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Global Behavior of a Second Order Difference Equation with Two-Period Coefficient

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Abstract. In this paper, we present a detailed study of the following difference equation

$$x_{n+1} = \frac{\alpha_n}{1 + x_n x_{n-1}},\ n \in \mathbb{N}_0,$$

where the sequence $(\alpha_n)_{n\geq 0}$ is positive, real, periodic with period two and the initial values $x_{-1},\ x_0$ are nonnegative real numbers. By this study, we determine global behavior of positive solutions of the above mentioned equation. We also give closed forms of its general solution.
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1 Introduction

Over the last two decades, many studies on nonlinear difference equations have been published (see, e.g., [5, 8, 9, 10, 11, 16, 17, 21, 22, 23, 25,

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26, 27, 28, 29, 30, 31, 32, 33, 34, 35] and the references therein). These equations play an important role in applications since they often arise as mathematical model of a problem (see, e.g., [1, 23]).

The solvability of nonlinear difference equations has been studied for the last fifteen years. As with differential equations, the solvability is a fundamental problem of the theory of difference equations. This problem is actually older (see, e.g., [1, 4, 20, 24]). The following difference equation

$$y_{n+1} = \frac{a + by_n}{c + dy_n},\ n \in \mathbb{N}_0, \tag{1}$$

where the parameters a, b, c, d and the initial value y_0 are real or complex numbers such that $ad \neq bc$, $d \neq 0$ and $y_0 \neq -c/d$, is a prototype for solvable difference equations. This equation is named bilinear difference equation or linear fractional difference equation. By taking $b = 0$ in Equation (1), we have the following equation

$$y_{n+1} = \frac{a}{c + dy_n},\ n \in \mathbb{N}_0. \tag{2}$$

Equation (2) can be reduced to the equation

$$x_{n+1} = \frac{\alpha}{1 + x_n},\ n \in \mathbb{N}_0,$$

where $\alpha = \frac{ad}{c^2}$, by change of variables $y_n = \frac{c}{d}x_n$. Equation (2) can be extended as the following

$$y_{n+1} = \frac{a}{c + dy_ny_{n-1}},\ n \in \mathbb{N}_0, \tag{3}$$

where the parameters a, c, d is positive real number and the initial values y_{-1}, y_0 are nonnegative real numbers such that the solution $(\ y_n)_{n\geq-1}$ exists. If c and d are positive, then Equation (3) can be also reduced to the equation

$$x_{n+1} = \frac{\alpha}{1 + x_nx_{n-1}},\ n \in \mathbb{N}_0, \tag{4}$$

where $\alpha = \sqrt{\frac{d}{c}}\frac{a}{c}$, by change of variables $y_n = \sqrt{\frac{c}{d}}x_n$, which posed by Amleh et al. [3]. They conjectured that every positive solution of Equation (4) has a finite limit. But they can only confirmed it when

$0 < \alpha \leq 2$ (for equations associated with (4), see, e.g., [6, 22]). Then, Drymonis et al. [7] showed that every positive solution of the following difference equation

$$x_{n+1} = \frac{\alpha_n}{1 + x_n x_{n-1}},\ n \in \mathbb{N}_0, \tag{5}$$

where

$$\alpha_n = \begin{cases} a, & \text{if } n \text{ is even} \\ b, & \text{if } n \text{ is odd} \end{cases} \text{ and } a > 0,\ b > 0,\ a \neq b$$

and the initial values $x_{-1},\ x_0$ are nonnegative real numbers, converges to a prime period two sequence.

In this paper, we handle Equation (5) and determine global behavior of positive solutions of Equation (5) by giving a closed form solution. That is, we here use an advanced method which is different than the method given in [7]. When $(\alpha_n)_{n\geq 0}$ is a constant sequence, we exactly confirm Conjecture 2.2. given for Equation (5) in [3].

Before our discussion, we present some definitions and the known results which will be used in this study. For the theory of difference equations, one can refer to the monograph of Kocic and Ladas [18].

2 Preliminaries

Consider the following system

$$\begin{cases} u_{n+1} = f(u_n, v_n), \\ v_{n+1} = g(u_n, v_n), \end{cases} n \in \mathbb{N}_0. \tag{6}$$

Let $\|\cdot\|$ be the norm of vector $(u, v) \in \mathbb{R}^2$. Then, we present the following definition and some useful lemmas which will serve to analyze equation (5).

