravelu, and S. J. Liao, A new method for homoclinic solutions of ordinary differential equations, *Chaos Solitons Fractals,* 39 (2009), 1073-1082.

[14] Y. Bouremel, Explicit series solution for the Glauert-jet problem by means of the homotopy analysis method, *Commun. Nonlinear. Sci. Numer. Simulat*, 12 (2007), 714-724.

[15] F. M. Allan, Construction of analytic solution to chaotic dynamical systems using the Homotopy analysis method, *Chaos Solitons Fractals,* 39 (2009), 1744-1752.

[16] A. Golbabai, M. Fardi, and K. Sayevand, Application of optimal homotopy asymptotic method for solving strongly nonlinear oscillatory system, *Mathematical and Computer Modelling*, in press.

[17] S. J. Liao, Notes on the homotopy analysis method: some definitions and theorems, *Commun. Nonlinear Sci. Numer. Simul*, 14 (2009), 983-997.

[18] Y. Cherruault, *Convergence of Adomians method,* Kybernetes, 18 (1989), 31-38.

[19] S. R. Seyed Alizadeh, G. G. Domairry, and S. Karimpour, An Approximation of the Analytical Solution of the Linear and Nonlinear Integro-Differential Equations by Homotopy Perturbation Method, *Acta Appl Math.*, 104 (2008), 355-366.

[20] M. Ghasemi, M. Tavassoli Kajani, and E. Babolian, Application of He's homotopy perturbation method to nonlinear integro-differential equations, *Applied Mathematics and Computation*, 188 (2007), 538-548.

[21] A. Shahsavaran and A. Shahsavaran, Application of Lagrange Interpolation for Nonlinear Integro Differential Equations, *Applied Mathematical Sciences*, 18 (2012), 887-892.

[22] K. Maleknejad and Y. Mahmoudi, Taylor polynomial solution of high-order nonlinear VolterraFredholm integro-differential equations, *Applied Mathematics and Computation*, 145 (2003), 641-653.

Mojtaba Fardi
Department of Mathematics
M.Sc. of Mathematics
Islamic Azad University, Najafabad Branch
Najafabad, Iran
E-mail: fardi_ mojtaba@yahoo.com

Mehdi Ghasemi
Department of Applied Mathematics
Assistant Professor of Mathematics
Faculty of Mathematical Science
Shahrekord University
P.O. Box 115
Shahrekord, Iran
E-mail: meh_ ghasemi@yahoo.com

Farshad Hemati Boroujeni
Department of Mathematics
M.Sc. of Mathematics
Islamic Azad University, Boroujen Branch
Boroujen, Iran
E-mail: farshad_86@yahoo.com

Journal of Mathematical Extension
Vol. 6, No. 2, (2012), 91-98

Numerical Range in C*-Algebras

M. T. Heydari
Yasouj University

Abstract. Let $\mathcal{A}$ be a C*-algebra with unit 1 and let $\mathcal{S}$ be the state space of $\mathcal{A}$, i.e., $\mathcal{S} = \{\varphi \in \mathcal{A}^* : \varphi \geqslant 0, \varphi(1) = 1\}$. For each $a \in \mathcal{A}$, the C*-algebra numerical range is defined by

$$V(a) := \{\varphi(a) : \varphi \in \mathcal{S}\}.$$

We prove that if $V(a)$ is a disc with center at the origin, then $\|a+a^*\| = \|a - a^*\|$.

