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Original Research Paper

S^{JS}-Metric and Topological Spaces

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Abstract. We introduce the idea of S^{JS}-metric spaces which is a generalization of S-metric spaces. Next we study the properties of S^{JS}-metric spaces and prove several theorems. We also deal with abstract S^{JS}-topological spaces induced by S^{JS}-metric and obtain several classical results including Cantor's intersection theorem in this setting.

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1 Introduction

In 1906 Maurice Fréchet [6] introduced metric spaces in his seminal work "Sur quelques points du calcul fonctionnel". A metric space is a set together with a metric (a real valued distance function between points of the set) on the set and this metric also induces topological

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properties like open and closed sets, which lead to the study of more abstract topological spaces [11]. However soon after the publication of Fréchet's paper, researchers have started to generalize/extend his idea. There are two types of generalizations/extensions of a metric; replace real number set R by some other larger set or relax one of the conditions in the definition of a metric. Menger [14] was the first to propose probabilistic metric spaces, a generalization of metric spaces. During the last six decades a lot of further generalizations/extension of metric spaces was introduced/proposed by the researchers; pseudometric spaces/dislocated metric spaces [7], partial metric spaces [3], modular metric space with the Fatou property [12], fuzzy metric spaces [10], cone metric spaces [8], $b-$metric spaces [4], generalized $D-$metric spaces [2, 5, 15], generalized cone metric spaces [1] and so on. Sedghi et al. [17] gave the concept of $S-$metric spaces by modifying $D-$metric and $G-$metric spaces. Following this Souayan and Mlaiki [18] proposed the concept of S_b- metric spaces as a generalization of $S-$metric spaces. Afterwards Rohen et al. [16] have given the definition of S_b- metric space in a more generalized way and they renamed the usual S_b- metric space as symmetric S_b-metric space. Recently Mehravaran et al. [13] have defined dislocated S_b-metric space and proved some fixed point theorems therein. In the year 2015, Jleli and Samet [9] introduced the idea of $JS-$metric spaces, which is one of the interesting generalization of usual metric spaces. They also showed that any standard metric space, $b-$metric space, dislocated metric space and modular metric space with the Fatou property are $JS-$metric space. In this paper we continue these efforts to further weaken the hypothesis of a metric. First we introduce S^{JS}- metric spaces with examples and study their properties. Next we discuss S^{JS}-topological spaces induced by S^{JS}-metric and prove several classical theorems including Cantor's intersection theorem in this setting.

2 Preliminaries

Let us recall some basic preliminaries here for subsequent use. Jleli and Samet [9] have given the following definitions regarding a generalized metric space.

Let A be a non-empty set and $d : A \times A \to [0, \infty]$ be a mapping. For any $a \in A$, define the set

$$C(d, A, a) = \{\{a_n\} \subset A : \lim_{n\to\infty} d(a_n, a) = 0\}.$$

Definition 2.1. [9] Let $d : A \times A \to [0, \infty]$ be a mapping which satisfies the following conditions:

(i) $d(a, b) = 0$ implies $a = b$ for all $a, b \in A$;

(ii) for every $(a, b) \in A \times A$, we have $d(a, b) = d(b, a)$;

(iii) if $(a, b) \in A \times A$ and $\{a_n\} \in C(d, A, a)$ then

$$d(a, b) \leq p \limsup_{n\to\infty} d(a_n, b), \text{ for some } p > 0.$$

The pair (A, d) is a generalized metric space, usually known as $JS-$metric space.

Jleli and Samet [9] observed that any metric space, $b-$metric space and dislocated metric space are $JS-$metric space. Our below example shows that a rectangular metric space [2] may not be a $JS-$metric space.