Definition 2.1. [18] The equilibrium point $(\bar{u}, \bar{v})$ is said to be:

(i) stable if given $\varepsilon > 0$ and $N > 0$ there exists $\delta > 0$ such that $\|(u_0, v_0) - (\bar{u}, \bar{v})\| < \delta$ implies that $\|(u_n, v_n) - (\bar{u}, \bar{v})\| < \varepsilon$ for all $n > N$, and unstable if it is not stable;

(ii) attracting if there exists $\eta > 0$ such that $\|(u_0, v_0) - (\bar{u}, \bar{v})\| < \eta$ implies that $\lim_{n\to\infty} (u_n, v_n) = (\bar{u}, \bar{v})$

(iii) asymptotically stable if it is stable and attracting.
(iv) globally asymptotically stable if (i) and (ii) with $\eta = \infty$ hold.

We quote the following lemma from [19].

Lemma 2.2. *Let $F = (f, g)$ be a continuously differentiable function defined on an open set $D \in \mathbb{R}^2$.*

(a) If the eigenvalues of the Jacobian matrix $J_F((\bar{u}, \bar{v}))$, that is, both roots of its characteristic equation

$$\lambda^2 - TrJ_F((\bar{u}, \bar{v}))\lambda + DetJ_F((\bar{u}, \bar{v})) = 0, \tag{7}$$

lie inside the unit disk, then the equilibrium point $(\bar{u}, \bar{v})$ of (6) is locally asymptotically stable.

(b) A necessary and sufficient condition for both roots of equation (7) to lie inside the unit disk is

$$|TrJ_F((\bar{u}, \bar{v}))| < 1 + DetJ_F((\bar{u}, \bar{v})) < 2.$$

The following lemma shows that all solutions of (5) are bounded. In [7], this result was obtained for the case the sequence $(\alpha_n)_{n \geq 0}$ is periodic with period p. But, we present it in the case $p = 2$ for the completeness.

Lemma 2.3. *Assume that $(\alpha_n)_{n \geq 0}$ is a periodic sequence of prime period 2. Then, every solution of Equation (5) is bounded.*

Proof. From (5), we have

$$x_{n+1} = \frac{\alpha_n}{1 + x_n x_{n-1}} \leq \alpha_n \tag{8}$$

for every $n \geq 0$. Hence we see that $x_{2n+1} \leq a$ and $x_{2n+2} \leq b$ for every $n \geq 0$. Also, from (5) and (8), we have

$$x_{n+1} = \frac{\alpha_n}{1 + x_n x_{n-1}} \geq \frac{\alpha_n}{1 + ab}$$

for every $n \geq 1$. Hence we see that $x_{2n+1} \geq \frac{a}{1+ab}$ and $x_{2n+2} \geq \frac{b}{1+ab}$ for every $n \geq 1$. Consequently, we have

$$\frac{a}{1 + ab} \leq x_{2n+1} \leq a, \ \frac{b}{1 + ab} \leq x_{2n+2} \leq b \tag{9}$$

for every $n \geq 0$. $\square$

Lemma 2.4. *[22] Consider the cubic equation*

$$P(z) = z^3 - \alpha z^2 - \beta z - \gamma = 0. \qquad (10)$$

Equation (10) has the discriminant

$$\Delta = -\alpha^2\beta^2 - 4\beta^3 + 4\alpha^3\gamma + 27\gamma^2 + 18\alpha\beta\gamma.$$

Then, the following statements are true;

(i) If $\Delta < 0$, then the polynomial P has three distinct real zeros ρ_1, ρ_2, ρ_3.

(ii) If $\Delta = 0$, then there are two subcases:

(a) if $\beta = \frac{-\alpha^2}{3}$ and $\gamma = \frac{\alpha^3}{27}$, then the polynomial P has the triple root $\rho = \frac{\alpha}{3}$,

(b) if $\beta \neq -\frac{\alpha^2}{3}$ or $\gamma \neq \frac{\alpha^3}{27}$, then the polynomial P has the double root r and the simple root ρ.

(iii) If $\Delta > 0$, then the polynomial P has one real root p and two complex roots $re^{\pm i\theta}$, $\theta \in (0, \pi)$.

3 Main Results

In this section, we prove our main results.

It is clear that Equation (5) can be written as follows:

$$x_{2n+1} = \frac{a}{1 + x_{2n}x_{2n-1}}, \; x_{2n+2} = \frac{b}{1 + x_{2n+1}x_{2n}}. \qquad (11)$$

To conduct a stability analysis, we set

$$x_{2n-1} = u_n \text{ and } x_{2n} = v_n, \; n \in \mathbb{N}_0. \qquad (12)$$

Then (11) can be written in the form

$$\begin{cases} u_{n+1} = & \frac{a}{1+u_n v_n} \\ v_{n+1} = & \frac{b(1+u_n v_n)}{1+u_n v_n + a v_n} \end{cases}, \; n \in \mathbb{N}_0. \qquad (13)$$

That is, since system (13) is equivalent to Equation (5), we simultaneously study the system.