AMS Subject Classification: 47A12; 46K10
Keywords and Phrases: Support line, numerical ranges, norm

1. Introduction

Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. We can write

$$T = A + iB, \tag{1}$$

where A and B are Hermitian operators. Such a decomposition is unique; we have

$$A = \frac{1}{2}(T + T^*), B = \frac{1}{2i}(T - T^*). \tag{2}$$

The elements A, B are called the real and imaginary parts of T, denoted by $Re(T)$ and $Im(T)$, respectively, and the decomposition (1) is called the Cartesian decomposition of T.
The *numerical range* of T is the set

Received: June 2011; Accepted: March 2012

$$W(T) := \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\},$$

in the complex plane, where $\langle ., .\rangle$ denotes the inner product in $\mathcal{H}$. In other words, $W(T)$ is the image of the unit sphere $\{x \in \mathcal{H} : \|x\| = 1\}$ of $\mathcal{H}$ under the (bounded) quadratic form $x \mapsto \langle Tx, x\rangle$.

Some properties of the numerical range follow easily from the definition. Recall that, the numerical range is unchanged under the unitary equivalence of operators: $W(T) = W(U^*TU)$ for any unitary operator U. It also behaves nicely under the operation of taking the adjoint of an operator: $W(T^*) = \{\overline{z} : z \in W(T)\}$. One of the most fundamental properties of the numerical range is it's convexity, stated by the famous Toeplitz-Hausdorff Theorem. Other important property of $W(T)$ is that its closure contains the spectrum of the operator. Also, $W(T)$ is a connected set and it is compact in the finite dimensional case.

2. Numerical Range and Norm

Suppose E is a bounded convex subset of the plane. For $0 \leqslant \theta < 2\pi$ define

$$p_E(\theta) := sup\{Re(e^{-i\theta}z) : z \in E\}. \tag{3}$$

Note that for $z \in \mathbb{C}$, the number $Re(e^{-i\theta}z)$ is the real dot product of the plane vectors $e^{i\theta}$ and z, i.e., the signed length of the projection of z in the direction of $e^{i\theta}$. Thus the set

$$\prod_\theta = \{z \in \mathbb{C} : Re(e^{-i\theta}z) \leqslant p_E(\theta)\},$$

is a closed half-plane that contains E and intersects ∂E. The boundary L_θ of $\prod_\theta$ is called the support line of E perpendicular to $e^{i\theta}$. The magnitude of $p_E(\theta)$ is the orthogonal distance from the origin to L_θ. The function $p_E(\theta) : [0, 2\pi) :\longrightarrow \mathbb{R}$ defined by (3) is called the support function of E. The Hahn-Banach theorem insures that the closure of E is the intersection of all the half-planes $\prod_\theta$ as θ runs from 0 to 2π. Hence two bounded convex sets with the same support function have the same closures(see [3]).

In our applications the set E will always contain the origin in its closure, in which case $p_E \geqslant 0$. We will be particularly interested in the support function of a standard ellipse.

Proposition 2.1. *Suppose* $a, b > 0$ *and* E *is the elliptical disc determined by the inequality* $\frac{x^2}{a^2}+\frac{y^2}{b^2} \leqslant 1$. *Then* $p_E(\theta) = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$. $(0 \leqslant \theta < 2\pi)$.

Proof. We parameterize the boundary of E by the complex equation $z(t) = a\cos t + ib\sin t$, with $0 \leqslant t < 2\pi$. So

$$\begin{aligned} p_E(\theta) &= sup\{Re(e^{-i\theta}z) : z \in E\} \\ &= sup\{a\cos\theta\cos t + b\sin\theta\sin t, 0 \leqslant t < 2\pi\}. \end{aligned}$$

Put $f(t) = a\cos\theta\cos t + b\sin\theta\sin t, 0 \leqslant t < 2\pi$. Since f is twice differentiable so, by second derivative test, it has a local maximum at $\tan t = \frac{b}{a}\tan\theta$. After a little calculation with right triangles this yields the equations

$$\cos t = \frac{a\cos\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \quad , \quad \sin t = \frac{b\sin\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}$$

and then by substituting, $p_E(\theta) = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$. $(0 \leqslant \theta < 2\pi)$. □

We note in closing that this result persists in the limiting case $b = 0$. In this case E is the real segment $[-a, a]$, for which the definition of support function yields $p_E(\theta) = a|\cos(\theta)|$. If $a = b$, then $p_E(\theta) = a$, indeed if E is a disc with center at the origin then the function $p_E(\theta)$ is constant for all θ.