Example 2.2. Let $X = \mathbb{R}$ and $d : X^2 \to [0, \infty)$ be defined as follows: $d(x, y) = d(y, x)$ for any $x, y \in X$, $d(x, y) = 0$ if $x = y$ and for $x \neq y$.

$$d(x,y) = \begin{cases} \frac{1}{n}, & \text{if } x = 1,\ y = 1 + \frac{1}{n} \text{ for any } n \geq 2 \\ \frac{1}{n^2}, & \text{if } x = 2,\ y = 1 + \frac{1}{n} \text{ for any } n \geq 2 \\ 3, & \text{otherwise} \end{cases}$$

Then it can be easily verified that (X, d) is a rectangular metric space but it is not a metric space, because

$$d(1, \frac{3}{2}) + d(\frac{3}{2}, 2) = \frac{3}{4} < 3 = d(1, 2)$$

Here we see that $\{1 + \frac{1}{n}\}_{n\geq 2} \in C(d, X, 1)$ but there exits no $p > 0$ for which

$$d(1, 2) \leq p \limsup_{n\to\infty} d(1 + \frac{1}{n}, 2).$$

Hence X is not a $JS-$metric space.

We now give the definitions of $S-$metric space, S_b-metric space and dislocated S_b-metric space.

Definition 2.3. [17] Let X be a non-empty set and $S : X^3 \to [0, \infty)$ be a function satisfying the following conditions, for each $x, y, z, w \in X$:

(i) $S(x, y, z) = 0$ if and only if $x = y = z$;

(ii) $S(x, y, z) \leq S(x, x, w) + S(y, y, w) + S(z, z, w)$.

The function S is called an $S-$metric and the pair (X, S) is called an $S-$metric space.

Definition 2.4. [16] Let X be a nonempty set and $s \geq 1$ be a given number. Also let a function $S_b : X^3 \to [0, \infty)$ satisfies the following conditions, for each $x, y, z, w \in X$:

(i) $S_b(x, y, z) = 0$ if and only if $x = y = z$;

(ii) $S_b(x, y, z) \leq s[S_b(x, x, w) + S_b(y, y, w) + S_b(z, z, w)]$.

The pair (X, S_b) is called an S_b-metric space.

A symmetric S_b-metric is a function which satisfies the conditions (i), (ii) and also the following condition:

$$S_b(x, x, y) = S_b(y, y, x)$$

for all $x, y \in X$.

Definition 2.5. [13] Let X be a non-empty set and $S_d : X^3 \to [0, \infty)$ be a mapping which satisfies the following conditions for all $x, y, z, w \in X$:

(i) $S_d(x, y, z) = 0$ implies $x = y = z$;

(ii) $S_d(x, y, z) \leq k[S_d(x, x, w) + S_d(y, y, w) + S_d(z, z, w)]$, where $k \geq 1$.

The function S_d is said to be a dislocated S_b-metric and the pair (X, S_d) is called a dislocated S_b-metric space. In the case when $k = 1$, S_d is known as the dislocated S-metric.

3 S^{JS}-metric spaces

Let X be a nonempty set and $J : X^3 \to [0, \infty]$ be a function. For any $x \in X$ define

$$S(J, X, x) = \{\{x_n\} \subset X : \lim_{n\to\infty} J(x, x, x_n) = 0\}$$

for all $x \in X$.

Definition 3.1. Let X be a nonempty set and $J : X^3 \to [0, \infty]$ satisfies the following conditions:

(J_1) $J(x, y, z) = 0$ implies $x = y = z$ for any $x, y, z \in X$;

(J_2) there exists some $b > 0$ such that for any $(x, y, z) \in X^3$ and $\{z_n\} \in S(J, X, z)$, we have

$$J(x, y, z) \leq b \limsup_{n\to\infty}(J(x, x, z_n) + J(y, y, z_n))$$

Then the pair (X, J) is called an S^{JS}-metric space.

Additionally if J also satisfies

(J_3) $J(x, x, y) = J(y, y, x)$ for all $x, y \in X$, then we call it a symmetric S^{JS}-metric space.

Example 3.2. Let $X = \mathbb{R} \cup \{-\infty, \infty\}$ and $J : X^3 \to [0, \infty]$ be defined by $J(x, y, z) = |x| + |y| + t|z|$; $t > 0$ with $t \neq 2$ for all $x, y, z \in X$, then clearly (J_1) is satisfied. For any $z \neq 0$, $S(J, X, z) = \emptyset$. If $z = 0$ then for $\{z_n\} \in S(J, X, 0)$, we have

$$J(x, y, 0) \leq \frac{1}{2} \limsup_{n\to\infty}(J(x, x, z_n) + J(y, y, z_n))$$

for all $x, y \in X$. Then (J_2) is also satisfied. So (X, J) is an $S^{JS}-$metric space but it is not symmetric.

