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Original Research Paper

A New Approach to the Study of Solutions for a Fractional Boundary Value Problem in Hölder Spaces

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Abstract. In this paper, we study the existence of nontrivial solutions for a fractional boundary value problem in Hölder spaces by a technical approach based on Leray-Schauder nonlinear alternative. Moreover, using the concept of orthogonal set on Banach fixed point theorem we obtain another existence result with weaker conditions. Also, recent results are extended and improved. In addition, we give some examples to illustrate the feasibility and effectiveness of our results.

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1 Introduction

Fractional differential equations(FDEs) are generalization of ordinary differential equations and integration to arbitrary non-integer orders. FDEs have gained importance due to their numerous applications in many fields of science and engineering. Indeed, there are a large number of phenomena including fluid flow, diffusive transport akin to diffusion, rheology, probability, electrical networks, etc, that are modeled by different equations involving fractional order derivatives, see for details [11, 14, 15, 16, 17, 19, 20, 23, 25, 26, 27, 28] and references therein. In recent decades, many researchers proved the existence and multiplicity of solution of nonlinear initial fractional differential equations by the use of some fixed point theorems, one can see [4, 5, 6, 8, 9, 13, 24]. Among them, Guo [13] with Leray-Schauder nonlinear alternative has given some sufficient conditions for the existence of nontrivial solutions to the following nonlinear fractional differential equation boundary value problem

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t)) + g(t) = 0, \qquad 0 < t < 1, \\ u(0) = 0, \quad u(1) = \beta u(\eta), \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \eta < 1$ and $\beta \in \mathbb{R}$, $\beta\eta^{\alpha-1} \neq 1$, and D^{α}_{0+} is the Riemann-Liouville differential operator of order α, $f[0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and $g(t) : [0,1] \longrightarrow [0, +\infty)$ is Lebesgue integrable. Especially, f does not have the nonnegative assumption and monotonicity which was essential for the technique used in almost all existed literature. Recently, Cabrera et al. [9] have studied the existence and uniqueness of solution for the following boundary value problem of fractional type with nonlocal integral boundary conditions

$$\begin{cases} {}^{c}D^{\alpha}_{0+}u(t) = \lambda f(t, u(t)), \qquad t \in [0,1], \\ u(0) = \gamma I^{\rho}_{0+}u(\eta) = \gamma \displaystyle\int_0^{\eta} \frac{(\eta - s)^{\rho-1}}{\Gamma(\rho)} u(s)ds, \end{cases} \tag{1}$$

where ${}^{c}D^{\alpha}_{0+}$ denotes the Caputo fractional derivative and $0 < \alpha \leq 1$, $0 < \eta < 1$ and $\gamma, \rho, \lambda \in \mathbb{R}$, as follows.

Theorem 1.1. *[9] Suppose that $0 < \alpha \leq 1$, $0 < \eta < 1$, $\lambda, \rho > 0$, $\gamma \in \mathbb{R}$ and $\gamma \neq \dfrac{\Gamma(\rho+1)}{\eta^\rho}$. Let $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that*

$$|f(t,x) - f(t,y)| \leq \phi(|x-y|)$$

for any $t \in [0,1]$ and $x, y \in \mathbb{R}$, where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is nondecreasing and $\phi(0) = 0$. Set

$$L = \left|\frac{\gamma\Gamma(\rho+1)}{\gamma\eta^\rho - \Gamma(\rho+1)}\right| \frac{1}{(\rho+\alpha)\Gamma(\rho+\alpha)}\eta^{\rho+\alpha} + \frac{2}{\alpha\Gamma(\alpha)}.$$

Under assumption that $\lambda \leq \dfrac{1}{L}$, Problem (1) has a unique solution in the space $\mathcal{H}_\alpha[0,1]$.

In present paper, in Section 2, as motivated by the work of Guo [13], we obtain sufficient conditions of the existence of nontrivial solutions for the following boundary value problem

$$\begin{cases} {}^cD^\alpha_{0+}u(t) + f(t,u(t)) + g(t) = 0, \qquad t \in [0,1], \\ u(0) = \gamma I^\rho_{0+}u(\eta) = \gamma \displaystyle\int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)} u(s)ds, \end{cases} \tag{2}$$

where $0 < \alpha \leq 1$, $0 < \eta < 1$ and $\gamma, \rho \in \mathbb{R}$, and $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g : [0,1] \longrightarrow [0,+\infty)$ are continuous. Also, some remarks and examples to support the results proved herein and to compare to the main results of Cabrera et al. [9]. Finally, in Section 3, we recall the definition of orthogonal set introduced in [1, 2, 12], and obtain another result for the existence of solutions of Problem (2) with weaker conditions. Some recent results are extended and improved.

2 Preliminaries

At first, we recall some important definitions, lemmas and propositions.

Definition 2.1. [23] For at least n-times continuously differentiable function $h : [0,\infty) \longrightarrow \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q_{0+}h(t) = \frac{1}{d\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} h^{(n)}(s)ds,$$

where $n = [q] + 1$ and $[q]$ denotes the integer part of q. Here, $\Gamma(\alpha)$ denotes the classical Gamma function.

Definition 2.2. [23] The Riemann-Liouville fractional integral of order q of a function $h : (0, \infty) \longrightarrow \mathbb{R}$ is defined by

$$I_{0+}^{q} h(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{h(s)}{(t-s)^{1-q}} ds,$$

provides that the right side is point wise defined on $(0, \infty)$.

Lemma 2.3. *[19] Suppose that $p, q \geq 0$ and $h \in L^1[0,1]$. Then*

$$I_{0+}^{p} I_{0+}^{q} h(t) = I_{0+}^{p+q} h(t) \quad \textit{and} \quad {}^{c}D_{0+}^{1} I_{0+}^{q} h(t) = h(t)$$

for any $t \in [0,1]$.

Lemma 2.4. *[19] Suppose that $\beta > \alpha > 0$ and $h \in L^1[0,1]$. Then*

$${}^{c}D_{0+}^{\alpha} I_{0+}^{\beta} h(t) = I_{0+}^{\beta-\alpha} h(t) \quad \textit{for any} \quad t \in [0,1].$$

Lemma 2.5. *[22] Suppose that $\gamma \neq \dfrac{\Gamma(\rho+1)}{\eta^{\rho}}$ and $\rho > 0$. Then for a function $h \in C[0,1]$, the solution of the fractional differential equation*

$${}^{c}D_{0+}^{\alpha} = h(t), \quad t \in [0,1],$$

with $0 < \alpha \leq 1$ and under the boundary condition

$$x(0) = \gamma I_{0+}^{\rho} x(\eta) = \gamma \int_0^{\eta} \frac{(\eta - s)^{\rho-1}}{\Gamma(\rho)} x(s) ds,$$

where $0 < \eta < 1$, is given by

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + \frac{\gamma \Gamma(\rho+1)}{\Gamma(\rho+1) - \gamma \eta^{\rho}} \int_0^{\eta} \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} h(s) ds.$$

