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New Results on the Minimum Edge Dominating Energy of a Graph

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Abstract. Let G be a graph with n vertices and m edges. The minimum edge dominating energy is defined as the sum of the absolute values of eigenvalues of the minimum edge dominating matrix of the graph G. In this paper, some lower and upper bounds for the minimum edge dominating energy of graph G are established.

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1 Introduction

In this paper, we consider G as a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. For vertex $v_i \in V$, the degree of v_i, written by d_i, is the number of edges incident with v_i. The maximum vertex degree is denoted by Δ. The Zagreb index was first introduced in where is one of the important molecular descriptor

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with many chemical properties [11]. The Zagreb index $M(G)$ is defined as $M(G) = \sum_{i=1}^{n} d_i^2$.
The adjacency matrix $A(G)$ of G is defined by its entries as $a_{ij} = 1$ if $v_i v_j \in E(G)$ and 0 otherwise. The eigenvalues of the graph G are the eigenvalues of its adjacency matrix $A(G)$. Graph energy is one of the most effective topological indices in chemical graph theory that is usable in chemistry. Gutman defined the energy of a graph G, in 1978. It was considered as the summation of the absolute eigenvalues of G [9]. The significant chemical applications for graph energy have been found in the molecular orbital theory of conjugated molecules [8, 10] and the details of the mathematical terms of graph spectra and graph theory have been explained in [5, 18].

A subset D of V is the dominating set of graph G if every vertex of $V \setminus D$ is adjacent to some vertex in D. Any dominating set with minimum cardinality is called a minimum dominating set. The minimum dominating matrix of the graph G is defined as following

$$A_D(G) := (a_{ij}) = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 1 & \text{if } i = j \, and \, v_i \in D \\ 0 & otherwise \end{cases}$$

The minimum dominating energy of the graph G is defined as the sum of the absolute values of eigenvalues of the matrix $A_D(G)$ [17].

The edge energy (called $EE(G)$) of a graph G is defined as the sum of the absolute values of eigenvalues of the adjacency matrix of the line graph of G [3]. The line graph $L(G)$ of G is the graph that each vertex of it represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are incident in G [12]. Let e be an edge in G. There are two vertices u and v in V such that $e = uv$. The degree of the edge e in G is defined to be $deg(e) = deg(u) + deg(v) - 2$.

Let G be a simple graph with edge set $\{e_1, e_2, \ldots, e_m\}$ and $F \subseteq E$ be the minimum edge dominating set of G or the minimum dominating set of $L(G)$. The minimum edge dominating matrix of G is the $m \times m$ matrix defined by $A_F(G) := (a_{ij})$ in which

$$a_{ij} = \begin{cases} 1 & \text{if } e_i \, and \, e_j \, are \, adjacent, \\ 1 & \text{if } i = j \, and \, e_i \in F, \\ 0 & otherwise. \end{cases}$$

The minimum edge dominating energy of G is introduced in [1] as following

$$EE_F(G) = \sum_{i=1}^{m} |\lambda_i|,$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ is a non-increasing sequence of the eigenvalues of $A_F(G)$.
In [1], the minimum edge dominating energy of some graphs is introduced and studied and some lower and upper bounds of this energy are obtained. Movahedi in [16] established relations between the minimum edge dominating energy of a graph G and the graph energy, the edge energy and signless Laplacian energy of G. In [14], some bounds for the minimum edge dominating energy of subgraphs of a graph are obtained. In [15], some results of eigenvalues and energy from minimum edge dominating matrix in caterpillars are obtained. In this paper, we investigate other bounds of the minimum edge dominating energy of graphs.

2 Main Results

In this section, some lower and upper bounds for the minimum edge dominating energy ($EE_F(G)$) of a graph are calculated. We first state some following results on $EE_F(G)$ in [1].

Lemma 2.1. *Let G be a simple graph with n vertices and m edges where vertices have degree d_i for $i = 1, 2, \ldots, n$. Let F be the minimum edge dominating set of G. If $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of the minimum edge dominating matrix $A_F(G)$, then*

(i) $\sum_{i=1}^{m} \lambda_i = |F|$,

(ii) $\sum_{i=1}^{m} \lambda_i^2 = |F| + \sum_{i=1}^{n} d_i^2 - 2m$.

