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Two Approximations for P-Norms in System of Independent Functions and Trigonometric Series

M. Ghanbari
Farahan Branch, Islamic Azad University

Abstract. The aim of this article is find approximations for p-norms of terms $f(x) = \sum_{n=1}^{\infty} \frac{1}{n} sin2^n x; x \in (0, 2\pi)$ and $P(x) = \sum_{k=1}^{\infty} \frac{1}{k}\psi_k(x); x$ in $(0, 1)$, in system of independent functions $\{\psi_n(x)\}_{n=1}^{\infty}$.

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1. Introduction

In this article we introduce some propositions on a sequence of independent functions that will be needed. In other hand we intend to study a particular series, which is not only interesting in itself, but provide examples illuminating many points of the general theory of trigonometric series.

Definition 1.1. ([3]) *A set of real measurable functions* $\{f_n(x)\}_{n=1}^{N}$ *with domain (0,1) is a set of independent functions if for every interval* $I_n, n = 1, ..., N$, *the following condition is satisfied:*

$$m\{x \in (0,1) : f_n(x) \in I_n, n = 1, ..., N\} = \Pi_{n=1}^{N} m\{x \in (0,1) : f_n(x) \in I_n\}. \quad (1)$$

An infinite sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ is a *sequence (or system) of independent functions* (S.I.F) if the set $\{f_n(x)\}_{n=1}^{N}$ is a set of independent

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functions for every N=1,2,3,. . . .
If the measure of the set G on which functions $f_n(x)$ are defined is not 1 (but finite and positive), the definition of independence takes the following form:
$\{f_n(x)\}_{n=1}^N$ is a set of independent functions if

$m\{x \in G; f_n(x) \in I_n, n = 1,2,...,N\} = [m(G)]^{-N+1} \prod_{n=1}^N m\{x \in G; f_n(x) \in I_n\}$,

for every interval $I_n, \quad n = 1,2,...,N$.

Theorem 1.2. ([3]) *An S.I.F.* $\{\psi_n(x)\}_{n=1}^{\infty}, x \in (0,1)$, *is an orthonormal system if it satisfies, for* $n = 1,2,...,$ *the conditions*

$$\int_0^1 (\psi_n(x))dx = 0; \quad \int_0^1 \psi_n^2(x)dx = 1.$$

Theorem 1.3. ([3]) *The following inequality holds for every set* $\{\psi_n(x)\}_{n=1}^N$ *of independent functions which satisfy*

$$\|\psi_n\|_2 = 1, \quad \|\psi_n\|_\infty \leqslant M, \quad \int_0^1 \psi_n(x)\ dx = 0; \quad n = 1,2,...,N, \quad (2)$$

$$\lambda(t) = m\{x \in (0,1) : |\sum_{n=1}^N a_n\psi_n(x)| > t(\sum_{n=1}^N a_n^2)^{1/2}\} \leqslant 2e^{-t^2/4M^2},$$

for every $t \geqslant 0$.

Theorem 1.4. ([3,5]) *(Khinchin's inequality). For all numbers* $p > 2$ *and* $M \geqslant 1$ *there exist constants* $C_{p,M}$ *such that, for every polynomial* $f(x) = \sum_{n=1}^N a_n\psi_n(x)$ *in an S.I.F.* $\{\psi_n(x)\}_{n=1}^{\infty}$ *that satisfies (2), the inequality*

$$\|f\|_p \leqslant C_{p,M}\|f\|_2 = C_{p,M}(\sum_{n=1}^N a_n^2)^{1/2},$$

will be satisfied.