Let $C[a,b]$ be the space of the continuous functions on closed interval $[a,b]$ with the sup-norm, i.e.,

$$\|x\|_{\infty} = \sup\{|x(t)| : t \in [a,b]\}.$$

Set $0 < \alpha \leq 1$ fixed, denoted by $\mathcal{H}_\alpha[a,b]$ the space of the real functions x defined on $[a,b]$ and satisfied in the Hölder condition, that is, all functions x so that there exists a constant $\mathcal{H}_x^\alpha$ such that

$$|x(t) - x(p)| \leq \mathcal{H}_x^\alpha |t-p|^\alpha \tag{3}$$

for any $t, p \in [a,b]$. Also, we define the least possible constant for inequality (3) is satisfied for $x \in \mathcal{H}_\alpha[a,b]$ by $\mathcal{H}_x^\alpha$ as follows:

$$\mathcal{H}_x^\alpha = \sup\left\{ \frac{|x(t)-x(p)|}{|t-p|^\alpha} : t, p \in [a,b], t \neq p \right\}.$$

In [7], the authors proved the spaces $\mathcal{H}_\alpha[a,b]$ with $0 < \alpha \leq 1$ endowed with the following norm

$$\|x\| = |x(a)| + \sup\left\{ \frac{|x(t)-x(p)|}{|t-p|^\alpha} : t, p \in [a,b], t \neq p \right\} = |x(a)| + \mathcal{H}_x^\alpha,$$

is a Banach space.

We need the following proposition to prove our main results.

Proposition 2.6. *Assume that* $0 < \alpha \leq 1$, $0 < \eta < 1$, $\rho > 0$, $\gamma \in \mathbb{R}$ *and* $\gamma \neq \dfrac{\Gamma(\rho+1)}{\eta^\rho}$. *Suppose that* $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ *and* $g : [0,1] \longrightarrow [0, +\infty)$ *are continuous functions and* $u \in \mathcal{H}^\alpha[0,1]$. *Let* T *be the function defined by*

$$\begin{aligned}(Tu)(t) = {} & \frac{\gamma\Gamma(\rho+1)}{\gamma\eta^\rho - \Gamma(\rho+1)} \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} [f(s,u(s)) + g(s)]ds \\ & - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s,u(s)) + g(s)]ds\end{aligned}$$

for any $t \in [0,1]$. *The operator* $T : \mathcal{H}^\alpha[0,1] \longrightarrow \mathcal{H}^\alpha[0,1]$ *is completely continuous.*

Proof. Let $\mathcal{M}$ be any bounded subset of $\mathcal{H}^\alpha[0,1]$. As f and g are continuous functions and $u \in \mathcal{H}^\alpha[0,1] \subset C[0,1]$, there exists $M = \sup\{|f(s,u) + q(s)| : s \in [0,1]; u \in [-\|u\|_\infty, \|u\|_\infty]\}$. By the same argument as given in Proposition 1 in [9], we have $Tu \in \mathcal{H}^\alpha[0,1]$ for all

$u \in \mathcal{H}^\alpha[0,1]$. In the following, without loss of generality, we can suppose that $t > p$. For any $u \in \mathcal{H}^\alpha[0,1]$,

$$\begin{aligned}
\|Tu\|_\alpha &= |Tu(0)| + \sup\left\{\frac{|Tu(t)-Tu(p)|}{|t-p|^\alpha} : t,p \in [a,b]\right\}\\
&\leq \left|\frac{\gamma\Gamma(\rho+1)}{\gamma\eta^\rho - \Gamma(\rho+1)}\right| \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}|f(s,u(s))+q(s)|ds\\
&+ \sup\left\{\frac{M(\int_0^p[(p-s)^{\alpha-1}-(t-s)^{\alpha-1}]ds + \int_p^t(t-s)^{\alpha-1}ds)}{\Gamma(\alpha)|t-p|^\alpha} :\right.\\
&\left. t,p \in [a,b]\right\}\\
&\leq \frac{M\Gamma(\alpha+1)}{\Gamma(\rho+\alpha)}\left|\frac{\gamma}{\gamma\eta^\rho - \Gamma(\rho+1)}\right| \int_0^\eta (\eta-s)^{\rho+\alpha-1}ds\\
&+ \sup\left\{\frac{M\left[\frac{(t-p)^\alpha}{\alpha} + \frac{p^\alpha}{\alpha} - \frac{t^\alpha}{\alpha} + \frac{(t-p)^\alpha}{\alpha}\right]}{\Gamma(\alpha)|t-p|^\alpha} : t,p \in [a,b]\right\}\\
&\leq \frac{M\Gamma(\alpha+1)}{\Gamma(\rho+\alpha)}\left|\frac{\gamma}{\gamma\eta^\rho - \Gamma(\rho+1)}\right| \frac{\eta^{\rho+\alpha}}{\rho+\alpha} + \frac{2M}{\Gamma(\alpha)}\frac{(t-p)^\alpha}{\alpha(t-p)^\alpha}\\
&\leq \frac{M\Gamma(\alpha+1)\eta^{\rho+\alpha}}{\Gamma(\rho+\alpha+1)}\left|\frac{\gamma}{\gamma\eta^\rho - \Gamma(\rho+1)}\right| + \frac{2M}{\Gamma(\alpha+\alpha)} < \infty,
\end{aligned}$$

and so all the functions in $T(\mathcal{M})$ are uniformly bounded. Also, we show that all the functions in $T(\mathcal{M})$ are equicontinuous. Let $h(t) = (t-s)^{\alpha-1}$, then $h(t)$ is continuously differentiable function. For any $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, there exist positive constants M_1 such that

$$|h(t_2) - h(t_1)| = |h(\mu)(t_2-t_1)| \leq M_1|t_2-t_1|, \ \ \mu \in [t_1,t_2].$$

Set $\vartheta = \dfrac{M(1+M_1)}{\Gamma(\alpha)}$. For any $u \in \mathcal{H}^\alpha[0,1]$, $\forall \epsilon > 0$, there exists $\delta = \dfrac{\epsilon}{\vartheta}$, such that when $|t_2 - t_1| < \delta$, we obtain

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| = &\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}[f(s,u(s))+g(s)]ds\right.\\
&\left. - \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}[f(s,u(s))+g(s)]ds\right|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \Big| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} [f(s, u(s)) + g(s)] ds \\
&+ \int_0^{t_1} (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} [f(s, u(s)) + g(s)] ds \\
&< \frac{1}{\Gamma(\alpha)} \left(\int_{t_1}^{t_2} M ds + \int_0^1 M M_1 |t_2 - t_1| \right) \\
&= \frac{M(1 + M_1)}{\Gamma(\alpha)} |t_2 - t_1| = \vartheta |t_2 - t_1| < \epsilon.
\end{aligned}
$$

An application of the Arzela-Ascoli theorem shows that T is completely continuous and the proof is complete. □

The following result is one of the pivotal results in fixed point theory which we use later.

Lemma 2.7. *[10] Let X be a real Banach space, Ω be a bounded open subset of X, $0 \in \Omega$, $T : \overline{\Omega} \longrightarrow X$ be a completely continuous operator. Then either there exists $x \in \partial\Omega$, $\mu > 1$ such that $T(x) = \mu x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

3 Main results

We formulate our main results as follows.