3.1 Locally asymptotically stability

In this subsection, we study locally asymptotically stability of the unique positive equilibrium $(\bar{u}, \bar{v}) = \left(\bar{u}, \frac{b}{a}\bar{u}\right)$ of system (13).

Lemma 3.1. *System (13) has the unique positive equilibrium point on* $\left(\frac{a}{1+ab}, a\right) \times \left(\frac{b}{1+ab}, b\right)$.

Proof. Equilibrium points of system (13) is solutions of the algebraic system

$$\bar{u} = \frac{a}{1+\bar{u}\bar{v}}, \;\; \bar{v} = \frac{b\left(1+\bar{u}\bar{v}\right)}{1+\bar{u}\bar{v}+a\bar{v}}. \tag{14}$$

From (14), we see that

$$\bar{v} = \frac{b}{a}\bar{u} \tag{15}$$

which implies

$$\bar{u}^3 + \frac{a}{b}\bar{u} - \frac{a^2}{b} = 0.$$

Now, we consider the polynomial

$$P\left(\bar{u}\right) = \bar{u}^3 + \frac{a}{b}\bar{u} - \frac{a^2}{b}. \tag{16}$$

From (9), (12) and (16), we have

$$P\left(a\right) = a^3 > 0$$

and

$$P\left(\frac{a}{1+ab}\right) = \frac{-a^2}{b}\frac{ab\left(\left(1+ab\right)^2 - 1\right)}{\left(1+ab\right)^3} < 0.$$

Also, since

$$P'\left(\bar{u}\right) = 3\bar{u}^2 + \frac{a}{b} > 0$$

the polynomial $P\left(\bar{u}\right)$ has the unique zero on $\left(\frac{a}{1+ab}, a\right)$. On the other hand, by taking into account (15), we have

$$\frac{b}{a}\bar{u} = \bar{v} \in \left(\frac{b}{a}\frac{a}{1+ab}, \frac{b}{a}a\right) = \left(\frac{b}{1+ab}, b\right).$$

Hence, system (13) has the unique positive equilibrium point on

$$\left(\frac{a}{1+ab}, a\right) \times \left(\frac{b}{1+ab}, b\right)$$

such that $(\bar{u}, \bar{v}) = \left(\bar{u}, \frac{b}{a}\bar{u}\right)$. $\square$

Theorem 3.2. *The unique equilibrium* $(\bar{u}, \bar{v}) = \left(\bar{u}, \frac{b}{a}\bar{u}\right)$ *of system (13) is locally asymptotically stable.*

Proof. We define the map

$$F : \left(\frac{a}{1+ab}, a\right) \times \left(\frac{b}{1+ab}, b\right) \to \left(\frac{a}{1+ab}, a\right) \times \left(\frac{b}{1+ab}, b\right)$$

associated to system (13), i.e.

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{a}{1+xy} \\ \frac{b(1+xy)}{1+xy+ay} \end{pmatrix}.$$

The Jacobian matrix of F evaluated at $\left(\bar{u}, \frac{b}{a}\bar{u}\right)$ is

$$J_F(\bar{u}, \bar{v}) = \begin{pmatrix} \frac{-b\bar{u}^3}{a^2} & \frac{-\bar{u}^3}{a} \\ \frac{b^3\bar{u}^6}{a^5} & \frac{-b\bar{u}^4}{a^3} \end{pmatrix}$$

and its characteristic equation associated with $\left(\bar{u}, \frac{b}{a}\bar{u}\right)$ is

$$\lambda^2 + \frac{a^4 b\bar{u}^3 + a^3 b\bar{u}^4}{a^6}\lambda + \frac{ab^2\bar{u}^7 + b^3\bar{u}^9}{a^6} = 0. \tag{17}$$

Therefore, from (17) and Lemma 2.2-(b), we have the inequalities

$$\left|\frac{a^4 b\bar{u}^3 + a^3 b\bar{u}^4}{a^6}\right| < 1 + \frac{ab^2\bar{u}^7 + b^3\bar{u}^9}{a^6} < 2$$

from which it follows that

$$(a - \bar{u})^2 + \bar{u}^2 > 0 \text{ and } 8ab + 1 > 0$$

which always hold. So, the proof is complete. $\square$

Theorem 3.3. *System (13) does not have positive periodic solutions with prime period two.*

Proof. First, we suppose that system (13) has positive periodic solutions with prime period two as follows:

$$\{\ldots, (\phi_1, \psi_1), (\phi_2, \psi_2), \ldots\},$$

where $\phi_1 \neq \phi_2$ and $\psi_1 \neq \psi_2$. Then, we have

$$\begin{cases} \phi_1 = \frac{a}{1+\phi_2\psi_2}, \ \psi_1 = \frac{b(1+\phi_2\psi_2)}{1+\phi_2\psi_2+a\psi_2}, \\ \phi_2 = \frac{a}{1+\phi_1\psi_1}, \ \psi_2 = \frac{b(1+\phi_1\psi_1)}{1+\phi_1\psi_1+a\psi_1} \end{cases} \tag{18}$$

from which it follows that

$$\psi_1 = \frac{b}{1+\phi_1\psi_2}, \ \psi_2 = \frac{b}{1+\phi_2\psi_1}. \tag{19}$$