Example 3.3. Let $X = \mathbb{R} \cup \{-\infty, \infty\}$ and $J : X^3 \to [0, \infty]$ be defined by $J(x, y, z) = |x| + |y| + 2|z|$ for all $x, y, z \in X$. Clearly the conditions (J_1) and (J_3) are satisfied. Also one can check that for any $x, y, z \in X$

$$J(x, y, z) \leq \limsup_{n\to\infty}(J(x, x, z_n) + J(y, y, z_n))$$

for any sequence $\{z_n\} \in S(J, X, z)$. Therefore (J_2) is also satisfied and hence X is a symmetric S^{JS}-metric space.

Remark 3.4. (1) Let (X, S) be an $S-$metric space (See Definition 2.3). Clearly S satisfies condition (J_1). Now let $(x, y, z) \in X^3$ and $\{z_n\}$ converges to z in (X, S), then $S(z, z, z_n) \to 0$ as $n \to \infty$ and from the condition (ii) we have

$$S(x, y, z) \leq \limsup_{n\to\infty}(S(x, x, z_n) + S(y, y, z_n))$$

Therefore S satisfies (J_2) also. Hence X is an S^{JS}-metric space. It is also symmetric.

(2) Let (X, S_b) be an S_b-metric space with coefficient $s \geq 1$ (See Definition 2.4). Then clearly S_b satisfies (J_1) and it also satisfies (J_2) for $b = s$. So an S_b-metric space is an S^{JS}-metric space.

(3) If (X, S_d) is a dislocated S_b-metric space with coefficient $k \geq 1$ (See Definition 2.5), then clearly S_d satisfies the condition (J_1) and condition (J_2) for $b = k$. So a dislocated S_b-metric space is an S^{JS}-metric space.

Definition 3.5. Let (X, J) be an S^{JS}-metric space, then a sequence $\{x_n\} \subset X$ is said to be convergent to an element $x \in X$ if $\{x_n\} \in S(J, X, x)$.

Definition 3.6. Let (X, J) be an S^{JS}-metric space. A sequence $\{x_n\} \subset X$ is said to be Cauchy if $\lim_{n,m\to\infty} J(x_n, x_n, x_m) = 0$.

Definition 3.7. An S^{JS}-metric space is said to be complete if every Cauchy sequence in X is convergent.

Definition 3.8. Let (X, J) be an S^{JS}-metric space and $T : X \to X$ be a self mapping. Then T is called continuous at $a \in X$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $x \in X$, $J(Ta, Ta, Tx) < \epsilon$ whenever $J(a, a, x) < \delta$.

Theorem 3.9. *In an S^{JS}-metric space (X, J) if $\{x_n\}$ converges to both x and y for $x, y \in X$, then $x = y$.*

Proof. Now,

$$J(x, x, y) \leq b \limsup_{n\to\infty}(2J(x, x, x_n)).$$

Since $x_n \to x$ then $\lim_{n\to\infty} J(x, x, x_n) = 0$, which implies $J(x, x, y) = 0$ that is $x = y$. □

Theorem 3.10. *Let (X, J) be an S^{JS}-metric space and $\{x_n\} \subset X$ converges to some $x \in X$. Then $J(x, x, x) = 0$.*

Proof. Since $\{x_n\}$ converges to x it follows that $\{x_n\} \in S(J, X, x)$ and thus

$$J(x, x, x) \leq b \limsup_{n\to\infty}(2J(x, x, x_n)),$$

which implies $J(x, x, x) = 0$. □

Theorem 3.11. *In a symmetric S^{JS}-metric space (X, J) if a Cauchy sequence $\{x_n\}$ has a convergent subsequence then $\{x_n\}$ is also convergent in X.*