Theorem 3.1. *Suppose that $f(t, 0) \neq 0$ for all $t \in [0, 1]$, $\gamma \neq \dfrac{\Gamma(\rho + 1)}{\eta^\rho}$, and there exist nonnegative and continuous functions p, r defined on $[0, 1]$ such that*

$$
\begin{cases}
|f(t, u(t))| \leq p(t)|u(t)| + r(t), \quad u \in \mathcal{H}^\alpha[0, 1], t \in (0, 1), \textit{almost every where}, \\
\left| \dfrac{\gamma \Gamma(p + 1)}{\gamma \eta^\rho - \Gamma(\rho + 1)} \right| \displaystyle\int_0^\eta \frac{(\eta - s)^{\rho + \alpha - 1}}{\Gamma(\rho + \alpha)} p(s) ds + \\
\sup \left\{ \dfrac{\displaystyle\int_0^{t'} [(t' - s)^{\alpha-1} - (t - s)^{\alpha-1}] p(s) ds + \int_{t'}^t (t - s)^{\alpha-1} p(s) ds}{\Gamma(\alpha) |t - t'|^\alpha} \right\} < 1.
\end{cases}
$$

Then Problem (2) has at least one nontrivial solution $u^* \in \mathcal{H}^\alpha[0, 1]$.

Proof. Arguing as in [13], let

$$\begin{aligned} A &= \left|\frac{\gamma\Gamma(\rho+1)}{\gamma\eta^{\rho}-\Gamma(\rho+1)}\right| \int_0^{\eta} \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} p(s)ds \\ &+ \sup\left\{ \frac{\int_0^{t'} [(t'-s)^{\alpha-1}-(t-s)^{\alpha-1}]p(s)ds + \int_{t'}^{t} (t-s)^{\alpha-1}p(s)ds}{\Gamma(\alpha)|t-t'|^{\alpha}} : \right. \\ & \left. t, t' \in [a,b],\ t \neq t' \right\} \end{aligned}$$

and

$$\begin{aligned} B &= \left|\frac{\gamma\Gamma(\rho+1)}{\gamma\eta^{\rho}-\Gamma(\rho+1)}\right| \int_0^{\eta} \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} k(s)ds \\ &+ \sup\left\{ \frac{\int_0^{t'} [(t'-s)^{\alpha-1}-(t-s)^{\alpha-1}]k(s)ds + \int_{t'}^{t} (t-s)^{\alpha-1}k(s)ds}{\Gamma(\alpha)|t-t'|^{\alpha}} : \right. \\ & \left. t, t' \in [a,b],\ t \neq t' \right\}, \end{aligned}$$

where $k(s) = r(s)+g(s)$. From our assumptions, we observe that $A < 1$. Since $f(t,0) \neq 0$ for all $t \in [0,1]$, there exists $[a,b] \subset [0,1]$ such that

$$\min_{a \le t \le b} |f(t,0)| > 0.$$

Taking into account that $r(t) \geq |f(t,0)|$, almost every where $t \in [0,1]$, we see that $B > 0$. Set $m = B(1-A)^{-1}$, $\Omega_m = \{u \in \mathcal{H}_\alpha[0,1] : \|u\| < m\}$. Lemma 2.5 follows that Problem (2) has a solution $u = u(t)$ if and only if u solves the operator equation

$$\begin{aligned} (Tu)(t) = \frac{\gamma\Gamma(\rho+1)}{\gamma\eta^{\rho}-\Gamma(\rho+1)} \int_0^{\eta} \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} [f(s,u(s))+g(s)]ds \\ - \frac{1}{\Gamma(\alpha)} \int_0^{t} (t-s)^{\alpha-1}[f(s,u(s))+g(s)]ds \end{aligned}$$

in $\mathcal{H}_\alpha[0,1]$. Therefore, we only need to find a fixed point of T in $\mathcal{H}_\alpha[0,1]$. Applying Proposition 2.6, we observe that the operator $T : \mathcal{H}_\alpha[0,1] \longrightarrow \mathcal{H}_\alpha[0,1]$ is completely continuous. Without loss of generality, we can suppose $t > t^{'}$ and $u \in \partial\Omega_m$, $\mu > 1$ satisfying $Tu = \mu u$. Then

$$\begin{aligned}
\mu m &= \mu\|u\| \\
&= \|Tu\| = |Tu(0)| + \sup\Big\{\frac{|Tu(t) - Tu(t^{'})|}{|t-t'|^{\alpha}} : \\
&\quad t, t^{'} \in [a,b], t \neq t^{'} Big\} \\
&\leqslant \left|\frac{\gamma\Gamma(\rho+1)}{\gamma\eta^{\rho} - \Gamma(\rho+1)}\right| \int_0^{\eta} \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} |f(s,u(s)) + g(s)| ds \\
&+ \sup\Bigg\{ \frac{\Big|\int_0^t (t-s)^{\alpha-1}[f(s,u(s)) + g(s)]ds}{\Gamma(\alpha)|t-t'|^{\alpha}} \\
&- \frac{\int_0^{t^{'}} (t^{'}-s)^{\alpha-1}[f(s,u(s)) + g(s)]ds\Big|}{\Gamma(\alpha)|t-t'|^{\alpha}} \Bigg\} \\
&\leqslant \left|\frac{\gamma\Gamma(\rho+1)}{\gamma\eta^{\rho} - \Gamma(\rho+1)}\right| \int_0^{\eta} \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} |f(s,u(s)) + g(s)| ds \\
&+ \sup\Bigg\{ \frac{\displaystyle\int_0^{t^{'}} |(t-s)^{\alpha-1} - (t^{'}-s)^{\alpha-1}||f(s,u(s)) + g(s)|ds}{\Gamma(\alpha)|t-t'|^{\alpha}} \\
&+ \frac{\displaystyle\int_{t^{'}}^{t} (t-s)^{\alpha-1}|f(s,u(s)) + g(s)|ds}{\Gamma(\alpha)|t-t'|^{\alpha}} \Bigg\} \\
&\leqslant \left|\frac{\gamma\Gamma(\rho+1)}{\gamma\eta^{\rho} - \Gamma(\rho+1)}\right| \int_0^{\eta} \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} p(s)ds\|u\| \\
&+ \left|\frac{\gamma\Gamma(\rho+1)}{\gamma\eta^{\rho} - \Gamma(\rho+1)}\right| \int_0^{\eta} \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} [r(s) + g(s)] ds \\
&+ \sup\left\{ \frac{\displaystyle\int_0^{t^{'}} [(t-s)^{\alpha-1} - (t^{'}-s)^{\alpha-1}]p(s)ds + \int_{t^{'}}^{t} (t-s)^{\alpha-1}p(s)ds}{\Gamma(\alpha)|t-t'|^{\alpha}} \right\} \|u\| \\
&+ \sup\Bigg\{ \frac{\displaystyle\int_0^{t^{'}} [(t-s)^{\alpha-1} - (t^{'}-s)^{\alpha-1}]g(s)ds}{\Gamma(\alpha)|t-t'|^{\alpha}} \\
&+ \frac{\displaystyle\int_{t^{'}}^{t} (t-s)^{\alpha-1}(r(s) + g(s))ds}{\Gamma(\alpha)|t-t'|^{\alpha}} \Bigg\} \\
&\leq A\|u\| + B = Am + B.
\end{aligned}$$

Then

$$\mu \leq A + \frac{B}{m} = A + \frac{B}{B(1-A)^{-1}} = A + (1-A) = 1.$$

This contradicts $\mu > 1$. Hence, Lemma 2.7 follows that T has a fixed point $u^* \in \overline{\Omega}$. Since $f(t,0) \neq 0$ for all $t \in [0,1]$, Problem (2) has a nontrivial solution $u^* \in \mathcal{H}_\alpha[0,1]$. This completes the proof. $\square$

In the sequel, we present some theorems with some sufficient conditions for the existence of a nontrivial solution to Problem (2).