Definition 1.5. ([1,3]) *For n=1,2,3,. . . , the nth Rademacher function is defined by*

$$r_n(x) = \begin{cases} 1, & \textit{if } i \textit{ odd and } x \in ((i-1)/2^n, i/2^n) = \Delta_n^i; \\ -1, & \textit{if } i \textit{ even and } x \in ((i-1)/2^n, i/2^n) = \Delta_n^i. \end{cases} \tag{3}$$

In addition, it will be convenient to suppose in what follows that $r_0(x) = 1$ *for* $x \in (0,1)$ *and that* $r_n(i/2^n) = 0$ *for* $i = 0, 1, \ldots, 2^n; \quad n = 0, 1, \ldots.$ *Then we can give a more compact definition of the Rademacher functions by the formula*

$$r_n(x) = sgnsin 2^n \pi x, x \in [0,1], n = 0, 1, \ldots \ . \tag{4}$$

Theorem 1.6. ([3]) *The functions* $\{r_n(x)\}_{n=0}^{\infty}; x \in [0,1]$, *form an S.I.F.*

Theorem 1.7. ([4]) *Let* $(\Omega; \upsilon), \upsilon(\Omega) \leqslant 1$, *be a measurable space, and let* $f \in L^1(\Omega)$ *satisfy the inequality*

$$\|f\|_{L^p(\Omega;\upsilon)} \leqslant c \ log(p+2), \quad p = 1, 2, \cdots, \quad c > 0.$$

Then the following inequality holds

$$\int_\Omega exp(exp(\frac{|f(x)|}{c\lambda_1}))d\upsilon(x) \leqslant \lambda_2.$$

Theorem 1.8. ([3]) *Let* $f \in L^1(0,1)$, *and for* $t \in R^1$ *let*

$$\lambda_f(t) = m\{x \in (0,1) : |f(x)| > t\}; \widetilde{\lambda}_f(t) = m\{x \in (0,1) : f(x) > t\}.$$

Then

$$\int_0^1 f(x)dx = -\int_{-\infty}^{\infty} t d\widetilde{\lambda}_f(t),$$

and if $f \in L^p(0,1), 0 < p < \infty$, *then*

$$\int_0^1 |f(x)|^p dx = -\int_0^{\infty} t^p d\lambda_f(t) = p\int_0^{\infty} t^{p-1}\lambda_f(t)dt.$$

2. Main Results

Theorem 2.1. *For any even number* $p > 2$ *and for every series* $P(x) = \sum_{k=1}^{\infty} \frac{1}{k}\psi_k(x); x \in (0,1)$, *in an S.I.F.* $\{\psi_n(x)\}_{n=1}^{\infty}$ *whose components satisfy the conditions (2), the inequalities*

$$\|P\|_p \leqslant 2M\sqrt{p}(\sum_{k=1}^{\infty}(1/k)^2)^{1/2} and \int_0^1 exp(exp(\lambda_1|P(x)|)dx \leqslant \lambda_2 \ ,$$

will be satisfied, that M, λ_1, λ_2 *are constants.*

Proof. Let $\lambda(t) = m\{x \in (0,1); |P(x)| > t\}$, by Theorem 1.2, $\lambda(t) \leqslant 2exp(-t^2/4M^2)$, therefore according to Theorem 1.6

$$\|P\|_p = \{p\int_0^{\infty} t^{p-1}\lambda(t)dt\}^{1/p} \leqslant \{2p\int_0^{\infty} t^{p-1}exp(-t^2/4M^2)dt\}^{1/p}.$$

In other hand

$$\int_0^{\infty} t^{p-1}exp(-t^2/4M^2)dt = 2^{p-1}M^p\int_0^{\infty} exp(-u)u^{p/2-1}du = 2^{p-1}M^p\Gamma(p/2),$$

then

$$\|P\|_p \leqslant \{2^pM^pp\Gamma(p/2)\}^{1/p} = \{2^pM^pp(p/2-1)!\}^{1/p} = \frac{1}{2}2M(p)^{1/p}(p-2)^{1/p}(p-4)^{1/p}...1 \ ,$$

and it is trivial that

$$\|P\|_p \leqslant M.2.(p^{1/p})^{p/2} = 2M\sqrt{p} \leqslant 2M\sqrt{p}(\sum_{k=1}^{N}(1/k)^2)^{1/2} \leqslant 2M\sqrt{p}(\sum_{k=1}^{\infty}(1/k)^2)^{1/2}.$$