From the first two equations of (18) and (19), we have

$$\phi_1\phi_2(\psi_2-\psi_1)+\phi_1-\phi_2=0, \ \psi_1\psi_2(\phi_1-\phi_2)+\psi_1-\psi_2=0. \tag{20}$$

(20) implies $\phi_1\phi_2\psi_1\psi_2 = -1$ which is a contradiction. So, the proof is completed. □

3.2 Closed form solution of Equation (5)

In this subsection, we obtain a closed form solution of Equation (5). By applying the change of variables

$$x_n = \frac{z_{n-1}}{z_n} \tag{21}$$

to Equation (5), we have third order linear equation

$$z_{n+1} - \frac{1}{\alpha_n}z_n - \frac{1}{\alpha_n}z_{n-2} = 0, \ n \in \mathbb{N}_0, \tag{22}$$

where $z_0 = 1$, $z_{-1} = x_0$, $z_{-2} = x_0x_{-1}$. We write Equation (22) as the following

$$z_{2n+1} - \frac{1}{a}z_{2n} - \frac{1}{a}z_{2n-2} = 0 \tag{23}$$

and

$$z_{2n+2} - \frac{1}{b} z_{2n+1} - \frac{1}{b} z_{2n-1} = 0$$

from which it follows that

$$z_{2n+4} - \frac{1}{ab} z_{2n+2} - \frac{2}{ab} z_{2n} - \frac{1}{ab} z_{2n-2} = 0 \tag{24}$$

for every $n \in \mathbb{N}_0$. Equation (24) has the characteristic equation

$$P_1(\lambda) = \lambda^6 - \frac{1}{ab}\lambda^4 - \frac{2}{ab}\lambda^2 - \frac{1}{ab} = 0.$$

Let

$$Q(\lambda) = \lambda^3 - \frac{1}{\sqrt{ab}}\lambda^2 - \frac{1}{\sqrt{ab}} \text{ and } R(\lambda) = \lambda^3 + \frac{1}{\sqrt{ab}}\lambda^2 + \frac{1}{\sqrt{ab}}. \tag{25}$$

Then, $P_1(\lambda) = Q(\lambda) R(\lambda)$. Note also that the polynomials Q and R satisfy the relation $Q(-\lambda) = -R(\lambda)$. That is, if λ is any zero of the polynomial R, then $-\lambda$ is a zero of the polynomial Q. On the other hand, we consider the linear equation

$$w_{n+1} - \frac{1}{ab} w_n - \frac{2}{ab} w_{n-1} - \frac{1}{ab} w_{n-2} = 0 \tag{26}$$

whose characteristic equation is

$$P_1(\sqrt{\mu}) = \mu^3 - \frac{1}{ab}\mu^2 - \frac{2}{ab}\mu - \frac{1}{ab} = 0.$$

We see from Lemma 2.4 that the equation $P_1(\sqrt{\mu}) = 0$ has one real root and two complex roots denoted by p^2 and $r^2 e^{\pm i2\theta}$, $\theta \in (0, \pi)$, respectively. These notations are valid, since $\mu = \lambda^2$. Also, note that since $ab > 0$ and $\mu^3 = \frac{1}{ab}(\mu+1)^2$, the unique real root of $P_1(\sqrt{\mu}) = 0$ is positive. So, we have the general solution of (26) as follows:

$$w_{n-1} = C_1 p^{2n} + r^{2n}(C_2 \cos 2n\theta + C_3 \sin 2n\theta), \; n \geq -1,$$

where C_1, C_2 and C_3 are arbitrary constats. A solution of Equation (24) satisfies Equation (26), that is $w_{n-1} = z_{2n}$, and so we can write the general solution of Equation (24) as follows:

$$z_{2n} = C_1 p^{2n} + r^{2n}(C_2 \cos 2n\theta + C_3 \sin 2n\theta), \; n \geq -1, \tag{27}$$