Lemma 2.2. *Let G be a simple graph with n vertices and m edges where vertices have degree d_i for $i = 1, 2, \dots, n$. Let F be the minimum edge dominating set of G with cardinality k. If $\lambda_1(G)$ is the largest minimum edge dominating eigenvalue of G, then*

$$\lambda_1(G) \geq \frac{k - 2m + \sum_{i=1}^{n} d_i^2}{m}.$$

In the following lemma, we only mention the lower bound of the largest minimum edge dominating eigenvalue in Theorem 11 of [1].

Lemma 2.3. *Let G be a simple graph with n vertices and m edges where vertices have degree d_i for $i = 1, 2, \dots, n$. Let F be the minimum edge dominating set of G with cardinality k. If $\lambda_1(G)$ is the largest minimum edge dominating eigenvalue of G, then*

$$\lambda_1(G) \geqslant \frac{k}{m} + \frac{1}{m}\sqrt{\frac{m\alpha - k^2}{m-1}},$$

where $\alpha = k - 2m + \sum_{i=1}^{n} d_i^2$.

The following result is a lower bound of the largest eigenvalue of the minimum edge dominating matrix $A_F(G)$ in terms of the maximum degree of G.

Theorem 2.4. *Let G be a simple graph with n vertices, m edges and the maximum degree Δ. Let F be the minimum edge dominating set of G with cardinality k. If $\lambda_1(G)$ is the largest eigenvalue of the minimum edge dominating matrix $A_F(G)$, then*

$$\lambda_1(G) \geqslant \frac{2\Delta + k}{m}.$$

Proof. Let $L(G)$ be the line graph of the graph G with m' edges and the maximum degree Δ'. For any graph G, $m \geqslant \Delta$ so, $m' \geqslant \Delta'$ in graph $L(G)$. We consider the vertex $x \in V(L(G))$ that $deg(x) = \Delta'$. Therefore, there are two vertices u and v in graph G such that $x = uv$ and

$$\Delta' = deg(x) = deg(u) + deg(v) - 2.$$

Since the edge x has the maximum edge degree in graph G, then at least one of the two vertices u and v have the maximum vertex degree. Therefore, we have $m' \geqslant \Delta' \geqslant \Delta$.
By Lemma 2.2 and since $m' = -m + \frac{1}{2}\sum_{i=1}^{n} d_i^2$, we get

$$\begin{aligned}\lambda_1(G) &\geq \frac{k - 2m + \displaystyle\sum_{i=1}^{n} d_i^2}{m} \\ &= \frac{2m' + k}{m} \geqslant \frac{2\Delta + k}{m}.\end{aligned}$$

Therefore, the result completes. □

We use the following known inequality that will be needed in the proof of the next results.

Cauchy-Schwarz inequality [2] For all sequences of real numbers a_i and b_i,

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big),$$

equality holds if and only if $a_i = kb_i$ for a non zero constant $k \in \mathbb{R}$.

In Theorem 6 of [1], authors obtained the upper bound for $EE_F(G)$ in terms of the square of the degrees of the vertices in graph G. In the following theorem, we reduce this condition to the maximum degree in G.

Theorem 2.5. *Let G be a simple graph with n vertices and m edges. Let the maximum degree of G be Δ. If F is the minimum edge dominating set of G and $|F| = k$, then*

$$EE_F(G) \leq \frac{2\Delta + k}{m} + \sqrt{(m-1)\Big(m^2 + k - \Big(\frac{2\Delta + k}{m}\Big)^2\Big)}.$$

Proof. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ be the non-increasing order of eigenvalues of the minimum edge dominating matrix $A_F(G)$. By considering

$a_i = 1$ and $b_i = |\lambda_i|$ in Cauchy-Schwarz inequality, we get

$$\Big(\sum_{i=2}^{m} |\lambda_i|\Big)^2 \leq \Big(\sum_{i=2}^{m} 1\Big)\Big(\sum_{i=2}^{m} |\lambda_i|^2\Big).$$