For other inequality can be written that

$$P(x) = \sum_{k=1}^{\infty}\frac{1}{k}\psi_k(x) = \sum_{n=0}^{\infty}\sum_{k=2^n}^{2^{n+1}-1}\frac{1}{k}\psi_k(x) \ ,$$

and let

$$P_n(x) = \frac{1}{2^n}\psi_{2^n}(x) + \cdots + \frac{1}{2^{n+1}-1}\psi_{2^{n+1}-1}(x).$$

It is clear that

$$\|P_n(x)\|_p \leqslant \|P_n(x)\|_\infty \leqslant \frac{M}{2^n}.2^n = M.$$

In other hand

$$\|P_n(x)\|_p \leqslant 2M\sqrt{p}.\sqrt{(\frac{1}{2^n})^2 + \cdots + (\frac{1}{2^{n+1}-1})^2} \leqslant 2M\sqrt{p}.\sqrt{(\frac{1}{2^n})^2.2^n} = \sqrt{\frac{c_1 p}{2^n}} \ ,$$

then $\|P_n(x)\|_p \leqslant \min\{M, \sqrt{\frac{c_1p}{2^n}}\}$ therefore

$$\|P(x)\|_p \leqslant \sum_{n=1}^{\infty} \min\{M, \sqrt{\frac{c_1p}{2^n}}\} = \sum_{n=1}^{\infty}(\frac{M + \sqrt{\frac{c_1p}{2^n}} - |M - \sqrt{\frac{c_1p}{2^n}}|}{2}) \ ,$$

with separate of last summation, it can be obtained

$$\|P_n(x)\|_p \leqslant \sum_{n=1}^{[log_2^{c_1p}]} \frac{M + \sqrt{\frac{c_1p}{2^n}} + M - \sqrt{\frac{c_1p}{2^n}}}{2} + \sum_{n=[log_2^{c_1p}]+1}^{\infty} \frac{M + \sqrt{\frac{c_1p}{2^n}} - M + \sqrt{\frac{c_1p}{2^n}}}{2} \ ,$$

and finally

$$\|P_n(x)\|_p \leqslant Mlog_2^{c_1p} + \sqrt{\frac{2}{c_1p}} \leqslant c_2log(p) < c_2log(p+2).$$

Now by Theorem 1.5 can be written that $\int_0^1 exp(exp(\lambda_1|P(x)|))dx \leqslant \lambda_2$. □

Corollary 2.2. *For series $P(x) = \sum_{k=1}^{\infty} \frac{1}{k} r_k(x)$ the following inequalities will be satisfied*

$$\|P\|_p \leqslant c\sqrt{p}(\sum_{k=1}^{\infty}(1/k)^2)^{1/2} \quad and \quad \int_0^1 exp(exp(\lambda_1|P(x)|))dx \leqslant \lambda_2 \ ,$$

that $\{r_n(t)\}$ is the Rademacher system, $p \geqslant 2$ is an even integer and c, λ_1, λ_2 are constants.
Now consider the system $\{\phi_n(x)\} = \{sin(2^n x\}$ over $[o, 2\pi]$. With a simple change of variable it can be obtained the system $\{\phi_n(t)\} = \{sin(2^{n+1}\pi t\}$ over $[o, 1], (x \to 2\pi t)$.

Theorem 2.3. *If the series $\Sigma a_n^2 < \infty$, the function*

$$f(t) = \sum_{k=0}^{\infty} a_k sin(2^{k+1}\pi t), t \in [0,1] \quad , \tag{5}$$

or equivalently

$$g(t) = \sum_{k=0}^{\infty} a_k sin(2^k t), t \in [0,2\pi] \quad , \tag{6}$$

belongs to L^q for every $q > 0$.