Theorem 3.2. *Suppose that $f(t,0) \neq 0$ for all $t \in [0,1]$, $t \in [0,1]$, $\gamma < \dfrac{\Gamma(\rho+1)}{\eta^\rho}$, and there exist nonnegative and continuous functions p, r defined on $[0,1]$ such that*

$$|f(t,u(t))| \leq p(t)|u(t)| + r(t), \qquad u \in \mathcal{H}^\alpha[0,1], t \in [0,1], \textit{almost every where},$$

and one of the following conditions holds:

(a_1) *The function $p(s)$ satisfies in*

$$\begin{cases} p(s) \leq \dfrac{|\gamma|\alpha\Gamma(\rho)\Gamma(\rho+1)\eta^{\rho+\alpha} + 2(\Gamma(\rho+1)-\gamma\eta^\rho)\Gamma(\alpha+\rho+1)}{(\alpha\rho+\alpha^2)\Gamma(\alpha)\Gamma(\alpha+\rho)(\Gamma(\rho+1)-\gamma\eta^\rho)}, \\ \qquad s \in [0,1], \textit{almost everywhere}, \\ mes\Bigg\{ s \in [0,1] : p(s) < \\ \dfrac{|\gamma|\alpha\Gamma(\rho)\Gamma(\rho+1)\eta^{\rho+\alpha} + 2(\Gamma(\rho+1)-\gamma\eta^\rho)(\alpha+\rho)\Gamma(\alpha+\rho)}{(\alpha\rho+\alpha^2)\Gamma(\alpha)\Gamma(\alpha+\rho)(\Gamma(\rho+1)-\gamma\eta^\rho)} \Bigg\} > 0; \end{cases}$$

(a_2) *There exist constants $k, \lambda > 1$ with $\dfrac{1}{\lambda} + \dfrac{1}{k} = 1$ such that if we set*

$$Z = \left[\frac{|\gamma|\Gamma(\rho+1)\eta^{\rho+\alpha-(\frac{k-1}{k})}}{(\Gamma(\rho+1)-\gamma\eta^\rho)(k\alpha+k\rho-k-1)^{\frac{1}{k}}\Gamma(\rho+\alpha)} \right.$$
$$\left. + \frac{1}{\Gamma(\alpha)} \sup \left\{ \frac{|(t')^{\alpha-(\frac{k-1}{k})} + 2(t-t')^{\alpha-(\frac{k-1}{k})} - t^{\alpha-(\frac{k-1}{k})}|}{(k\alpha-k+1)^{\frac{1}{k}}(t-t')^\alpha} \right\} \right],$$

we have $\displaystyle\int_0^1 p(s)^\lambda ds < \frac{1}{Z^\lambda}$.

Then Problem (2) has at least one nontrivial solution $u^* \in \mathcal{H}^\alpha[0,1]$.

Proof. We only need to show that $A < 1$, where A is defined in Theorem 3.1. We consider the following cases:

(a_1) Here, taking the condition $\gamma < \dfrac{\Gamma(\rho+1)}{\eta^n}$ into account, we obtain

$$
\begin{aligned}
A &= \left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^\rho}\right| \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} p(s)ds \\
&\quad + \sup\left\{ \frac{\int_0^{t'} [(t'-s)^{\alpha-1} - (t-s)^{\alpha-1}]p(s)ds + \int_{t'}^t (t-s)^{\alpha-1}p(s)ds}{\Gamma(\alpha)|t-t'|^\alpha} : \right. \\
&\quad \left. t,t' \in [a,b],\ t \neq t' \right\} \\
&\leq \left[\frac{|\gamma|\alpha\Gamma(\rho)\Gamma(\rho+1)\eta^{\rho+\alpha} + 2(\Gamma(\rho+1)-\gamma\eta^\rho)\Gamma(\alpha+\rho+1)}{(\alpha\rho+\alpha^2)\Gamma(\alpha)\Gamma(\alpha+\rho)(\Gamma(\rho+1)-\gamma\eta^\rho)}\right] \\
&\quad \times \left(\frac{|\gamma|\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^\rho} \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} ds \right. \\
&\quad + \sup\left\{ \frac{\int_0^{t'} [(t'-s)^{\alpha-1} - (t-s)^{\alpha-1}]ds + \int_{t'}^t (t-s)^{\alpha-1}ds}{\Gamma(\alpha)|t-t'|^\alpha} : \right. \\
&\quad \left.\left. t,t' \in [a,b],\ t \neq t' \right\}\right) \\
&\leq \left[\frac{|\gamma|\alpha\Gamma(\rho)\Gamma(\rho+1)\eta^{\rho+\alpha} + 2(\Gamma(\rho+1)-\gamma\eta^\rho)\Gamma(\alpha+\rho+1)}{(\alpha\rho+\alpha^2)\Gamma(\alpha)\Gamma(\alpha+\rho)(\Gamma(\rho+1)-\gamma\eta^\rho)}\right] \\
&\quad \times \left(\frac{|\gamma|\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^\rho} \frac{\eta^{\rho+\alpha}}{(\rho+\alpha)\Gamma(\rho+\alpha)} \right. \\
&\quad \left. + \sup\left\{ \frac{\left[\frac{(t-t')^\alpha}{\alpha} + \frac{t'^\alpha}{\alpha} - \frac{t^\alpha}{\alpha} + \frac{(t-t')^\alpha}{\alpha}\right]}{|t-t'|^\alpha} :\ t,t' \in [a,b],\ t \neq t' \right\} \right) \\
&\leq \left[\frac{|\gamma|\alpha\Gamma(\rho)\Gamma(\rho+1)\eta^{\rho+\alpha} + 2(\Gamma(\rho+1)-\gamma\eta^\rho)\Gamma(\alpha+\rho+1)}{(\alpha\rho+\alpha^2)\Gamma(\alpha)\Gamma(\alpha+\rho)(\Gamma(\rho+1)-\gamma\eta^\rho)}\right] \\
&\quad \times \left(\frac{|\gamma|\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^\rho} \frac{\eta^{\rho+\alpha}}{\Gamma(\rho+\alpha+1)} + \frac{2}{\alpha\Gamma(\alpha)} \right) \\
&= 1.
\end{aligned}
$$