Using Lemma 2.1, we have

$$\Big(EE_F(G) - \lambda_1\Big)^2 \leq (m-1)\Big(k - 2m + \sum_{i=1}^{n} d_i^2 - \lambda_1^2\Big),$$

where d_i is the degree of the vertex i for $i = 1, 2, \ldots, n$.
By putting $\alpha = k - 2m + \sum_{i=1}^{n} d_i^2$ and rearranging, we have

$$EE_F(G) \leq \lambda_1 + \sqrt{(m-1)\big(\alpha - \lambda_1^2\big)}.$$

Let $f(x) = x + \sqrt{(m-1)\big(\alpha - x^2\big)}$. Since $f(x)$ is a decreasing function and using Lemma 2.2, $f'(\lambda_1) \leq f\big(\frac{2\Delta+k}{m}\big)$.
So,

$$EE_F(G) \leq f(\lambda_1) \leq f\big(\frac{2\Delta+k}{m}\big).$$

Therefore, we get

$$EE_F(G) \leq \frac{2\Delta+k}{m} + \sqrt{(m-1)\Big(\alpha - \Big(\frac{2\Delta+k}{m}\Big)^2\Big)}.$$

For the Zagreb index $M(G)$ of a graph G, we have $M(G) = \sum_{i=1}^{n} d_i^2 \leqslant m(m+1)$ [6]. Thus, $\alpha = \sum_{i=1}^{n} d_i^2 - 2m + k \leqslant m^2 + k$. Therefore, we get

$$EE_F(G) \leq \frac{2\Delta+k}{m} + \sqrt{(m-1)\Big(m^2 + k - \Big(\frac{2\Delta+k}{m}\Big)^2\Big)}.$$

□

Theorem 2.6. *Let G be a graph with n vertices and m edges where vertices have degree d_i for $i = 1, 2, \ldots, n$. If F is the minimum edge dominating set of G with cardinality k, then*

$$EE_F(G) \leq \frac{\beta}{m\sqrt{m-1}} + \sqrt{\alpha(m-1) - \Big(\frac{\beta}{m}\Big)^2},$$

where $\alpha = k - 2m + \sum_{i=1}^{n} d_i^2$ and $\beta = k\sqrt{m-1} + \sqrt{m\alpha - k^2}$.

Proof. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ be the non-increasing order of eigenvalues of the minimum edge dominating matrix $A_F(G)$. Using Cauchy-Schwarz inequality and putting $a_i = 1$ and $b_i = |\lambda_i|$, we get

$$\Big(\sum_{i=2}^{m} |\lambda_i|\Big)^2 \leq \Big(\sum_{i=2}^{m} 1\Big)\Big(\sum_{i=2}^{m} \lambda_i^2\Big),$$

Using Lemma 2.1, we have

$$\Big(EE_F(G) - \lambda_1\Big)^2 \leq (m-1)\big(\alpha - \lambda_1^2\big),$$

in which $\alpha = k - 2m + \sum_{i=1}^{n} d_i^2$.
Note that the function $f(x) = x + \sqrt{(m-1)\big(\alpha - x^2\big)}$ decreases for $x \geqslant \sqrt{\frac{\alpha}{m}}$ with conditions $\alpha \geqslant m$ and $k \geqslant m$. Using Lemma 2.3, we have

$$\lambda_1 \geqslant \frac{k}{m} + \frac{1}{m}\sqrt{\frac{m\alpha - k^2}{m-1}} \geqslant \sqrt{\frac{\alpha}{m}}.$$

So, $f(\lambda_1) \leqslant f\Big(\frac{k}{m} + \frac{1}{m}\sqrt{\frac{m\alpha - k^2}{m-1}}\Big)$, which implies that

$$\begin{aligned} EE_F(G) &\leq \frac{k}{m} + \frac{1}{m}\sqrt{\frac{m\alpha - k^2}{m-1}} + \sqrt{(m-1)\Big(\alpha - \Big(\frac{k}{m} + \frac{1}{m}\sqrt{\frac{m\alpha - k^2}{m-1}}\Big)^2\Big)} \\ &= \frac{k\sqrt{m-1} + \sqrt{m\alpha - k^2}}{m\sqrt{m-1}} + \sqrt{\alpha(m-1) - \Big(\frac{k\sqrt{m-1} + \sqrt{m\alpha - k^2}}{m}\Big)^2}. \end{aligned}$$

By putting $\beta = k\sqrt{m-1} + \sqrt{m\alpha - k^2}$, the result completes. □

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be eigenvalues of the minimum edge dominating matrix $A_F(G)$. According to the definition of the variance of eigenvalues in [19], we have

$$\begin{aligned} s^2 &= \frac{1}{m}\Big[\sum_{i=1}^{m} \lambda_i^2 - \frac{1}{m}\Big(\sum_{i=1}^{m} \lambda_i\Big)^2\Big] \\ &= \frac{tr(A_F)^2 m - k^2}{m^2}, \end{aligned}$$

where $tr(A_F)^2$ is the trace of the square matrix A_F. The positive square root of the variance is the standard deviation that is denoted by s. We consider the following result of [20].