Proof. Let $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to $x \in X$. Now since (X, J) is symmetric, we have

$$J(x, x, x_n) = J(x_n, x_n, x) \leq b \limsup_{k\to\infty}(2J(x_n, x_n, x_{n_k})).$$

Taking $n, k \to \infty$ we have $\lim_{n\to\infty} J(x, x, x_n) = 0$. So $\{x_n\}$ converges to x. □

Theorem 3.12. *In an S^{JS}-metric space (X, J) if T is continuous at $a \in X$ then for any sequence $\{x_n\} \in S(J, X, a)$ implies $\{Tx_n\} \in S(J, X, Ta)$.*

Proof. Let $\epsilon > 0$ be given. Since T is continuous at a then for $\epsilon > 0$ there exists $\delta > 0$ such that $J(a, a, x) < \delta$ implies $J(Ta, Ta, Tx) < \epsilon$.

As $\{x_n\}$ converges to a, so for $\delta > 0$ there exists $N \in \mathbb{N}$ such that $J(a, a, x_n) < \delta$ for all $n \geq N$. Therefore for any $n \geq N$, $J(Ta, Ta, Tx_n) < \epsilon$ and thus $Tx_n \to Ta$ as $n \to \infty$. □

4 S^{JS}-topological spaces

Definition 4.1. Let (X, J) be an S^{JS}-metric space. The open and closed ball of center $x \in X$ and radius $r > 0$ in X are defined as follows:

$$\begin{aligned} B_J(x, r) &= \{y \in X : J(x, x, y) < r\}; \\ B_J[x, r] &= \{y \in X : J(x, x, y) \leq r\}. \end{aligned}$$

Remark 4.2. It may happen that in an S^{JS}-metric space X, $x \notin B_J(x, r)$ for some $r > 0$ and $x \in X$. In Example 3.2 if we take $x = 1$, $r = 2$ and $t = 1$ then $J(1, 1, 1) = 3$ and therefore $1 \notin B_J(1, 2)$.

Theorem 4.3. *Let (X, J) be an S^{JS}-metric space. Let $\tau = \{\emptyset\} \cup \{U(\neq \emptyset) \subset X$: for any $x \in U$ there exists $r > 0$ such that $B_J(x, r) \subset U\}$. Then τ forms a topology on X, called the topology induced by J and (X, τ) is said to be a S^{JS}- topological space.*

Proof. Clearly $X \in \tau$. Now let $\{G_\alpha\}_{\alpha\in\Lambda}$, Λ being an indexing set, be a collection of members of τ and $G = \cup_{\alpha\in\Lambda} G_\alpha$. If $x \in G$ then there exists some $\beta \in \Lambda$ such that $x \in G_\beta$. So there exists $r > 0$ such that $B_J(x, r) \subset G_\beta \subset G$. Hence $G \in \tau$.

Also let $G, H \in \tau$ and $y \in G \cap H$. Then there exist $r_1, r_2 > 0$ such that $B_J(y, r_1) \subset G$ and $B_J(y, r_2) \subset H$. If we take $r = \min\{r_1, r_2\}$ then we have $B_J(y, r) \subset G \cap H$ and so $G \cap H \in \tau$. Therefore τ forms a topology on X. □

Definition 4.4. Let (X, J) be an S^{JS}-topological space. and $F \subset X$. Then F is said to be closed if there exists an open set $U \subset X$ such that $F = U^c$.

Theorem 4.5. *Let (X, J) be an S^{JS}-topological space and $F \subset X$ be closed. Let $\{x_n\} \subset F$ be such that $\{x_n\} \in S(J, X, x)$, then $x \in F$.*

Proof. If possible let $x \notin F$. Then $x \in F^c = U$, where U is open. So there exists $r > 0$ such that $B_J(x, r) \subset U$. Now $\lim_{n\to\infty} J(x, x, x_n) = 0$ so for $r > 0$ there exists $N \in \mathbb{N}$ such that $J(x, x, x_n) < r$ whenever $n \geq N$. Thus $x_n \in B_J(x, r) \subset U$ for all $n \geq N$, a contradiction. Hence $x \in F$. □