(a_2) Using the Hölder inequality and bearing the condition $\gamma < \dfrac{\Gamma(\rho+1)}{\eta^n}$ in mind, we obtain

$$
\begin{aligned}
A &= \left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^\rho}\right| \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} p(s)ds \\
&+ \sup\left\{ \frac{\int_0^{t'} [(t'-s)^{\alpha-1}-(t-s)^{\alpha-1}]p(s)ds + \int_{t'}^t (t-s)^{\alpha-1}p(s)ds}{\Gamma(\alpha)|t-t'|^\alpha} : \right. \\
&\left. t, t' \in [a,b],\, t \neq t' \right\} \\
&\leq \frac{|\gamma|\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^\rho} \left[\int_0^\eta \left(\frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\right)^k ds\right]^{\frac{1}{k}} \left[\int_0^\eta p(s)^\lambda ds\right]^{\frac{1}{\lambda}} \\
&+ \sup\left\{ \frac{[\int_0^{t'} [(t'-s)^{\alpha-1}]^k ds]^{\frac{1}{k}} - [\int_0^{t'} [(t-s)^{\alpha-1}]^k ds]^{\frac{1}{k}}}{\Gamma(\alpha)|t-t'|^\alpha} [\int_0^{t'} p(s)^\lambda ds]^{\frac{1}{\lambda}} \right. \\
&+ \left. \frac{[\int_{t'}^t ((t-s)^{\alpha-1})^k ds]^{\frac{1}{k}} [\int_{t'}^t p(s)^\lambda ds]^{\frac{1}{\lambda}}}{\Gamma(\alpha)|t-t'|^\alpha} \right\} \\
&\leq \frac{|\gamma|\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^\rho} \frac{\eta^{\rho+\alpha-(\frac{k-1}{k})}}{(k\alpha+k\rho-k-1)^{\frac{1}{k}}\Gamma(\rho+\alpha)} \left[\int_0^\eta p(s)^\lambda ds\right]^{\frac{1}{\lambda}} \\
&+ \frac{1}{\Gamma(\alpha)} \sup\left\{ [\frac{t'^{\alpha-(\frac{k-1}{k})} + (t-t')^{\alpha-(\frac{k-1}{k})} - t^{\alpha-(\frac{k-1}{k})}}{(k\alpha-k+1)^{\frac{1}{k}}(t-t')^\alpha} [\int_0^{t'} p(s)^\lambda ds]^{\frac{1}{\lambda}}] \right. \\
&+ \left. [\frac{(t-t')^{\alpha-(\frac{k-1}{k})}}{(k\alpha-k+1)^{\frac{1}{k}}|t-t'|^\alpha} \int_{t'}^t p(s)^\lambda ds]^{\frac{1}{\lambda}}] \right\} \\
&\leq \left[\frac{|\gamma|\Gamma(\rho+1)\eta^{\rho+\alpha-(\frac{k-1}{k})}}{(\Gamma(\rho+1)-\gamma\eta^\rho)(k\alpha+k\rho-k-1)^{\frac{1}{k}}\Gamma(\rho+\alpha)} \right. \\
&+ \left. \frac{1}{\Gamma(\alpha)} \sup\left\{ \frac{|t'^{\alpha-(\frac{k-1}{k})} + 2(t-t')^{\alpha-(\frac{k-1}{k})} - t^{\alpha-(\frac{k-1}{k})}|}{(k\alpha-k+1)^{\frac{1}{k}}(t-t')^\alpha} \right\} \right] \\
&\times \left[\int_0^1 p(s)^\lambda ds\right]^{\frac{1}{\lambda}} \\
&< Z \times \frac{1}{Z} = 1.
\end{aligned}
$$

□

Here, we give an example to illustrate Theorem 3.2.

Example 3.3. Let $\gamma = -2$, $\eta = \frac{1}{4}$, $\rho = 2$, $\alpha = \frac{1}{2}$, $k = 2$, $f(t,u) = \frac{u^3(1-\tanh^2(t))}{u^2+1}$ and $g(t) = \frac{t}{t+1}$ for all $t \in [0,1]$. Owing to Theorem 3.2, the problem

$$\begin{cases} {}^cD^{\frac{1}{2}}_{0^+}u(t) + \dfrac{u^3(1-\tanh^2(t))}{u^2+1} + \dfrac{t}{t+1} = 0, \qquad t \in [0,1], \\ u(0) = -2I^2_{0^+}u(\frac{1}{4}) = -2\int_0^{\frac{1}{4}} \dfrac{(\frac{1}{4}-s)^{\frac{3}{2}}}{\Gamma(2)}u(s)ds, \end{cases}$$

admits at least one nontrivial solution $u^* \in \mathcal{H}^\alpha[0,1]$. Indeed, simple calculations show that

$$\frac{|\gamma|\alpha\Gamma(\rho)\Gamma(\rho+1)\eta^{\rho+\alpha} + 2(\Gamma(\rho+1) - \gamma\eta^\rho)\Gamma(\alpha+\rho+1)}{(\alpha\rho+\alpha^2)\Gamma(\alpha)\Gamma(\alpha+\rho)(\Gamma(\rho+1)-\gamma\eta^\rho)} \cong 2.28,$$

and

$$|\frac{u^3(1-\tanh^2(t))}{u^2+1}| \leq (1-\tanh^2(t))|u(t)| + r(t)$$

for all positive function $r(t)$. Then $p(t) = 1-\tanh^2(t)$ and

$$p(s) \leq 1 < \frac{|\gamma|\alpha\Gamma(\rho)\Gamma(\rho+1)\eta^{\rho+\alpha} + 2(\Gamma(\rho+1) - \gamma\eta^\rho)\Gamma(\alpha+\rho+1)}{(\alpha\rho+\alpha^2)\Gamma(\alpha)\Gamma(\alpha+\rho)(\Gamma(\rho+1)-\gamma\eta^\rho)} \cong 2.28.$$

On the other hand, we observe that Banach fixed point theorem can not be applied to our example because f is not Lipschitz contraction mappings, but we have

$$|f(t,u(t))| \leq p(t)u(t) + r(t).$$

Now, we give a variant of Theorem 3.2.

Theorem 3.4. *Suppose that $f(t,0) \neq 0$, $t \in [0,1]$, $\gamma > \frac{\Gamma(\rho+1)}{\eta^\rho}$, and there exist nonnegative and continuous functions p, r defined on $[0,1]$ such that*

$$|f(t,u(t))| \leq p(t)|u(t)| + r(t), \qquad u \in \mathcal{H}^\alpha[0,1], t \in [0,1], \textit{almost everywhere},$$

and one of the following conditions holds:

(b_1) *The function $p(s)$ satisfies*

$$\begin{cases} p(s) \leq \dfrac{|\gamma|\alpha\Gamma(\rho)\Gamma(\rho+1)\eta^{\rho+\alpha} + 2(\gamma\eta^\rho - \Gamma(\rho+1))\Gamma(\alpha+\rho+1)}{(\alpha\rho+\alpha^2)\Gamma(\alpha)\Gamma(\alpha+\rho)(\gamma\eta^\rho - \Gamma(\rho+1))}, \\ \qquad s \in [0,1], \textit{almost every where}, \\ \\ mes\Bigg\{ s \in [0,1]; p(s) \leq \\ \dfrac{|\gamma|\alpha\Gamma(\rho)\Gamma(\rho+1)\eta^{\rho+\alpha} + 2(\gamma\eta^\rho - \Gamma(\rho+1))\Gamma(\alpha+\rho+1)}{(\alpha\rho+\alpha^2)\Gamma(\alpha)\Gamma(\alpha+\rho)(\gamma\eta^\rho - \Gamma(\rho+1))} \Bigg\} > 0; \end{cases}$$