where

$$\begin{aligned} C_1 &= \frac{p^2}{ab}\frac{\left(abr^4+1\right)x_0x_{-1}+ax_0-2abr^2\cos 2\theta+1}{p^4+r^4-2p^2r^2\cos 2\theta}, \\ C_2 &= \frac{-p^2\left(\left(abr^4+1\right)x_0x_{-1}+ax_0\right)+abp^4+abr^4-p^2}{ab\left(p^4+r^4-2p^2r^2\cos 2\theta\right)}, \\ C_3 &= \frac{\left(\cos\left(2\,t\right)abp^2r^4-abp^4r^2-\cos\left(2\,t\right)p^2+r^2\right)x_0x_{-1}}{ab\sin 2\theta\left(p^4+r^4-2p^2r^2\cos 2\theta\right)} \\ &\quad -\frac{\left(-\cos\left(2\,t\right)ap^2+ar^2\right)x_0}{ab\sin 2\theta\left(p^4+r^4-2p^2r^2\cos 2\theta\right)} \\ &\quad -\frac{\cos\left(2\,t\right)abp^4-\cos\left(2\,t\right)abr^4-\cos\left(2\,t\right)p^2+r^2}{ab\sin 2\theta\left(p^4+r^4-2p^2r^2\cos 2\theta\right)}. \end{aligned}$$

On the other hand, by (23) and some operations, for every $n \geq -1$, we have

$$\begin{aligned} z_{2n+1} &= \frac{1}{a}z_{2n}+\frac{1}{a}z_{2n-2} \\ &= C_1\frac{p^2+1}{a}p^{2n-2}+\frac{r^{2n}}{a}\left(C_2'\cos 2n\theta+C_3'\sin 2n\theta\right), \end{aligned} \tag{28}$$

where

$$C_2' = C_2+\frac{C_2\cos 2\theta-C_3\sin 2\theta}{r^2} \text{ and } C_3' = C_3+\frac{C_3\cos 2\theta+C_2\sin 2\theta}{r^2}.$$

Also, the relations $P_1(\lambda) = Q(\lambda)R(\lambda)$ and $Q(-\lambda) = -R(\lambda)$ imply that if p is a root of $Q(\lambda)$, then $-p$ is a root of $R(\lambda)$. Hence, p satisfies the relation

$$\frac{p^2+1}{a} = \sqrt{\frac{b}{a}}p^3.$$

From this and (28), it follows that

$$z_{2n+1} = C_1\sqrt{\frac{b}{a}}p^{2n+1}+\frac{r^{2n}}{a}\left(C_2'\cos 2n\theta+C_3'\sin 2n\theta\right),\ n \geq -1, \tag{29}$$

where

$$C_2' = C_2+\frac{C_2\cos 2\theta-C_3\sin 2\theta}{r^2} \text{ and } C_3' = C_3+\frac{C_3\cos 2\theta+C_2\sin 2\theta}{r^2}.$$

Therefore, from (21), (27) and (29), we have the closed form solution of Equation (5) as follows:

$$x_{2n} = \frac{C_1\sqrt{\frac{b}{a}}p^{2n-1} + \frac{r^{2n-2}}{a}\left(C_2'\cos\left(2n-2\right)\theta + C_3'\sin\left(2n-2\right)\theta\right)}{C_1p^{2n} + r^{2n}\left(C_2\cos 2n\theta + C_3\sin 2n\theta\right)} \tag{30}$$

and

$$x_{2n+1} = \frac{C_1p^{2n} + r^{2n}\left(C_2\cos 2n\theta + C_3\sin 2n\theta\right)}{C_1\sqrt{\frac{b}{a}}p^{2n+1} + \frac{r^{2n}}{a}\left(C_2'\cos 2n\theta + C_3'\sin 2n\theta\right)}, \tag{31}$$

where

$$\begin{aligned}
C_1 &= \frac{p^2}{ab}\frac{\left(abr^4+1\right)x_0x_{-1} + ax_0 - 2abr^2\cos 2\theta + 1}{p^4 + r^4 - 2p^2r^2\cos 2\theta}, \\
C_2 &= \frac{-p^2\left(\left(abr^4+1\right)x_0x_{-1} + ax_0\right) + abp^4 + abr^4 - p^2}{ab\left(p^4 + r^4 - 2p^2r^2\cos 2\theta\right)}, \\
C_3 &= \frac{\left(\cos\left(2\,t\right)abp^2r^4 - abp^4r^2 - \cos\left(2\,t\right)p^2 + r^2\right)x_0x_{-1}}{ab\sin 2\theta\left(p^4 + r^4 - 2p^2r^2\cos 2\theta\right)} \\
&\quad - \frac{\left(-\cos\left(2\,t\right)ap^2 + ar^2\right)x_0}{ab\sin 2\theta\left(p^4 + r^4 - 2p^2r^2\cos 2\theta\right)} \\
&\quad - \frac{\cos\left(2\,t\right)abp^4 - \cos\left(2\,t\right)abr^4 - \cos\left(2\,t\right)p^2 + r^2}{ab\sin 2\theta\left(p^4 + r^4 - 2p^2r^2\cos 2\theta\right)}.
\end{aligned}$$

and

$$C_2' = C_2 + \frac{C_2\cos 2\theta - C_3\sin 2\theta}{r^2} \text{ and } C_3' = C_3 + \frac{C_3\cos 2\theta + C_2\sin 2\theta}{r^2}.$$

for every $n \geq -1$.