Proof. It is sufficient to prove the theorem for $q = 2, 4, 6, \cdots$. We shall show that

$$\int_0^1 f^{2k}(t)dt \leqslant M_k(\sum_{n=0}^{\infty} a_n^2)^k; k = 1, 2, 3, \cdots \quad , \tag{7}$$

or equivalently

$$\int_0^{2\pi} g^{2k}(t)dt \leqslant M_k(\sum_{n=0}^{\infty} a_n^2)^k; k = 1, 2, 3, \cdots \quad , \tag{8}$$

where M_k is a constant depending only on k.
Denoting by $S_n(t)$ and $S_n^*(t)$, the partial sums of the series (5) and the partial sums of the series (6) respectively.
So

$$\int_0^1 S_n^{2k}(t)dt = \sum A_{\alpha_1,\alpha_2,\cdots,\alpha_r} a_{m1}^{\alpha_1} \cdots a_{m_r}^{\alpha_r} \int_0^1 sin^{\alpha_1}(2^{m_1+1}\pi t) \cdots sin^{\alpha_r}(2^{m_r+1}\pi t)dt \ ,$$

or equivalently

$$\int_0^{2\pi} (S_n^*)^{2k}(t)dt = \sum A_{\alpha_1,\alpha_2,\cdots,\alpha_r} a_{m1}^{\alpha_1} \cdots a_{m_r}^{\alpha_r} \int_0^{2\pi} sin^{\alpha_1}(2^{m_1}t) \cdots sin^{\alpha_r}(2^{m_r}t)dt \ ,$$

where

$$A_{\alpha_1,\alpha_2,\cdots,\alpha_r} = \frac{(\alpha_1 + \alpha 2 + \cdots + \alpha_r)!}{\alpha_1!\alpha_2! \cdots \alpha_r!}$$

and the summations on the right are taken over the set

$$\{m_1, m_2, \cdots m_r, \alpha_1, \alpha_2, \cdots, \alpha_r\} \quad ,$$

defined by the relations:

$0 \leqslant m_i \leqslant n, 0 \leqslant \alpha_i \leqslant 2k; i = 1, 2, \cdots, r; 1 \leqslant r \leqslant 2k; \alpha_1 + \alpha_2 + \cdots + \alpha_r = 2k.$

Now it is easily verified that the integrals on the right vanish unless $\alpha_1, \alpha_2, \cdots, \alpha_r$ are all even, that in this case they are less than or equal 1. Thus the right side of above relations are less than or equal of the following term

$$\sum A_{2\beta_1,\cdots,2\beta_r} a_{n1}^{2\beta_1} \cdots a_{n_r}^{2\beta_r}.$$

Observing that

$$\sum A_{\beta_1,\beta_2,\cdots,\beta_r} a_{m1}^{2\beta_1} a_{m2}^{2\beta_2} \cdots a_{m_r}^{2\beta_r} = (a_0^2 + a_1^2 + \cdots + a_n^2)^k.$$

It can be obtained (7) and (8) with $S_n(t)$ and $S_n^*(t)$ replaced by $f(t)$ and $g(t)$ respectively, M_k being now the upper bound of the ratio

$$A_{2\beta_1,\cdots,2\beta_r} / A_{\beta_1,\beta_2,\cdots,\beta_r}.$$

Notice

$$M_k \leqslant (2k)!/2^k k! = (k+1) \cdots 2k/2^k \leqslant k^k \tag{9}$$

Since $S_n(t) \to f(t)$ and $S_n^*(t) \to g(t)$ for almost every t, finally with use Fatous lemma the proof is complete. □

Corollary 2.4. *The function $exp(\mu f^2(t))$ is integrable for $\mu > 0$.*

Corollary 2.5. *For any even number $p \geqslant 2$ and series*

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} sin 2^n x; \quad x \in (0, 2\pi),$$

the inequalities

$$\|f\|_p \leqslant C\sqrt{p}(\sum_{n=1}^{\infty} (1/n)^2)^{1/2} \quad and \quad \int_0^{2\pi} exp(exp(\lambda_1 |f(x)|))dx \leqslant \lambda_2 \quad ,$$

will be satisfied, that C, λ_1, λ_2 are constants.