Proposition 2.2. *If* T *is a bounded linear operator on a Hilbert space* $\mathcal{H}$ *such that* $\overline{W}(T)$ *is a disc with center at the origin. Then*

$$\|Re(T)\| = \|Im(T)\|.$$

Proof. We compute the support function p_T of $W(T)$ in this standard fashion:

$$\begin{aligned} p_T(\theta) : &= sup\{Re(e^{-i\theta}z) : z \in W(T)\} \\ &= sup\{Re(e^{-i\theta} < Tf, f >) : f \in \mathcal{H}, \|f\| = 1\} \\ &= sup\{< H_\theta f, f >: f \in \mathcal{H}, \|f\| = 1\} \end{aligned}$$

where $H_\theta := Re(e^{-i\theta}T) = \frac{1}{2}(e^{-i\theta}T + e^{i\theta}T^*)$.
Since H_θ is a self-adjoint operator on $\mathcal{H}$ and $W(T)$ is a disc with center at the origin, then the last calculation show that for each $0 \leqslant \theta < 2\pi$,

$$p_T(\theta) = sup\{| < H_\theta f, f > | : f \in \mathcal{H}, \|f\| = 1\} = \|H_\theta\|.$$

Now, Proposition 2. implies that, $p_T(\theta)$ and also $\|H_\theta\|$ is constant for all θ. In particular, $\|H_0\| = \|H_{\frac{\pi}{2}}\|$ or

$$\|T + T^*\| = \|T - T^*\|.$$

This completes the proof. $\square$

Let $\mathcal{A}$ be a C*-algebra with unit 1 and let $\mathcal{S}$ be the state space of $\mathcal{A}$, i.e., $\mathcal{S} = \{\varphi \in \mathcal{A}^* : \varphi \geqslant 0, \varphi(1) = 1\}$. For each $a \in \mathcal{A}$, the C*-algebra numerical range is defined by

$$V(a) := \{\varphi(a) : \varphi \in \mathcal{S}\}.$$

It is well known that $V(a)$ is non empty, compact and convex subset of the complex plane and $V(\alpha 1 + \beta a) = \alpha + \beta V(a)$ where $a \in \mathcal{A}$, $\alpha, \beta \in \mathbb{C}$, and if $z \in V(a)$, $|z| \leqslant \|a\|$ (for further details see [2]).
As an example, let $\mathcal{A}$ be the C*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ and $A \in \mathcal{A}$. It is well known that $V(A)$ is the closure of $W(A)$, where

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\},$$

is the usual numerical range of the operator A.

Theorem 2.3. *Let $a \in \mathcal{A}$ be such that $V(a)$ be a disc with center at the origin. Then*

$$\|Re(a)\| = \|Im(a)\|$$

.

Proof. Let ρ be a state of $\mathcal{A}$. Then there exists a cyclic representation φ_ρ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\rho$ and a unit cyclic vector x_ρ for φ_ρ such that

$$\rho(a) = \langle \varphi_\rho(a)x_\rho, x_\rho \rangle, \ a \in \mathcal{A}.$$

By Gelfand-Naimark Theorem, the direct sum $\varphi : a \mapsto \sum_{\rho \in \mathcal{S}} \oplus \varphi_\rho(a)$ is a faithful representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H} = \sum_{\rho \in \mathcal{S}} \oplus \mathcal{H}_\rho$ (see [6]). Therefore, for each $\rho \in \mathcal{S}, \rho(a) \in W(\varphi_\rho(a)) \subset W(\varphi(a))$, and hence $V(a)$ is contained in $W(\varphi(a))$. On the other hand if x is a unit vector of $\mathcal{H}$, then the formula $\rho(b) = \langle \varphi(b)x, x\rangle, b \in \mathcal{A}$ defines a state on $\mathcal{A}$ and hence $\rho(a) = \langle \varphi(a)x, x \rangle \in V(a)$. So it follows that

$$W(T_a) = V(a), \tag{4}$$

where $T_a = \varphi(a)$. (see also Theorem 3 of [1]).