Theorem 4.6. *Let (X, J) be an S^{JS}- topological space and $F \subset X$ be closed. If X is complete then (F, J_F) is also complete.*

Proof. Let $\{x_n\} \subset F$ be Cauchy in F. Since X is complete and $\{x_n\}$ is Cauchy in X also, there exists $z \in X$ such that $\{x_n\} \in S(J, X, z)$. As F is closed then by Theorem 4.5 we have $z \in F$. Thus $\{x_n\}$ is convergent in F. Therefore F is complete. □

Theorem 4.7. *Let (X, J) be an S^{JS}-topological space and T be continuous self mapping on X. Then for any open set U, $T^{-1}(U)$ is open.*

Proof. Let U be any open set in X, if $T^{-1}(U) = \emptyset$ then we are done. So let $T^{-1}(U) \neq \emptyset$ and $a \in T^{-1}(U)$. Then $Ta \in U$ and since U is open there exists $\epsilon > 0$ such that $B_J(Ta, \epsilon) \subset U$. T is continuous at $'a'$ so there exists $\delta > 0$ such that $J(x, x, a) < \delta$ implies $J(Tx, Tx, Ta) < \epsilon$. Therefore $T(B_J(a, \delta)) \subset B_J(Ta, \epsilon) \subset U$ implying that $B_J(a, \delta) \subset T^{-1}(U)$. Hence $T^{-1}(U)$ is open. □

Definition 4.8. Let (X, J) be an S^{JS}-metric space and $A \subset X$. Then $diam(A) = \sup\{J(a, a, b) : a, b \in X\}$.

Definition 4.9. In an S^{JS}-topological space (X, J), a sequence $\{F_n\}$ of subsets of X is said to be decreasing if $F_1 \supset F_2 \supset F_3 \supset ...$.

The following theorem gives conditions under which the intersection of such a sequence is non empty.

Theorem 4.10. *[Cantor's intersection property] Let (X, J) be a complete S^{JS}-metric space and $\{F_n\}$ be a decreasing sequence of nonempty closed subsets of X such that $diam(F_n) \to 0$ as $n \to \infty$. Then the intersection $\cap_{n=1}^{\infty} F_n$ contains exactly one point.*

Proof. Let $x_n \in F_n$ be arbitrary for all $n \in \mathbb{N}$. Since $\{F_n\}$ is decreasing, we have $\{x_n, x_{n+1}, ...\} \subset F_n$ for all $n \in \mathbb{N}$.

Now for any $n, m \in \mathbb{N}$ with $n, m \geq k$ we have $J(x_n, x_n, x_m) \leq diam(F_k)$, $k \geq 1$. Let $\epsilon > 0$ be given. Then there exists some $p \in \mathbb{N}$ such that $diam(F_p) < \epsilon$ since $diam(F_n) \to 0$ as $n \to \infty$. From this it follows that $J(x_n, x_n, x_m) < \epsilon$ whenever $n, m \geq p$. So $\{x_n\}$ is Cauchy in X. By the completeness of X there exists $z \in X$ such that $\{x_n\} \in S(J, X, z)$. Since $\{x_n, x_{n+1}, ...\} \subset F_n$ and F_n is closed for each $n \in \mathbb{N}$, using Theorem 4.5 we have $z \in \cap_{n=1}^{\infty} F_n$.

Next we prove the uniqueness of z. Let $y \in \cap_{n=1}^{\infty} F_n$ be another point, then $J(z, z, y) > 0$. As $diam(F_n) \to 0$, there exists $N_0 \in \mathbb{N}$ such that

$$diam(F_n) < J(z, z, y) \leq diam(F_n)$$

for all $n \geq N_0$, a contradiction. Hence $\cap_{n=1}^{\infty} F_n = \{z\}$ and this completes the proof of our theorem. $\square$

Definition 4.11. Let (X, J) be an S^{JS}-metric space and $A(\neq \emptyset) \subset X$. Then a closed set F (if exists) is said to be the closure of A if it is largest which satisfies

$A \subset F \subset A \cup \{x \in X$: there exists $\{x_n\} \subset A$ such that $\{x_n\} \in S(X, J, x)\}$.