(b_2) *There exist constants $k, \lambda > 1$ with $\dfrac{1}{\lambda} + \dfrac{1}{k} = 1$ such that if we set*

$$Z_1 = \Bigg[\frac{|\gamma|\Gamma(\rho+1)\eta^{\rho+\alpha-(\frac{k-1}{k})}}{(\gamma\eta^\rho - \Gamma(\rho+1))(k\alpha + k\rho - k - 1)^{\frac{1}{k}}\Gamma(\rho+\alpha)} + \frac{1}{\Gamma(\alpha)} \sup \left\{ \frac{|t'^{\alpha-(\frac{k-1}{k})} + 2(t-t')^{\alpha-(\frac{k-1}{k})} - t^{\alpha-(\frac{k-1}{k})}|}{(k\alpha - k + 1)^{\frac{1}{k}}(t-t')^\alpha} \right\} \Bigg],$$

we have $\int_0^1 p(s)^\lambda ds \leq \dfrac{1}{Z_1^\lambda}$.

Then Problem (2) has at least one nontrivial solution $u^* \in \mathcal{H}^\alpha[0,1]$.

Proof. By the same arguments as given in the proof of Theorem 3.2, the conclusion follows by applying Theorem 3.1. □

Here, we use Banach fixed point theorem to give an extension of Theorem 3.1.

Theorem 3.5. *Suppose that $f(t,0) \not\cong 0$, $t \in [0,1]$, $\gamma \neq \dfrac{\Gamma(\rho+1)}{\eta^\rho}$ and there exists nonnegative and continuous function p defined on $[0,1]$ such that*

$$\begin{cases} |f(t,u_1(t)) - f(t,u_2(t))| \leq p(t)|u_1(t) - u_2(t)|, \quad u_1, u_2 \in \mathcal{H}^\alpha[0,1], \\ t \in [0,1], \textit{almost every where}, \\ \\ M \leq \dfrac{1}{N}, \end{cases}$$

where $M = \sup\{|p(s)| : s \in [0,1]\}$ *and*

$$N = \frac{2}{\alpha\Gamma(\alpha)} + \left[\left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^{\rho}}\right|\frac{\eta^{\alpha+\rho}}{(\alpha+\rho)\Gamma(\alpha+\rho)}\right].$$

Then Problem (2) *has at least one nontrivial solution* $u^* \in \mathcal{H}^{\alpha}[0,1]$.

Proof. If $u_2 = 0$, then $|f(t,u_1(t))| \leq p(t)|u_1(t|+|f(t,0))|$, $u_1 \in \mathcal{H}^{\alpha}[0,1]$, $t \in [0,1] \times \mathbb{R}$ almost every where. Applying Theorem 3.1, it follows that Problem (2) admits a nontrivial solution $u^* \in \mathcal{H}^{\alpha}[0,1]$. If $u_2 \neq 0$, we show that the function T given in the proof of Theorem 3.1 is a contraction. We have

$$\begin{aligned}
\|Tu_1 - Tu_2\| &= |Tu_1(0) - Tu_2(0)| \\
&+ \sup\left\{\frac{|[Tu_1(t) - Tu_2(t)] - [Tu_1(t^{'}) - Tu_2(t^{'})]|}{|t-t'|^{\alpha}} :\right. \\
&\left. t, t^{'} \in [0,1], t \neq t^{'}\right\} \\
&\leq \left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^{\rho}}\int_0^{\eta}\frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}[f(s,u_1(s)) - f(s,u_2(s)]ds\right| \\
&+ \sup\frac{\left|\int_0^t (t-s)^{\alpha-1}[f(s,u_1(s)) - f(s,u_2(s))]ds\right.}{\Gamma(\alpha)|t-t'|^{\alpha}} \\
&- \frac{\left.\int_0^{t^{'}} (t^{'}-s)^{\alpha-1}[f(s,u_1(s)) - f(s,u_2(s))]ds\right|}{\Gamma(\alpha)|t-t'|^{\alpha}} \\
&\leq \left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^{\rho}}\int_0^{\eta}\frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}[f(s,u_1(s)) - f(s,u_2(s)]ds\right| \\
&+ \sup\frac{\int_0^{t^{'}} |(t-s)^{\alpha-1} - (t^{'}-s)^{\alpha-1}||f(s,u_1(s)) - f(s,u_2(s))|ds}{\Gamma(\alpha)|t-t'|^{\alpha}} \\
&+ \frac{\int_{t^{'}}^t (t-s)^{\alpha-1}|f(s,u_1(s)) - f(s,u_2(s))|ds}{\Gamma(\alpha)|t-t'|^{\alpha}} \\
&\leq \left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^{\rho}}\int_0^{\eta}\frac{(\eta-s)^{\rho+\alpha-1}p(t)|u_1(s) - u_2(s)|}{\Gamma(\rho+\alpha)}ds\right|
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \sup \frac{\int_0^{t'}((t'-s)^{\alpha-1}-(t-s)^{\alpha-1})p(t)|u_1(s)-u_2(s)|ds}{\Gamma(\alpha)|t-t'|^{\alpha}} \\
&+ \quad \frac{\int_{t'}^{t}(t-s)^{\alpha-1}p(t)|u_1(s)-u_2(s)|ds}{\Gamma(\alpha)|t-t'|^{\alpha}} \\
&\leq \quad M\|u_1-u_2\|\Bigg[\left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^{\rho}}\int_0^{\eta}\frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}ds\right| \\
&+ \quad \sup \frac{[\frac{(t-t')^{\alpha}}{\alpha}+\frac{t'^{\alpha}}{\alpha}+\frac{t^{\alpha}}{\alpha}+\frac{(t-t')^{\alpha}}{\alpha}]}{\Gamma(\alpha)|t-t'|^{\alpha}}\Bigg] \\
&\leq \quad \left[\frac{2}{\alpha\Gamma(\alpha)}+\left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^{\rho}}\int_0^{\eta}\frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}ds\right|\right]M\|u_1-u_2\|.
\end{aligned}
$$

This proves that T is contraction. Applying Banach fixed point theorem, Problem (2) has a unique solution in $\mathcal{H}^{\alpha}[0,1]$. □

Here, we give the following example as an application of Theorem 3.5.