Since $\varphi(Re(a)) = Re(T_a)$, $\varphi(Im(a)) = Im(T_a)$ and φ is isometry, thus by equation (4) and Proposition (2.) the proof is completed. □

Example 2.4. Let $\mathbb{U}$ denote the open unit disc in the complex plane. Recall that the *Hardy space* H^2 consists the functions $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ holomorphic in $\mathbb{U}$ such that $\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 < \infty$, with $\widehat{f}(n)$ denoting the n-th Taylor coefficient of f. The inner product inducing the norm of H^2 is given by $< f, g > := \sum_{n=0}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)}$. The inner product of two functions f and g in H^2 may also be computed by integration:

$$< f, g >= \frac{1}{2\pi i} \int_{\partial \mathbb{U}} f(z) \overline{g(z)} \frac{dz}{z},$$

where $\partial\mathbb{U}$ is positively oriented and f and g are defined a.e. on $\partial\mathbb{U}$ via radial limits.

Each holomorphic self map φ of $\mathbb{U}$ induces on H^2 a *composition operator* C_φ defined by the equation $C_\varphi f = f \circ \varphi (f \in H^2)$. A consequence of a famous theorem of J. E. Littlewood [7] asserts that C_φ is a bounded operator. (see also [9] and [4]). In fact,

$$\sqrt{\frac{1}{1-|\varphi(0)|^2}} \leqslant \|C_\varphi\| \leqslant \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

In the case $\varphi(0) \neq 0$ Joel H. Shapiro has been shown that the second inequality changes to equality if and only if φ is an inner function.
A *conformal automorphism* is a univalent holomorphic mapping of $\mathbb{U}$ onto itself. Each such map is linear fractional, and can be represented as a product $w.\alpha_p$, where

$$\alpha_p(z) := \frac{p-z}{1-\overline{p}z}, (z \in \mathbb{U}),$$

for some fixed $p \in \mathbb{U}$ and $w \in \partial\mathbb{U}$ (see [8]).
The map α_p interchanges the point p and the origin and it is a self-inverse automorphism of $\mathbb{U}$.

Each conformal automorphism is a bijection map from the sphere $\mathbb{C}\bigcup\{\infty\}$ to itself with two fixed points (counting multiplicity). An automorphism is called:

- *elliptic* if it has one fixed point in the disc and one outside the closed disc,
- *hyperbolic* if it has two distinct fixed point on the boundary $\partial\mathbb{U}$, and
- *parabolic* if there is one fixed point of multiplicity 2 on the boundary $\partial\mathbb{U}$.

For $r \in \mathbb{U}$, a r-*dilation* is a map of the form $\delta_r(z) = rz$ and we call r the dilation parameter of δ_r and in the case that $r > 0, \delta_r$ is called *positive dilation*. A *conformal* r-*dilation* is a map that is conformally conjugate to an r-dilation, i.e., a map $\varphi = \alpha^{-1} \circ \delta_r \circ \alpha$, where $r \in \mathbb{U}$ and α is a conformal automorphism of $\mathbb{U}$.
For $w \in \partial\mathbb{U}$, an w-*rotation* is a map of the form $\rho_w(z) = wz$. We call w the rotation parameter of ρ_w. A straightforward calculation shows that

every elliptic automorphism φ of $\mathbb{U}$ must have the form

$$\varphi = \alpha_p \circ \rho_w \circ \alpha_p,$$

for some $p \in \mathbb{U}$ and some $w \in \partial\mathbb{U}$.
In [3] the shapes of the numerical range for composition operators induced on H^2 by some conformal automorphisms of the unit disc specially parabolic and hyperbolic are investigated. The authors proved, among other things, the following results:

1. If φ is a conformal automorphism of $\mathbb{U}$ is either parabolic or hyperbolic then $W(C_\varphi)$ is a disc centered at the origin.