Lemma 2.7. *Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of the matrix A. Let m and s be the mean and standard deviation of eigenvalues, respectively. Then,*

$$\lambda_1 \geqslant m + \frac{s}{(n-1)^{\frac{1}{2}}}.$$

Theorem 2.8. *Let G be a graph with n vertices and m edges where vertices have degree d_i for $i = 1, 2, \ldots, n$. Let $\bar{\lambda}$ and s be he mean and standard deviation of eigenvalues of the minimum edge dominating matrix $A_F(G)$, respectively. If F is the minimum edge dominating set of G with cardinality k, then*

$$EE_F(G) \leqslant \bar{\lambda} + \frac{s}{\sqrt{m-1}} + \sqrt{\alpha(m-1) - \left(\bar{\lambda}\sqrt{m-1} + s\right)^2},$$

where $\alpha = k - 2m + \sum_{i=1}^{n} d_i^2$.

Proof. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ be the non-increasing order of eigenvalues of the minimum edge dominating matrix $A_F(G)$. Using Lemma 2.7, we have

$$\lambda_1 \geqslant \bar{\lambda} + \frac{s}{\sqrt{m-1}},$$

where $\bar{\lambda} = \frac{\sum_{i=1}^{m} \lambda_i}{m}$ and $s = \sqrt{\frac{1}{m}\left[\sum_{i=1}^{m} \lambda_i^2 - \frac{1}{m}\left(\sum_{i=1}^{m} \lambda_i\right)^2\right]}$.

Using Lemma 2.1, we have $\bar{\lambda} = \frac{k}{m}$ and

$$\begin{aligned} s^2 &= \frac{1}{m^2}\left[k - 2m + \sum_{i=1}^{n} (d_i)^2 - \frac{1}{m}k^2\right] \\ &= \frac{1}{m^2}\left[m\alpha - k^2\right], \end{aligned}$$

where $\alpha = k - 2m + \sum_{i=1}^{n} (d_i)^2$. Therefore, $s = \frac{\sqrt{m\alpha - k^2}}{m}$. Thus, we have $\lambda_1 \geqslant \frac{k}{m} + \frac{\sqrt{m\alpha - k^2}}{m\sqrt{m-1}}$. By Theorem 2.6, we have

$$EE_F(G) \leqslant \frac{k}{m} + \frac{\sqrt{m\alpha - k^2}}{m\sqrt{m-1}} + \sqrt{\alpha(m-1) - \left(\frac{k}{m}\sqrt{m-1} + \frac{\sqrt{m\alpha - k^2}}{m}\right)^2}.$$

By putting $s = \frac{\sqrt{m\alpha - k^2}}{m}$ and $\bar{\lambda} = \frac{k}{m}$ in the above inequality, the result holds. $\square$

We arrive at the following corollary of the definition of the variance of eigenvalues in terms of the trace of A_F and Theorem 2.8.

Corollary 2.9. *Let G be a graph with n vertices and m edges where vertices have degree d_i for $i = 1, 2, \ldots, n$. If F is the minimum edge dominating set of G with cardinality k, then*

$$EE_F(G) \leq \frac{\beta}{m\sqrt{m-1}} + \frac{1}{m}\sqrt{\alpha m^2 \sqrt{m-1} - \beta^2},$$

where $\beta = tr\big(A_F(G)\big)\sqrt{m-1} + \sqrt{tr\big(A_F(G)\big)^2 m - k^2}$.

We obtain the results for bounds of $EE_F(G)$ in terms of the largest and smallest absolute of the minimum edge dominating eigenvalues of the graph G. To do this, we need some previously know inequalities.

Lemma 2.10. (See [4]) *Suppose a, b, A and B are real constants and for $1 \leq i \leq n$, a_i and b_i are positive real numbers. Let $a \leq a_i \leq A$ and $b \leq b_i \leq B$, for $1 \leq i \leq n$. Then,*

$$\Big| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \Big| \leq \beta(n)\big(A - a\big)\big(B - b\big),$$

where $\beta(n) = n\lfloor \frac{n}{2} \rfloor \big(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor\big)$.