3.3 Globally asymptotically stability

In this subsection, we study globally asymptotically stability of the unique positive equilibrium $(\bar{u}, \bar{v}) = \left(\bar{u}, \frac{b}{a}\bar{u}\right)$ of system (13).

Lemma 3.4. *Consider the cubic polynomial $S\left(\lambda\right) = \lambda^3 - c\lambda^2 - c$. Then, zeros of the polynomial S satisfy the relation $|\sigma| = \frac{\rho}{\sqrt{1+\rho^2}}$, where ρ is the unique real zero of the polynomial S and σ is one of complex conjugate ones.*

Proof. Suppose that

$$S(\lambda) = (\lambda - \rho)\left(\lambda^2 - c_1\lambda - c_2\right),$$

where

$$c_1 = \frac{-\rho}{1+\rho^2} \text{ and } c_2 = \frac{-\rho^2}{1+\rho^2}.$$

Then, σ is one of complex conjugate zeros of the quadratic polynomial $\lambda^2 - c_1\lambda - c_2$. Therefore, the proof follows from absolute value of σ. ☐

It is clear from Lemma 3.4 that $|\sigma| = \frac{\rho}{\sqrt{1+\rho^2}} < \rho$.

Theorem 3.5. *The unique equilibrium* $(\bar{u}, \bar{v}) = \left(\bar{u}, \frac{b}{a}\bar{u}\right)$ *of system (13) is globally asymptotically stable.*

Proof. We know from Theorem 3.2 that the unique equilibrium $(\bar{u}, \bar{v}) = \left(\bar{u}, \frac{b}{a}\bar{u}\right)$ of system (13) is locally asymptotically stable. Hence, it is enough to show that

$$\lim_{n\to\infty} u_n = \bar{u} \text{ and } \lim_{n\to\infty} v_n = \bar{v}.$$

or

$$\lim_{n\to\infty} x_{2n} = \bar{v} \text{ and } \lim_{n\to\infty} x_{2n+1} = \bar{u}$$

by taking into account (12). We also know that $\bar{u}$ is the unique real zero of the polinomial P in (16). On the other hand, p is the unique real zero of the polinomial Q in (25). We claim that the zeros of the polinomials P and Q are of the relation

$$\sqrt{\frac{a}{b}}\frac{1}{p} = \bar{u}. \tag{32}$$

That is, we have

$$\begin{aligned} P\left(\sqrt{\frac{a}{b}}\frac{1}{p}\right) &= \left(\sqrt{\frac{a}{b}}\frac{1}{p}\right)^3 + \frac{a}{b}\sqrt{\frac{a}{b}}\frac{1}{p} - \frac{a^2}{b} \\ &= -\frac{a^2}{b}\frac{1}{p^3}\left(p^3 - \frac{1}{\sqrt{ab}}p^2 - \frac{1}{\sqrt{ab}}\right) \\ &= -\frac{a^2}{b}\frac{1}{p^3}Q(p) \\ &= 0. \end{aligned}$$

By taking limits of (30) and (31) as $n \to \infty$, by using (32) and the result of Lemma 3.4, we have

$$\begin{aligned}
\lim_{n\to\infty} x_{2n} &= \lim_{n\to\infty} \frac{C_1\sqrt{\frac{b}{a}}p^{2n-1} + \frac{r^{2n-2}}{a}\left(C_2' \cos(2n-2)\theta + C_3' \sin(2n-2)\theta\right)}{C_1 p^{2n} + r^{2n}\left(C_2 \cos 2n\theta + C_3 \sin 2n\theta\right)} \\
&= \lim_{n\to\infty} \frac{p^{2n-1}}{p^{2n}} \frac{C_1\sqrt{\frac{b}{a}} + \left(\frac{r}{p}\right)^{2n-1} \frac{1}{ar}\left(C_2' \cos(2n-2)\theta + C_3' \sin(2n-2)\theta\right)}{C_1 + \left(\frac{r}{p}\right)^{2n}\left(C_2 \cos 2n\theta + C_3 \sin 2n\theta\right)} \\
&= \sqrt{\frac{b}{a}}\frac{1}{p} \\
&= \frac{b}{a}\bar{u} \\
&= \bar{v}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} x_{2n+1} &= \lim_{n\to\infty} \frac{C_1 p^{2n} + r^{2n}\left(C_2 \cos 2n\theta + C_3 \sin 2n\theta\right)}{C_1\sqrt{\frac{b}{a}}p^{2n+1} + \frac{r^{2n}}{a}\left(C_2' \cos 2n\theta + C_3' \sin 2n\theta\right)} \\
&= \lim_{n\to\infty} \frac{p^{2n}}{p^{2n+1}} \frac{C_1 + \left(\frac{r}{p}\right)^{2n}\left(C_2 \cos 2n\theta + C_3 \sin 2n\theta\right)}{C_1\sqrt{\frac{b}{a}} + \left(\frac{r}{p}\right)^{2n+1} \frac{1}{ar}\left(C_2' \cos 2n\theta + C_3' \sin 2n\theta\right)} \\
&= \sqrt{\frac{a}{b}}\frac{1}{p} \\
&= \bar{u}.
\end{aligned}$$