2. If φ is a hyperbolic automorphism of $\mathbb{U}$ with antipodal fixed points and it is conformally conjugate to a positive dilation $z \mapsto rz$ $(0 < r < 1)$ then $W(C_\varphi)$ is the open disc of radius $1/\sqrt{r}$ centered at the origin.

3. If φ is elliptic and conformally conjugate to a rotation $z \mapsto \omega z$ $(|\omega| = 1)$ and ω is not a root of unity then $\overline{W}(C_\varphi)$ is a disc centered at the origin.

So, we have the following consequences:

Proposition 2.5. *If φ is a conformal automorphism of $\mathbb{U}$, except finite order elliptic automorphism, then*

$$\|C_\varphi + C_\varphi^*\| = \|C_\varphi - C_\varphi^*\|.$$

Also C_φ is not self adjoint. If φ is a finite order elliptic automorphism with rotation parameter w of order k, then

$$\sigma(C_\varphi) = \{1, w, w^2, ..., w^{k-1}\}.$$

If $w \neq \pm 1$, then $\sigma(C_\varphi)$ is not a subset of $\mathbb{R}$ and so C_φ is not self adjoint.

Corollary 2.6. *C_φ is Hermitian if and only if $\varphi(z) = z$ or $-z$.*

References

[1] S. K. Berberian and G. H. Orland, On the closure of the numerical range of an operator, *Proc. Amer. Math. Soc.*, 18 (1967), 499-503.

[2] F. F. Bonsall and J. Duncan, *Numerical Ranges of Operators on normed Spaces and of Elements of Normed Algebras,* London-New York: Cambridge University Press, 1971.

[3] P. S. Bourdon and J. H. Shapiro, The numerical range of automorphic composition operators, *J. Math. Analysis and application,* 251 (2000), 839-854.

[4] C. C. Cowen and B. D. Maccluer, *Composition operators on spaces of analytic functions*, CRC Press, Boca Raton, 1995.

[5] M. T. Heydari, C*-Algebra numerical range of quadratic elements, *Iranian Journal of Mathematical Sciences and Informatics*, 5(1) (2010), 49-53.

[6] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras,* Vol, I. Elementary Theory. Pure and Applied Mathematics 100, New York: Academic Press, 1983.

[7] J. E. Littlewood, On inequalities in the theory of functions, Proc. *London Math. Soc.*, 23 (1925), 481-519.

[8] W. Rudin, *Real and complex analysis,* Third edition, McGraw-Hill Book Co., New York, 1987.

[9] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, 1993.

Mohammd Taghi Heydari
Department of Mathematics
College of Sciences
Assistant Professor of Mathematics
Yasouj University
Yasouj, Iran
E-mail: heydari@mail.yu.ac.ir

به نام خدا

مجله گستره ریاضی

این مجله در پایگاه‌های استنادی علوم جهان اسلام (ISC)، جهاد دانشگاهی(SID)، پایگاه‌های داده ای Mathematical Reviews، MathSciNet و Zentralblatt Math نمایه گردیده است.
همچنین بر اساس شماره ۱۷/۱۳۸۰۴۲ ، حائز رتبه **علمی-پژوهشی** می باشد.

صاحب امتیاز: دانشگاه آزاد اسلامی ـ واحد شیراز

ISSN: 1735-8299

نشانی: شیراز، شهر جدید صدرا، بلوار ایران، دانشکده علوم پایه دانشگاه آزاد اسلامی ـ واحد شیراز
دفتر مجله گستره ریاضی، صندوق پستی ۵—۷۱۹۹۳
تلفن و فاکس: ۶۴۱۰۰۶۳ — ۷۱۱ ۹۸+

www.ijmex.com