Example 3.6. Let $\alpha = \frac{1}{2}$, $\eta = \frac{1}{2}$, $\rho = \frac{1}{2}$ and $\gamma = \frac{1}{100}$. Consider the following problem

$$
\begin{cases}
{}^{c}D_{0^+}^{\frac{1}{2}} + \dfrac{1}{10}sin(2(t+1))sin(u(t)-1) + cost = 0, \qquad t \in [0,1], \\
u(0) = \dfrac{1}{100}I_{0^+}^{\frac{1}{2}}u(\frac{1}{2}) = \dfrac{1}{100}\displaystyle\int_0^{\frac{1}{2}}\frac{(\frac{1}{2}-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}u(s)ds.
\end{cases} \tag{4}
$$

Clearly, we see that $\gamma \neq \dfrac{\Gamma(\rho+1)}{\eta^{\rho}}$ and

$$
\begin{aligned}
|f(t,x_1(t)) - f(t,x_2(t))| \quad &= \quad |\frac{1}{10}\sin(2(t+1))\sin(x_1(t)-1) \\
&- \quad \frac{1}{10}\sin(2(t+1))\sin(x_2(t)-1)|
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{10}\sin(2(t+1))|\sin(x_1(t)-1)-\sin(x_2(t)-1)| \\
&\leq \frac{1}{5}\cos(t+1)\sin(t+1)|(x_1(t)-1)-(x_2(t)-1)| \\
&\leq \frac{1}{5}\cos(t+1)|x_1(t)-x_2(t)|.
\end{aligned}
$$

Namely, the first condition of Theorem 3.5 is satisfied with $f(t,x) = \frac{1}{10}sin(2(t+1))sin(x-1)$ for each $t \in [0,1]$, $x \in \mathbb{R}$ and $p(t) = \frac{1}{5}cos(t+1)$ for each $t \in [0,1]$. Moreover, we have

$$
N = \frac{4}{\sqrt{\pi}} + [\frac{2}{599} \times \frac{200}{101\sqrt{2}}] \cong 2.27 \ \text{ and } \ M \cong= 0.19.
$$

This concludes that $M \leq \frac{1}{N}$. Hence, applying Theorem 3.5, Problem (4) admits a unique nontrivial solution $x^* \in H^\alpha[0,1]$. It is simple to verify that Theorem 1.1 [9] can not be applied to our example. In fact, it is just enough to put $g(t,x) = \frac{1}{10}sin(2(t+1))sin(1-x) - cos(t)$ and $\lambda = 1$. Therefore, we have the following problem equivalent to Problem (4), that is

$$
\begin{cases}
{}^cD^{\frac{1}{2}}_{0^+} = \lambda g(t,u(t)), \qquad t \in [0,1], \\
u(0) = \frac{1}{100} I^{\frac{1}{2}}_{0^+} u(\frac{1}{2}) = \frac{1}{100}\int_0^{\frac{1}{2}} \frac{(\frac{1}{2}-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} u(s)ds,
\end{cases}
$$

Moreover, an easy computation shows

$$
|f(t,x_1(t)) - f(t,x_2(t))| \leq \phi(|x_1(t) - x_2(t|),
$$

where $\phi(x) = \frac{1}{5}\cos(t+1)x$ for $t \in [0,1]$. But, clearly

$$
L = \left|\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1)-\gamma\eta^\rho}\right| \frac{1}{(\rho+\alpha)\Gamma(\rho+\alpha)}\eta^{\rho+\alpha} + \frac{2}{\alpha\Gamma(\alpha)} \cong 2.27,
$$

and so

$$
\lambda = 1 \not\leq \frac{1}{2.27} \cong \frac{1}{L} < 1.
$$

In the following, by setting some conditions for f and γ in Theorem 3.5, we give a theorem for the existence of a negative solution to Problem (2).

Theorem 3.7. *Suppose that* $f : [0,1] \times \mathbb{R} \longrightarrow [0,\infty)$*,* $f(t,0) \neq 0$*,* $t \in [0,1]$*,* $0 \leq \gamma < \dfrac{\Gamma(\rho+1)}{\eta^{\rho}}$ *and there exists nonnegative and continuous function* p *defined on* $[0,1]$ *such that*

$$\begin{cases} |f(t,u_1(t)) - f(t,u_2(t))| \leq p(t)|u_1(t) - u_2(t)|, \\ u_1, u_2 \in \mathcal{H}^{\alpha}[0,1], t \in [0,1], \textit{almost every where}, \\ \\ M \leq \dfrac{1}{N}, \end{cases}$$

where M *and* N *are defined in Theorem 3.5. Then Problem (2) has at least one negative solution* $u^* \in \mathcal{H}^{\alpha}[0,1]$*.*

Proof. As the same proof of Theorem 3.5, we can show that T is contraction. Then, Problem (2) has at least one solution $u^* \in \mathcal{H}^{\alpha}[0,1]$. On the other hand, since f and g are nonnegative functions and $0 \leq \gamma < \dfrac{\Gamma(\rho+1)}{\eta^{\rho}}$, the function $Tu(t)$ for each $t \in [0,1]$ admits negative value, that is

$$Tu(t) \leq 0 \text{ for all } t \in [0,1].$$

Hence, u^* is negative. $\square$

4 Another results for the existence of solution of Problem (2)

In this section, we present some theorems that condition

$$|f(t,u_1(t)) - f(t,u_2(t))| \leq p(t)|u_1(t) - u_2(t)|$$

is not necessarily needed for all u_1 and u_2. In fact, it is sufficient to satisfy just in a limited number of u_1 and u_2 in $\mathcal{H}^{\alpha}[0,1]$. Very recently, Eshaghi Gordji et al. [12] introduced the notation of the orthogonal sets and gave a real generalization of Banach fixed point theorem. For the depth of the subject, we refer to [1, 2]. In below, we remind some useful definitions to prove some fixed point theorems with weaker conditions.

Definition 4.1. [12] Let $X \neq \emptyset$, and $\perp \subseteq X \times X$ be a binary relation. If " $\perp$ " satisfies the following condition:

$$\exists x_0 \text{: } (\forall y,\ y \perp x_0) \text{ or } (\forall y,\ x_0 \perp y),$$

then " $\perp$ " is called an orthogonality relation and the pair $(X, \perp)$ an orthogonal set(briefly, *O-set*).

Note that in the above definition, we say that x_0 is an orthogonal element. Also, we say that elements $x, y \in X$ are $\perp$-comparable either $x \perp y$ or $y \perp x$.

Definition 4.2. [1] Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}$ is called a *strongly orthogonal sequence*(briefly, *SO-sequence*) if

$$(\forall n, k; \quad x_n \perp x_{n+k}) \quad \text{or} \quad (\forall n, k; \quad x_{n+k} \perp x_n).$$

Let $(X, \perp)$ be an O-set and "d" be a metric on X. The triplet $(X, \perp, d)$ is called an orthogonal metric space.

Definition 4.3. [1] Let $(X, \perp, d)$ be an orthogonal metric space. X is said to be *strongly orthogonal complete*(briefly, *SO-complete*) if every Cauchy SO-sequence is convergent.

Definition 4.4. [1] Let $(X, \perp, d)$ be an orthogonal metric space. A mapping $f : X \to X$ is *strongly orthogonal continuous* (briefly, *SO-continuous*) *in* $a \in X$ *if for each SO-sequence* $\{a_n\}$ *in* X, $a_n \to a$, *then* $f(a_n) \to f(a)$. *Also,* f *is SO-continuous on* X *if* f *is SO-continuous in each* $a \in X$.

