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Original Research Paper

Some New Results on Regular Module

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Abstract. The aim of this paper is to study unitary regular modules on commutative rings with identity. Regularity accompanied by cocyclic property results in some prime-related conclusions on both modules and rings. Further to this, regularity addresses also radical property of submodules and they are related closely. This property not only affects the modules on ring R but also restricts R to totally idempotent one.

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1 Introduction

Regularity concept started with a paper [7] of Von Neumann for rings with the regular property for elements and then was extended to modules by [14, 15]. This concept has been studied widely in papers and is in interest [2, 4, 9, 10].

In [3], it was proved that every projective module on von Neumann regular ring is a regular module. Moreover, it was proved a projective

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module M is regular if and only if every cyclic submodule of M is a direct summand of M. Also, they considered both regularity and projectivity properties and concluded $Z(M) = J(M) = 0$, where $J(M)$ and $Z(M)$ are the Jacobson and singular submodules of M [3]. Moreover, they proved M is regular and indecomposable if and only if M is projective and hollow.

In this paper, we would like to investigate regular modules with respect to prime, primary, idempotent and radical properties which lead us to interesting results as follows:

Regular submodule not only has effects on its submodules but also it affects the ring; in regular R-module M, R is simple if and only if every proper ideal is prime if and only if every submodule of R-module M is prime. In every regular R-module, R is fully idempotent. We know that zero submodule can characterize many thing, so knowing that it is prime, is important to us and in the realm of regularity, we get if M is regular cocyclic R-module then 0 is a prime submodule of M. Thinking of prime and primary properties is important because of their oneway relation but here we can give an equivalency between them; if M is torsion-free and regular, then every proper submodule N of M is prime if and only if N is a primary submodule of M.

Radical submodules are other ones to study under regularity. Here, we only give one result on them; every submodule of a regular R-module M is a radical submodule of M.

We may like to go deeply and study submodules of submodules. In this investigation we get that for all proper submodule contained in N there exists a prime submodule of N containing it.

Moreover, by assuming that R is regular we will get different results; on every regular ring R, every submodule $N \neq M$ is primary submodule if and only if it is a prime submodule of M.

Here are some basic definitions and results required for the rest of paper.

Definition 1.1. [3, Definition 2.1] Let M be an R-module. An element $x \in M$ is called regular if there exists $\varphi \in \mathrm{Hom}_R(M, R)$ such that $\varphi(x)x = x$. The R-module M is called regular if every element $x \in M$ is regular.

Definition 1.2. An R-submodule N of M is called pure if $N \cap IM = IN$ for every ideal I of R.

Lemma 1.3. *[2, Lemma 1] If M is a regular R-module, then every submodule of M is pure.*

Lemma 1.4. *[1] A ring R is Von Neumann regular if and only if R is regular as an R-module.*

Definition 1.5. [13] A submodule L of an R-module M is called isolated if $rad_M L \neq rad_M N$ for every proper submodule N of L.

Theorem 1.6. *[13, Theorem 2.6] The following statement are equivalent.*

- *Every submodule is an isolated submodule of M.*
- *For any given every submodule N of M we have $N \cap IM = IN$ where I is an arbitrary ideal of R.*

2 Main results

This section is devoted to proving the results stated in the previous section. First of all, we bring a result on the isolated submodules.

Proposition 2.1. *[13] Suppose R is an arbitrary ring. Then for any proper submodule N, we define it to be isolated if and only if every submodule H of N is contained in a prime submodule L of M such that $H \subseteq L$ but $N \nsubseteq L$.*

Here, we start with some results making the way to our results. Every submodule is pure and the converse is also true, [13, Theorem 1.9]. So, every submodule of regular module is isolated. It is worth mentioning that submodules of R-module R which are prime, are also prime as ideals of R [13].

Theorem 2.2. *Let M be a regular R-module and N a submodule of M. Then every proper submodule of N is contained in a prime submodule of N.*

Proof. Let N be a submodule of a regular R-module M. Then from lemma 1.3 N is pure and by Theorem 1.6 N is an isolated submodule of M. Proposition 2.1 implies that for every submodule H contained in N, there exists a prime submodule L of M in which L involves H but N is out of L and so by [13, Lemma 1.2] $L \cap N$ is prime. $\square$

The converse of above theorem dose not hold. Suppose I is a proper ideal of R which is not direct summand of R. We know that every proper ideal J of I is contained in a maximal ideal and so in a prime ideal of R. However, it does not imply that R is a regular R-module. In [13], the authors brought an example of an ideal contained in a prime one but is not isolated and so is not pure claiming non-regularity of R.