Lemma 2.11. (See [7]) *Let a_i and b_i, $1 \leq i \leq n$, are non-negative real numbers and r and R are real constants. Then,*

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i^2 \leq (r + R)\big(\sum_{i=1}^{n} a_i b_i\big),$$

where $ra_i \leq b_i \leq Ra_i$, *for* $1 \leq i \leq n$.

Lemma 2.12. (See [13]) *Let* $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$ *be real non-negative numbers such that* $P = \sum_{i=1}^{n} a_i^2$ *and* $Q = \sum_{i=1}^{n} a_i$*. Then for arbitrary real numbers* k_1 *and* k_2 *with the properties*

$$a_1 \geq k_1 \geq \sqrt{\frac{P}{n}} \quad \text{and} \quad \sqrt{\frac{P}{n}} \geq k_2 \geq a_n$$

the following is valid

$$Q \leq min\Big\{k_1 + \sqrt{(n-1)(P-k_1^2)},\\ k_2 + \sqrt{(n-1)(P-k_2^2)},\ \sqrt{nP - \frac{n}{2}(a_i - a_n)^2}\Big\}.$$

Equality holds if and only if $a_1 = a_2 = \ldots = a_n$*.*

Theorem 2.13. *Let* G *be a graph with* n *vertices and* m *edges where vertices have degree* d_i *for* $i = 1, 2, \ldots, n$*. Let* F *be the minimum edge dominating set of* G *with cardinality* k*. If* $|\lambda_1^*|$ *and* $|\lambda_m^*|$ *are the largest and the smallest absolute values of eigenvalues of matrix* $A_F(G)$*, respectively, then*

$$EE_F(G) \geq \sqrt{m\alpha - \beta(m)\Big(|\lambda_1^*| - |\lambda_m^*|\Big)^2},$$

where $\alpha = k - 2m + \sum_{i=1}^{n} d_i^2$*,* $\beta(m) = m\lfloor \frac{m}{2} \rfloor \big(1 - \frac{1}{m}\lfloor \frac{m}{2} \rfloor\big)$*.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be eigenvalues of the minimum edge dominating matrix $A_F(G)$ where F is the minimum edge dominating set.
By considering $a_i = b_i = |\lambda_i|$ and $a = b = |\lambda_m^*|$ and $A = B = |\lambda_1^*|$ for $1 \leq i \leq m$, in Lemma 2.10,

$$\left| m\sum_{i=1}^{m} |\lambda_i|^2 - \Big(\sum_{i=1}^{m} |\lambda_i|\Big)^2 \right| \leq \beta(m)\Big(|\lambda_1^*| - |\lambda_m^*|\Big)^2,$$

where $|\lambda_1^*| = \max_{1 \leq i \leq m}\{|\lambda_i|\}$ and $|\lambda_m^*| = \min_{1 \leq i \leq m}\{|\lambda_i|\}$.
Using Lemma 2.1(i) and considering $\alpha = k - 2m + \sum_{i=1}^{n} d_i^2$, we get

$$\Big| m\alpha - EE_F(G)^2 \Big| \leq \beta(m)\Big(|\lambda_1^*| - |\lambda_m^*|\Big)^2,$$

therefore,

$$m\alpha - EE_F(G)^2 \le \beta(m)\Big(|\lambda_1^*| - |\lambda_m^*|\Big)^2.$$

By rearranging, the result holds. $\square$

It is easy to see that $\beta(m) = m\lfloor \frac{m}{2} \rfloor \big(1 - \frac{1}{m}\lfloor \frac{m}{2} \rfloor\big) \le \frac{m^2}{4}$. So using Theorem 2.13, we have

$$EE_F(G) \ge \sqrt{m\alpha - \beta(m)\Big(|\lambda_1^*| - |\lambda_m^*|\Big)^2},$$

$$\ge \sqrt{m\alpha - \frac{m^2}{4}\Big(|\lambda_1^*| - |\lambda_m^*|\Big)^2}$$

Therefore, the bound of Theorem 2.13 is an improvement of the bound given in Theorem 9 of [1].