We denote F as $\overline{A}$.

Remark 4.12. If (X, J) is an S^{JS}-metric space and $A(\neq \emptyset) \subset X$ is closed then by Theorem 4.5 we have $\overline{A} = A$.

Theorem 4.13. *Let (X, J) be an S^{JS}-metric space and $A(\neq \emptyset) \subset X$. Then $\{x \in X :$ there exists $\{x_n\} \subset A$ such that $\{x_n\} \in S(X, J, x)\} = \{x \in X :$ for all $r > 0, B_J(x, r) \cap A \neq \emptyset\}$.*

Proof. Let $y \in \{x \in X :$ for all $r > 0, B_J(x, r) \cap A \neq \emptyset\}$. Then $B_J(y, \frac{1}{n}) \cap A \neq \emptyset$ for all $n \in \mathbb{N}$. So there exists $y_n \in B_J(y, \frac{1}{n}) \cap A$ for all $n \in \mathbb{N}$ and we have, $\{y_n\} \in S(X, J, y)$. Thus $y \in \{x \in X :$ there exists $\{x_n\} \subset A$ such that $\{x_n\} \in S(X, J, x)\}$.

Conversely let, $z \in \{x \in X :$ there exists $\{x_n\} \subset A$ such that $\{x_n\} \in S(X, J, x)\}$. Then there exists $\{z_n\} \subset A$ such that $J(z, z, z_n) \to 0$ as $n \to \infty$. Let us choose a $r > 0$. Then there exists $m \in \mathbb{N}$ such that $z_n \in B_J(z, r)$ for all $n \geq m$. So $B_J(z, r) \cap A \neq \emptyset$. Hence $z \in \{x \in X :$ for all $r > 0, B_J(x, r) \cap A \neq \emptyset\}$. □

Remark 4.14. Clearly from Theorem 4.3 we have $A \subset \overline{A} \subset A \cup \{x \in X :$ for all $r > 0, B_J(x, r) \cap A \neq \emptyset\}$.

Theorem 4.15. *Let (X, J) be an S^{JS}-metric space and A, B be two nonempty subsets of X with $A \subset B$. Then $\overline{A} \subset \overline{B}$.*

Proof. Clearly $\overline{A}$ and $\overline{B}$ are largest closed sets respectively satisfying the followings

$A \subset \overline{A} \subset A \cup \{x \in X :$ there exists $\{x_n\} \subset A$ such that $\{x_n\} \in S(X, J, x)\}$,

$B \subset \overline{B} \subset B \cup \{x \in X :$ there exists $\{x_n\} \subset B$ such that $\{x_n\} \in S(X, J, x)\}$.

Now, $A \cup B \subset \overline{A} \cup \overline{B} \subset (A \cup B) \cup (\{x \in X :$ there exists $\{x_n\} \subset A$ such that $\{x_n\} \in S(X, J, x)\} \cup \{x \in X :$ there exists $\{x_n\} \subset B$ such that $\{x_n\} \in S(X, J, x)\})$ implies that $B \subset \overline{A} \cup \overline{B} \subset B \cup \{x \in X :$ there exists $\{x_n\} \subset B$ such that $\{x_n\} \in S(X, J, x)\}$. Since $\overline{A} \cup \overline{B}$ is closed, it follows that $\overline{A} \cup \overline{B} \subset \overline{B}$. Therefore we have $\overline{A} \cup \overline{B} = \overline{B}$ and thus $\overline{A} \subset \overline{B}$. □

Theorem 4.16. *Let (X, J) be a symmetric S^{JS}-metric space and $A(\neq \emptyset) \subset X$ for which $\overline{A}$ exists. Then $diam(\overline{A}) \leq L\, diam(A)$, where $L = \max\{1, 2b, 4b^2\}$.*