Proposition 2.3. *In reguar R-module M, every proper submodule H of submodule N is contained in a prime submodule $K \subseteq N$, in which K has the property that can be lifted to M.*

Proof. We know that every submodule of regular module M is pure, and so is isolated, according to Theorem 1.6. Let H be the stated submodule in the proposition. By using [13, Proposition 1.6], the existence of such prime submodule K of N is trivial such that $H \subseteq K$ and satisfies the stated property. $\square$

Now, one can try to understand under what conditions, a prime submodule K can be lifted to M. Suppose L is a prime submodule of an R-module M. Set $P = (L : M)$ which is clearly a prime ideal of R. In this case, L is called P-prime submodule M. McCasland and Smith [13] (Lying Over Theorem) proved that a P-prime submodule K of M can be lifted to M if and only if $N \cap PM \subseteq K$.

Now, if we consider Lying Over with regularity property plus the fact that in regular modules we have $N \cap PM = PN$ we get the following corollary.

Corollary 2.4. *Let M be a regular R-module and N be a R-submodule of M. Let K be a P-prime submodule of N. Then $PN \subseteq K$ if and only if K can be lifted to M.*

Corollary 2.5. *For every pure submodule N of M, we have $IN = N$ if and only if $N \subseteq IM$, where I is an arbitrary ideal of R.*

Proof. Suppose $N \subseteq IM$. Then $N = N \cap IM$. On the other hand, N is pure and we have $IN = N \cap IM$, where we $N = IN$. Conversely, let $N = IN$. Then $N \cap IM = IN = N$ and we get $N \subset IM$. $\square$

Theorem 2.6. *Suppose M is torsion-free and regular module. Then every proper submodule N of M is prime if and only if N is a primary submodule of M.*

Proof. The proof of necessary condition is clear. For conversely, suppose N is primary and $rx \in N$ for some $r \in R$, and $x \in M \setminus N$. It is enough to prove that $rM \subseteq N$. Since N is primary, then $r^k M \subseteq N$ which accompanied by regularity of M yields $\langle r \rangle M \cap N = \langle r \rangle N$. Suppose $y \in M$ then $r^k y = srn$ for some $s \in R, n \in N$, that gives us $r(r^{k-1}y - sn) = 0$, then $r^{k-1}y = sn \in N$, hence $r^{k-1}M \subseteq N$, so $r^{k-1}y = srn$, then $r^{k-2}y = sn \in N$ and by above argument $ry \in N$, then $rM \subseteq N$ so N is a prime submodule of M. $\square$

It is worthy of investigation relation between isolated submodule N of M and radical submodules of M which are submodules of N, we know if it is a radical submodule of M then $rad_M H = H \neq rad_M N$. It follows that M is isolated if all proper submodules of N are radical ones, also.

Corollary 2.7. *Let R be a regular ring. The followings are equivalent.*

(1) R is simple.

(2) Every proper submodule N of every non-zero R-module M is prime.

(3) Every proper submodule N of every non-zero R-module M is primary.

Proof. (1) $\Leftrightarrow$ (2) According to [11, Theorem 4.2].
(2) $\Leftrightarrow$ (3) [12, Corollary] gives this equivalency. $\square$

Lemma 2.8. *Let M be a regular R-module which contains an essential simple submodule, i.e., M is cocyclic module. Then 0 is a prime submodule of M.*

Proof. Let $IN = 0$ and $N \neq 0$. We prove that $IM = 0$ resulting 0 is a prime submodule of M. Let $L \neq 0$ be an essential simple submodule of M. Put $P = \{r \in R | rL = 0\}$, since M is regular then $L \cap PM = PL = 0$.

So $PM = 0$. On the other hand, $L \subseteq N$ and $IN = 0$. So, $IL = 0$ which gives us $I \subseteq P$ hence $IM \subseteq PM = 0$ and so $IM = 0$ and we get the result. □

Let R be a regular R-module. Then $I \cap J = I \cap RJ = IJ$. Put $I = J$. Then we have $I = I \cap I = II = I^2$. Hence, R is fully idempotent (for definition of fully idempotent see [5, 6]). So, in regular R-module R, R is fully idempotent. In [13, Propositions 2.2], the author proved that the ring R is fully idempotent if and only if every submodule N is pure submodule of R-module M.

Hansen [8, Lemma 1] proved that the fully idempotent ring R is equivalent with the property that every proper ideals are radical submodules of R-module R. These explanations lead us to the following corollary.

Corollary 2.9. *Let R be a regular R-module. Then every proper submodule of R is a radical submodule.*

McCasland and Smith [13, Propositions 2.6] proved that N is radical submodule of M if and only if N is a pure submodule of M. Since every submodule of a regular module is pure , so in every regular module, every submodule is both radical and pure. As we see, the following result can be achieved.

Corollary 2.10. *In every regular R-module M, all proper submodules of M are radical and furthermore, R is a fully idempotent ring.*

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