So, the proof is completed. □

Theorem 3.6. *Equation (5) has positive periodic solutions with prime period two which is given by*

$$\left\{\ldots, \bar{u}, \frac{b}{a}\bar{u}, \bar{u}, \frac{b}{a}\bar{u}, \ldots\right\}. \tag{33}$$

Proof. First, we suppose that Equation (5) has positive periodic solutions with prime period two as follows:

$$\{\ldots, \phi, \psi, \phi, \psi, \ldots\}$$

From (11), we have

$$\phi = \frac{a}{1+\phi\psi}, \;\; \psi = \frac{b}{1+\phi\psi} \tag{34}$$

from which it follows that

$$\psi = \frac{b}{a}\phi. \tag{35}$$

By using (34) and (35), we have

$$P(\phi) = \phi^3 + \frac{a}{b}\phi - \frac{a^2}{b} = 0,$$

which has the unique real root $\phi = \bar{u}$. Hence, the result follows by (35). □

The following corollary is a straightforward result of Theorem 3.5

Corollary 3.7. *Every positive solution of Equation (5) tends to its periodic solution with prime period two which is given by (33).*

3.4 More on Equation (4)

In this subsection, we confirm Conjecture 2.2. given in [3]. We know from [3] that Equation (4) has a unique equilibrium $\bar{x}$ which is the unique positive root of the cubic

$$P_2(\bar{x}) = \bar{x}^3 + \bar{x} - \alpha = 0$$

and $\bar{x}$ is locally asymptotically stable for all values of the parameter α.

Theorem 3.8. *Every positive solution of Equation (4) tends to a finite limit.*

Proof. By applying the change of variables (21) to Equation (4), we have third order linear equation

$$z_{n+1} - \frac{1}{\alpha}z_n - \frac{1}{\alpha}z_{n-2} = 0, \ n \in \mathbb{N}_0. \tag{36}$$

Equation (36) has the characteristic equation

$$P_3(\lambda) = \lambda^3 - \frac{1}{\alpha}\lambda^2 - \frac{1}{\alpha} = 0,$$

which has one real root and two complex roots denoted by $\hat{p}$ and $\hat{r}e^{\pm i\theta}$, $\theta \in (0, \pi)$, respectively. Hence, Equation (36) has the general solution

$$z_n = \hat{C}_1\hat{p}^n + \hat{r}^n\left(\hat{C}_2\cos n\theta + \hat{C}_3\sin n\theta\right), \ n \geq -2, \tag{37}$$

where $\hat{C}_1$, $\hat{C}_2$ and $\hat{C}_3$ are arbitrary constats. From (21) and (37), it follows that

$$x_n = \frac{\hat{C}_1 \hat{p}^{n-1} + \hat{r}^{n-1}\left(\hat{C}_2 \cos(n-1)\theta + \hat{C}_3 \sin(n-1)\theta\right)}{\hat{C}_1 \hat{p}^{n} + \hat{r}^{n}\left(\hat{C}_2 \cos n\theta + \hat{C}_3 \sin n\theta\right)}, \quad n \geq -1, \tag{38}$$

which is the general solution of Equation (4). We claim that the zeros $\bar{x}$ and $\hat{p}$ of the polinomials P_2 and P_3 are of the relation

$$\frac{1}{\hat{p}} = \bar{x}. \tag{39}$$

That is, we have

$$\begin{aligned} P_2(\bar{x}) &= \bar{x}^3 + \bar{x} - \alpha \\ &= \frac{1}{\hat{p}^3} + \frac{1}{\hat{p}} - \alpha \\ &= -\frac{\alpha}{\hat{p}^3}\left(-\frac{1}{\alpha} - \frac{1}{\alpha}\hat{p}^2 + \hat{p}^3\right) \\ &= -\frac{\alpha}{\hat{p}^3} P_3(\hat{p}) \\ &= 0. \end{aligned}$$