Proof. Let $x, y \in \overline{A}$. Then we have to consider three cases.
Case 1. If $x, y \in A$ then

$$J(x, x, y) \leq diam(A). \tag{1}$$

Case 2. If $x \in A$ and $y \in \{x \in X :$ there exists $\{x_n\} \subset A$ such that $\{x_n\} \in S(X, J, x)\}$ then there exists a sequence $\{y_n\} \subset A$ such that $\{y_n\} \in S(X, J, y)$ and we have

$$\begin{aligned} J(x,x,y) &\leq 2b \limsup_{n\to\infty} J(x,x,y_n) \\ &\leq 2b\, diam(A). \end{aligned} \tag{2}$$

Case 3. If $x, y \in \{p \in X :$ there exists $\{p_n\} \subset A$ such that $\{p_n\} \in S(X, J, p)\}$ then there exists sequences $\{x_n\}, \{y_n\} \subset A$ such that $\{x_n\} \in S(X, J, x)$, $\{y_n\} \in S(X, J, y)$ and we have

$$\begin{aligned} J(x,x,y) &\leq 2b \limsup_{n\to\infty} J(x,x,y_n) \\ &= 2b \limsup_{n\to\infty} J(y_n,y_n,x) \\ &\leq 2b \limsup_{n\to\infty} (2b \limsup_{m\to\infty} J(y_n,y_n,x_m)) \\ &\leq 4b^2 diam(A). \end{aligned} \tag{3}$$

Therefore from (1), (2) and (3) we get $diam(\overline{A}) \leq L\, diam(A)$, $L = \max\{1, 2b, 4b^2\}$. □

Theorem 4.17. *(Converse of Theorem 4.10) Let (X, J) be a symmetric S^{JS}-metric space in which every nonempty subset has a closure and $\{F_n\}$ be a decreasing sequence of nonempty closed subsets of X with $diam(F_n) \to 0$ as $n \to \infty$. If $\cap_{n=1}^{\infty} F_n$ contains exactly one point then X is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X. Let us choose $G_n = \{x_n, x_{n+1}, x_{n+2}, ...\}$ for all $n \in \mathbb{N}$. Since $\{x_n\}$ is a Cauchy sequence therefore $diam(G_n) \to 0$ as $n \to \infty$. Also $\{\overline{G_n}\}$ is a decreasing sequence of nonempty closed subsets of X (using Theorem 4.15) such that $diam(\overline{G_n}) \to 0$ as $n \to \infty$ (from Theorem 4.16). Hence from the given condition we see that $\cap_{n=1}^{\infty} \overline{G_n} = \{z\}$, $z \in X$.

Now $J(z, z, x_n) \leq diam(\overline{G_n}) \to 0$ as $n \to \infty$. So $\{x_n\}$ is convergent and X is complete. □

Example 4.18. Let us consider the symmetric S^{JS}-metric space given in Example 3.3. Then we have for any $x \in X$ and for any $r > 0$

$$B_J(x,r) = \begin{cases} \emptyset, & \text{if } |x| \geq \frac{r}{2} \\ (-(\frac{r}{2} - |x|), (\frac{r}{2} - |x|)), & \text{if } |x| < \frac{r}{2} \end{cases}$$

and

$$B_J[x,r] = \begin{cases} \emptyset, & \text{if } |x|\,.>\frac{r}{2} \\ [-(\frac{r}{2}-|x|),(\frac{r}{2}-|x|)], & \text{if } |x| \le \frac{r}{2} \end{cases}$$

Here we see that the topology τ is given by

$$\tau = \{\emptyset\} \cup \{B(\neq \emptyset) : B \subset X\backslash\{0\}\} \cup \{B(\neq \emptyset) : 0 \in B$$

and there exists $r > 0$ such that $\left(-\frac{r}{2}, \frac{r}{2}\right) \subset B\}$.

Clearly any nonempty subset of X containing 0 is closed.

If $A(\neq \emptyset) \subset X$, $0 \notin A$ and there does not exist a sequence $\{x_n\} \subset A$ converging to 0 in X then there must exists some $r > 0$ such that $0 \in \left(-\frac{r}{2}, \frac{r}{2}\right) \subset X\backslash A$ and therefore we have A is closed. If $A(\neq \emptyset) \subset X$ is not closed, $0 \notin A$ and there exists a sequence $\{x_n\} \subset A$ converging to 0 in X then $\overline{A} = A \cup \{0\}$. So in (X, J) any nonempty subset of X has closure.