Theorem 2.14. *Let G be a graph with n vertices and m edges where vertices have degree d_i for $i = 1, 2, \ldots, n$. Let F be the minimum edge dominating set of G with cardinality k. If $|\lambda_1^*|$ and $|\lambda_m^*|$ are the largest and the smallest absolute values of eigenvalues of matrix $A_F(G)$, respectively, then*

$$EE_F(G) \ge \frac{\alpha + m|\lambda_1^*||\lambda_m^*|}{|\lambda_1^*| + |\lambda_m^*|},$$

where $\alpha = k - 2m + \sum_{i=1}^{n} d_i^2$.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be eigenvalues of the minimum edge dominating matrix $A_F(G)$ where F is the minimum edge dominating set.
By putting $a_i = 1$, $b_i = |\lambda_i|$, $r = |\lambda_m^*|$ and $R = |\lambda_1^*|$ for $1 \le i \le m$, in Lemma 2.11, we have

$$\sum_{i=1}^{m} |\lambda_i|^2 + |\lambda_1^*||\lambda_m^*| \sum_{i=1}^{m} 1 \le \Big(|\lambda_1^*| + |\lambda_m^*|\Big) \sum_{i=1}^{m} |\lambda_i|,$$

where $|\lambda_1^*| = \max\limits_{1\le i\le m}\{|\lambda_i|\}$ and $|\lambda_m^*| = \min\limits_{1\le i\le m}\{|\lambda_i|\}$.
Using Lemma 2.1(ii), we get

$$\alpha + m|\lambda_1^*||\lambda_m^*| \le \Big(|\lambda_1^*| + |\lambda_m^*|\Big) EE_F(G),$$

where $\alpha = k - 2m + \sum_{i=1}^{n} d_i^2$. By rearranging, the result holds. $\square$

By a simple calculation, one can easily show that the bound of Theorem 2.14 is an improvement of the bound given in Theorem 8 of [1]. The following results are obtained directly from Theorem 2.13 and Theorem 2.14.

Corollary 2.15. *Let G be a graph with m edges and F be the minimum edge dominating set of the graph G with cardinality k. If $|\lambda_1^*|$ and $|\lambda_m^*|$ are the largest and the smallest absolute values of eigenvalues of matrix $A_F(G)$, respectively, then*

(i) $EE_F(G) \geq \sqrt{m\big(k - 2m + M(G)\big) - \beta(m)\Big(|\lambda_1^*| - |\lambda_m^*|\Big)^2}$,
where $\beta(m) = m\lfloor \frac{m}{2} \rfloor \big(1 - \frac{1}{m}\lfloor \frac{m}{2} \rfloor\big)$ *and* $M(G)$ *is the Zagreb index of a graph* G.

(ii) $EE_F(G) \geq \frac{(k+M(G))+m\Big(|\lambda_1^*||\lambda_m^*|-2\Big)}{|\lambda_1^*|+|\lambda_m^*|}$.

Remark 2.16. Let G be a graph with n vertices and m edges. Assume that $|\lambda_1^*| = \max\limits_{1\leq i\leq m}\{|\lambda_i|\}$ and $|\lambda_m^*| = \min\limits_{1\leq i\leq m}\{|\lambda_i|\}$. According to Theorem 5 of [1], the largest minimum edge dominating eigenvalue of matrix $A_F(G)$ holds in $\lambda_1 \geq \frac{\alpha}{m}$. Therefore, we get $|\lambda_1^*| \geq \lambda_1 \geq \frac{\alpha}{m} \geq \sqrt{\frac{\alpha}{m}}$. According to Theorem 1 of [1], we have $\sum_{i=1}^{m} \lambda_i^2 = \alpha$. Since $|\lambda_1^*| = |\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_m^*|$. Thus, using Theorem 1 of [1], we get

$$\alpha = \sum_{i=1}^{m} \lambda_i^2 \geq \sum_{i=1}^{m} |\lambda_m|^2 = m|\lambda_m|^2.$$

Therefore, $\frac{\alpha}{m} \geq |\lambda_m|^2$. So, we have $\sqrt{\frac{\alpha}{m}} \geq |\lambda_m| \geq |\lambda_m^*|$.