Definition 4.5. [12] Let $(X, \perp)$ be an O-set. A mapping $T : X \to X$ is said to be $\perp$-preserving if $x \perp y$ implies $T(x) \perp T(y)$.

Definition 4.6. [2] Let $(X, \perp, d)$ be an orthogonal metric space and $0 < \lambda < 1$. A mapping $T : X \longrightarrow X$ is called an orthogonally contraction(briefly, $\perp$-contraction) with Lipschitz constant λ if for all $x, y \in X$ with $x \perp y$,

$$d(Tx, Ty) \leq \lambda d(x, y).$$

Theorem 4.7. *[3] Let $(X, \perp, d)$ be an SO-complete metric space(not necessarily complete metric space) and $0 < \lambda < 1$. Let $T : X \longrightarrow X$ be SO- continuous, $\perp$-contraction with Lipschitz constant λ and $\perp$-preserving. Then T has a unique fixed point $x^* \in X$.*

We state the main result of this section as extension of Theorem 3.7, in which the condition $|f(t, u_1(t)) - f(t, u_2(t))| \leq p(t)|u_1(t) - u_2(t)|$ only in u_1, u_2 with $u_1(t)u_2(t) > 0$ is assumed.

Theorem 4.8. *Suppose that $0 < \alpha \leq 1$, $0 < \eta < 1$, $\rho > 0$ and $0 \leq \gamma < \dfrac{\Gamma(\rho+1)}{\eta^\rho}$. Assume that $f : [0, 1] \times \mathbb{R} \longrightarrow [0, \infty)$ is a function satisfying the following assumptions:*

- (e_1) *f is continuous and $f(t, 0) \neq 0$ for all $t \in [0, 1]$;*
- (e_2) *there exists nonnegative and continuous function p defined on $[0, 1]$ such that*

$$\begin{cases} |f(t, u_1(t)) - f(t, u_2(t))| \leq p(t)|u_1(t) - u_2(t)|, \\ u_1(t)u_2(t) \geq 0, t \in [0, 1], \ \ \textit{almost every where}, \\ \\ p(s) \leq \dfrac{1}{N}, \end{cases}$$

where $N = \dfrac{2}{\alpha\Gamma(\alpha)} + \left[\dfrac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1) - \gamma\eta^\rho} \dfrac{\eta^{\alpha+\rho}}{(\alpha+\rho)\Gamma(\alpha+\rho)}\right].$

Then, Problem (2) has a unique negative solution $u^* \in \mathcal{H}^\alpha[0, 1]$.

Proof. Consider the following orthogonality relation in X:

$$u \perp v \ \ \Leftrightarrow \ \ u(t)v(t) \geq 0 \ \text{ for all } \ t \in [0, 1] \ \text{ and } \ u, v \in \mathcal{H}^\alpha[0, 1].$$

Then $(X, \perp)$ is an O-set with orthogonal element $x_0 = 0$. Since (X, d) is a complete metric space, then $(X, \perp, d)$ is SO-complete. Clearly, T is SO-continuous and $\perp$-contraction. Now, we prove T is $\perp$-preserving. Let $u, v \in \mathcal{H}^\alpha[0, 1]$ with $u \perp v$. We must show that

$$Tu(t)Tv(t) \geq 0 \ \text{ for all } \ t \in [0, 1].$$

Since f and g are positive and $0 \leq \gamma < \dfrac{\Gamma(\rho+1)}{\eta^\rho}$, by definition of $Tu(t)$, we have

$$Tu(t) \leq 0 \quad \text{for all} \quad t \in [0,1] \quad \text{and} \quad u \in \mathcal{H}^\alpha[0,1]. \tag{5}$$

This concludes that $Tu(t)Tv(t) \geq 0$, and so T is $\perp$-preserving. Applying Theorem 4.7, there exists unique nontrivial solution $u^* \in \mathcal{H}^\alpha[0,1]$ to Problem (2). Hence, using (5), we observe that u^* is negative. This completes the proof. $\square$

We end this paper by giving a variant version of Theorem 4.8.

Theorem 4.9. *Suppose that $0 < \alpha \leq 1$, $0 < \eta < 1$, $\rho > 0$ and $0 \leq \gamma < \dfrac{\Gamma(\rho+1)}{\eta^\rho}$. Assume that $f : [0,1] \times \mathbb{R} \longrightarrow [c, \infty)$ is a function satisfying the following assumptions:*

($h1$) *f is continuous;*

($h2$) *$f(t,0) \neq 0$ and $f(t,x(t))$ is decreasing respect to the second argument for any $t \in [0,1]$;*

(h_3) *there exists nonnegative and continuous function p defined on $[0,1]$ such that*

$$\begin{cases} |f(t,u_1(t)) - f(t,u_2(t))| \leq p(t)|u_1(t) - u_2(t)|, \\ u_1(t) \leq u_2(t) \leq c, t \in [0,1], \ \text{almost every where}, \\ \\ p(s) \leq \dfrac{1}{N}, \end{cases}$$

where c is arbitrary nonnegative value and

$$N = \frac{2}{\alpha\Gamma(\alpha)} + \left[\frac{\gamma\Gamma(\rho+1)}{\Gamma(\rho+1) - \gamma\eta^\rho} \frac{\eta^{\alpha+\rho}}{(\alpha+\rho)\Gamma(\alpha+\rho)} \right].$$

Then, Problem (2) has a unique negative solution $u^* \in \mathcal{H}^\alpha[0,1]$.

Proof. We consider the following orthogonality relation in X:

$$u \perp v \quad \Leftrightarrow \quad u(t) \leq v(t) \leq c \quad \text{for all} \quad t \in [0,1] \quad \text{and} \quad u, v \in H^\alpha[0,1].$$

Then $(X, \perp)$ is an O-set with orthogonal element $x_0 = c$. Now, it is enough to show that T is $\perp$-preserving. Let $u, v \in \mathcal{H}^\alpha[0,1]$ with $u \perp v$. We must show that

$$Tu(t) \leq Tv(t) \leq c \quad \text{for all} \quad t \in [0,1].$$

Applying (h_2), we have

$$\begin{aligned}
(Tu)(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[f(s,u(s)) + g(s)]ds \\
&+ \frac{\gamma\Gamma(\rho+1)}{\gamma\eta^\rho - \Gamma(\rho+1)} \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}[f(s,u(s)) + g(s)]ds \\
&\leq -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[f(s,v(s)) + g(s)]ds \\
&+ \frac{\gamma\Gamma(\rho+1)}{\gamma\eta^\rho - \Gamma(\rho+1)} \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}[f(s,v(s)) + g(s)]ds \\
&= (Tv)(t).
\end{aligned}$$

On the other hand, since f and g are positive value and $0 \leq \gamma < \dfrac{\Gamma(\rho+1)}{\eta^\rho}$, by definition of $Tu(t)$, we have

$$Tu(t) \leq 0 \quad \text{for all} \quad t \in [0,1]. \tag{6}$$

Therefore, since c is nonnegative value, we conclude that $Tu(t) \leq Tv(t) \leq c$. Applying (6) and Theorem 4.7, Problem (2) has a unique negative solution $u^* \in \mathcal{H}^\alpha[0,1]$. $\square$

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