Consequently, from (38), (39) and Lemma 3.4, we have

$$\begin{aligned} \lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{\hat{C}_1 \hat{p}^{n-1} + \hat{r}^{n-1}\left(\hat{C}_2 \cos(n-1)\theta + \hat{C}_3 \sin(n-1)\theta\right)}{\hat{C}_1 \hat{p}^{n} + \hat{r}^{n}\left(\hat{C}_2 \cos n\theta + \hat{C}_3 \sin n\theta\right)} \\ &= \lim_{n\to\infty} \frac{\hat{p}^{n-1}}{\hat{p}^n} \frac{\hat{C}_1 + \left(\frac{\hat{r}}{\hat{p}}\right)^{n-1}\left(\hat{C}_2 \cos(n-1)\theta + \hat{C}_3 \sin(n-1)\theta\right)}{\hat{C}_1 + \left(\frac{\hat{r}}{\hat{p}}\right)^{n}\left(\hat{C}_2 \cos n\theta + \hat{C}_3 \sin n\theta\right)} \\ &= \frac{1}{\hat{p}} \\ &= \bar{x}. \end{aligned}$$

□

The following corollary is a straightforward result of that $\bar{x}$ is locally asymptotically stable and Theorem 3.8.

Corollary 3.9. *The unique equilibrium $\bar{x}$ of Equation (4) is globally asymptotically stable for all positive values of the parameter α.*

4 Numerical Examples

In this section, we give some numerical examples to support our theoretical results.

Example 4.1. In Figure 1-3, we illustrate the solutions which corresponds to some special values of the initial conditions u_0, v_0 and the parameters a, b of (13).

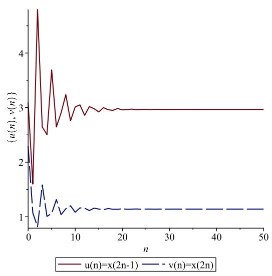

Figure 1: $a = 13$, $b = 5$.

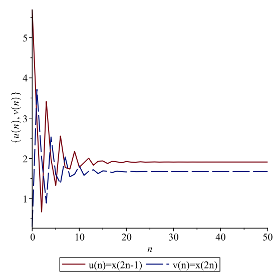

Figure 2: $a = 8$, $b = 7$.

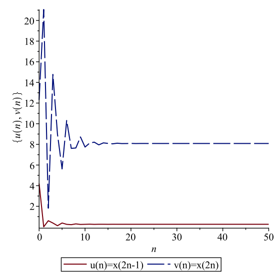

Figure 3: $a = 0.8$, $b = 25$.

Example 4.2. In Figure 4-6, we illustrate the solutions which corresponds to some special values of the initial conditions x_{-1}, x_0 and the parameters a, b of (5).

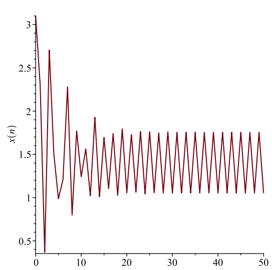

Figure 4: $a = 3$, $b = 5$.

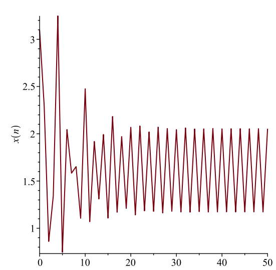

Figure 5: $a = 7$, $b = 4$.

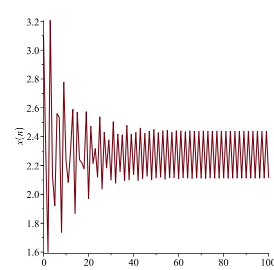

Figure 6: $a = 13$, $b = 15$.

Example 4.3. In Figure 7-9, we illustrate the solutions which corresponds to some special values of the initial conditions x_{-1}, x_0 and the parameter α of (4).

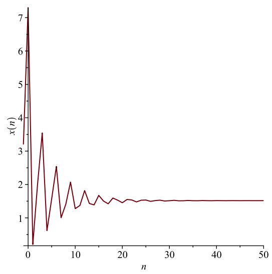

Figure 7: $\alpha = 5$.

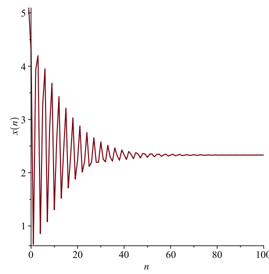

Figure 8: $\alpha = 15$.

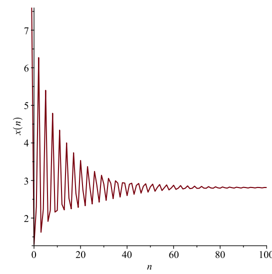

Figure 9: $\alpha = 25$.

5 Conclusion

In this study we mainly show that every positive solution of Equation (5) tends to a two periodic solution of the equation. To conduct a stability analysis, we handle system (13) which is equivalent to Equation (5) and so show that the unique positive equilibrium point of system (13) is globally asymptotically stable. Finally, we confirm Conjecture 2.2. given in [3].

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