Theorem 2.17. *Let G be a graph with n vertices and m edges where vertices have degree d_i for $i = 1, 2, \ldots, n$. Let F be the minimum edge dominating set of G with cardinality k. Assume $|\lambda_1^*|$ and $|\lambda_m^*|$ are the largest and the smallest absolute values of eigenvalues of matrix $A_F(G)$,* *respectively. If k_1 and k_2 are the arbitrary real numbers with the properties $|\lambda_1^*| \geq k_1 \geq \sqrt{\frac{\alpha}{m}}$ and $\sqrt{\frac{\alpha}{m}} \geq k_2 \geq |\lambda_m^*|$, then*

$$EE_F(G) \leq min\Big\{k_1 + \sqrt{(m-1)(\alpha - k_1^2)}, \qquad (1)$$
$$k_2 + \sqrt{(m-1)(\alpha - k_2^2)}, \sqrt{m\alpha - \frac{m}{2}\big(|\lambda_1^*| - |\lambda_m^*|\big)^2}\Big\}$$

where $\alpha = k - 2m + \sum_{i=1}^{n} d_i^2$. Equality holds if and only if $G \simeq nK_2$.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be eigenvalues of the minimum edge dominating matrix $A_F(G)$ where F is the minimum edge dominating set.
By setting $a_i = |\lambda_i|$ for $1 \leq i \leq m$ such that $|\lambda_1^*| = |\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_m^*| = |\lambda_m|$ in Lemma 2.12, the inequality holds.
Since the line graph of nK_2 is $\bar{K_n}$, we have $\lambda_1 = \ldots = \lambda_m = 1$. Therefore, using Lemma 2.12 the result easily follows. □

Remark 2.18. For $k_1 = \frac{\alpha}{m} \geq \sqrt{\frac{\alpha}{m}}$, it immediately follows.

$$EE_F(G) \leq \frac{\alpha}{m} + \sqrt{(m-1)\big(\alpha - \frac{\alpha^2}{m^2}\big)},$$

where was proven in [1]. The inequality (1) is stronger than the obtained inequlities in Theorem 6 and Theorem 10 of [1].

Theorem 2.19. *Let G be a graph with n vertices and m edges where Δ is the maximum degree of graph G. If F is the minimum edge dominating set of G with cardinality k, then*

$$EE_F(G) \geq \frac{2\Delta + k}{m} + (m-1) + \ln\Big(\frac{m|P|}{2\Delta + k}\Big),$$

where $P = |\det(A_F(G))|$.

Proof. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ be the non-increasing order of eigenvalues of the minimum edge dominating matrix $A_F(G)$. Consider $f(x) = x - 1 - \ln x$ for $x \geq 0$. It is easy to prove that $f(x)$ is increasing for $x \geq 0$ and decreasing for $0 < x \leq 1$. Therefore, $f(x) \geq f(1) = 0$, implying that $x \geq 1 + \ln x$ for $x > 0$, with equality holds if and only if $x = 1$. According to this, we get

$$\begin{aligned} EE_F(G) &= \lambda_1 + \sum_{i=2}^{m} |\lambda_i| \\ &\geq \lambda_1 + (m-1) + \sum_{i=2}^{m} \ln |\lambda_i| \\ &= \lambda_1 + (m-1) + \ln \prod_{i=2}^{m} |\lambda_i| \\ &= \lambda_1 + (m-1) + \ln |\det(A_F(G))| - \ln \lambda_1. \end{aligned} \tag{2}$$

By putting $P = \det A_F(G)$ and considering the function $g(x) = x + (m - 1) + \ln P - \ln x$, one can easily show that $g(x)$ is an increasing function on $1 \leq x \leq m$. Using Lemma 2.2, we have $\lambda_1 \geq \frac{2\Delta+k}{m}$. consequently, we have

$$\begin{aligned} g(x) &\geq g\big(\frac{2\Delta+k}{m}\big) \\ &= \frac{2\Delta+k}{m} + (m-1) + \ln\Big(\frac{m|P|}{2\Delta+k}\Big), \end{aligned}$$

for $x \geq \frac{2\Delta+k}{m}$.
Combing the above result with (2), the result completes. □

Considering a similar proof of theorem 2.19 and using Lemma 2.2 and the definition of the Zagreb index $M(G)$, we have the following result.

Corollary 2.20. *Let G be a graph with n vertices and m edges where Δ is the maximum degree of graph G. If F is the minimum edge dominating set of G with cardinality k, then*

$$EE_F(G) \geq \frac{M(G) - 2m + k}{m} + (m-1) + \ln\Big(\frac{mP}{M(G) - 2m + k}\Big),$$

where $P = |\det(A_F(G))|$.

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