Proof. According to (9) the following inequality is satisfied:

$$\|f\|_p \leqslant \{M_{\frac{p}{2}}(\sum_{n=1}^{\infty}(\frac{1}{n})^2)^{\frac{p}{2}}\}^{1/p} \leqslant \{(\frac{p}{2})^{\frac{p}{2}}(\sum_{n=1}^{\infty}(\frac{1}{n})^2)^{\frac{p}{2}}\}^{1/p} = C\sqrt{p}(\sum_{n=1}^{\infty}(\frac{1}{n})^2)^{\frac{1}{2}}.$$

For other inequality it can be written $f(x) = \sum_{n=0}^{\infty} f_n(x)$ such that $f_n(x) = \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k} sin2^k x$ therefore

$$\|f_n(x)\|_p \leqslant \|f_n(x)\|_\infty \leqslant \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k}|sin2^k x| \leqslant \frac{2^n}{2^n} = 1 \quad .$$

In other hand

$$\|f_n(x)\|_p \leqslant C\sqrt{p}\sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k^2} \leqslant C\sqrt{p}\frac{2^n}{(2^n)^2} = \frac{C\sqrt{p}}{2^n} \quad ,$$

then $\|f_n(x)\|_p \leqslant \min\{1, \frac{C\sqrt{p}}{2^n}\}$ therefore

$$\|f(x)\|_p \leqslant \sum_{n=0}^{\infty} \min\{1, \frac{C\sqrt{p}}{2^n}\} = \sum_{n=0}^{\infty}(\frac{1 + \frac{C\sqrt{p}}{2^n} - |1 - \frac{C\sqrt{p}}{2^n}|}{2}).$$

With separate of last summation, it can be obtained

$$\|f(x)\|_p \leqslant \sum_{n=0}^{[log_2^{C\sqrt{p}}]} \frac{1 + \frac{C\sqrt{p}}{2^n} + 1 - \frac{C\sqrt{p}}{2^n}}{2} + \sum_{n=[log_2^{C\sqrt{p}}]+1}^{\infty} \frac{1 + \frac{C\sqrt{p}}{2^n} - 1 + \frac{C\sqrt{p}}{2^n}}{2} \quad ,$$

and finally

$$\|f(x)\|_p \leqslant log_2^{C\sqrt{p}} + 1 + C\sqrt{p}(\frac{1}{2C\sqrt{p}}) = log_2^{C\sqrt{p}} + \frac{3}{2} \leqslant \alpha log(p+2) \quad ,$$

that α is a constant. Now by Theorem 1.5 can be written

$$\int_0^{2\pi} exp(exp(\lambda_1|f(x)|))dx \leqslant \lambda_2. \quad \square$$

References

[1] G. A. Karagulyan, On the order of growth $o(loglogn)$ of the partial sums of Fourier-Stieltjes series of random measures, in: *Russian Acad. Sci. sb. Math., 78 (1994),* 11-33.

[2] B. S. Kashin and A. A. Saakyan, *Orthogonal Series*, Transl. Mth. Monographs, 75, 2005.

[3] E. M. Stein, Harmonic Analysis: *Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, New Jersey, 1993.

[4] F. Sukochev and D. Zanin, Khinchin inequality and Banach-Saks type properties in rearrangement-invariant spaces, *Studia Math.,* 191 (2009), 101-122.

[5] A. zygmund, *Triginimetrical series, .3rd. Ed., Vol 2.* New York, Cambridge university press, 2002.

Mojtaba Ghanbari
Department of Mathematics
Assistant Professor of Mathematics
Farahan Branch, Islamic Azad University
Farahan, Iran
E-mail: m.ghanbari@iau-farahan.ac.ir