Example 4.19. (Supporting example for Theorem 4.17) Let us consider the symmetric S^{JS}-metric space given in Example 3.3. Also let $\{F_n\}$ be a decreasing sequence of nonempty closed subsets of X such that $diam(F_n) \to 0$ as $n \to \infty$. If $0 \notin F_m$ for some $m \in \mathbb{N}$. Then $0 \notin F_k$ for all $k \geq m$. Now let $x_k \in F_k$ for all $k \geq m$. Then $\{x_m, x_{m+1}, ...\} \subset F_m$ and also $J(x_k, x_k, x_k) \leq diam(F_k) \to 0$ as $m \leq k \to \infty$. Thus $|x_k| \to 0$ as $k \to \infty$ and we get $J(0, 0, x_k) = 2|x_k| \to 0$ as $m \leq k \to \infty$. Since F_m is closed so by Theorem 4.15 we get $0 \in F_m$, a contradiction.

Therefore $0 \in F_n$ for all $n \in \mathbb{N}$. Now if $t(\neq 0) \in \cap_{n=1}^{\infty} F_n$ then $J(t, t, t) \leq diam(F_n) \to 0$ as $n \to \infty$ implying that $t = 0$, a contradiction. Therefore $\cap_{n=1}^{\infty} F_n = \{0\}$. Here we see that (X, J) is complete.

The condition, S^{JS}-metric space X is symmetric is a sufficient condition in Theorem 4.17. Which can be shown from our next example.

Example 4.20. If we consider the S^{JS}-metric space given in Example 3.2 then it is not symmetric and the topology τ is given by $\tau = \{\emptyset\} \cup \{B(\neq \emptyset) : B \subset X\backslash\{0\}\} \cup \{B(\neq \emptyset) : 0 \in B$ and there exists $r > 0$ such that $0 \in (-r, r) \subset B\}$. Clearly any nonempty subset of X containing 0 is closed. If $A(\neq \emptyset) \subset X$, $0 \notin A$ and there does not exist a sequence $\{x_n\} \subset A$ converging to 0 in X then there must exists some $r > 0$

such that $0 \in (-r, r) \subset X \backslash A$ and therefore we have A is closed. If $A(\neq \emptyset) \subset X$ is not closed, $0 \notin A$ and there exists a sequence $\{x_n\} \subset A$ converging to 0 in X then $\overline{A} = A \cup \{0\}$.
So in (X, J) any nonempty subset of X has closure and we can prove that for any decreasing sequence $\{F_n\}$ of nonempty closed subsets of X such that $diam(F_n) \to 0$ as $n \to \infty$, $\cap_{n=1}^{\infty} F_n = \{0\}$, in a similar way as in Example 4.18.

Definition 4.21. Let (X, J) be an S^{JS}-metric space and $A(\neq \emptyset) \subset X$. Then $int(A)$ is the largest open set contained in A.

Definition 4.22. Let (X, J) be an S^{JS}-metric space. A subset A of X is said to be nowhere dense in X if $\overline{A}$ exists and $int(\overline{A}) = \emptyset$.

Theorem 4.23. *Let (X, J) be an S^{JS}-metric space and $A(\neq \emptyset) \subset X$. If $\overline{A}$ exists then $int(X \backslash A) = X \backslash \overline{A}$.*

Proof. Since $\overline{A}$ exists then $A \subset \overline{A} \subset A \cup \{x \in X :$ there exists $\{x_n\} \subset A$ such that $\{x_n\} \in S(X, J, x)\}$. Let us denote the set $\{x \in X :$ there exists $\{x_n\} \subset A$ such that $\{x_n\} \in S(X, J, x)\}$ by A'. Then $(X \backslash A) \cap (X \backslash A') \subset X \backslash \overline{A} \subset X \backslash A$. Now $X \backslash \overline{A}$ is open so $X \backslash \overline{A} \subset int(X \backslash A)$. If $int(X \backslash A) = \emptyset$ then we are done. So let $int(X \backslash A) \neq \emptyset$ and $x \in int(X \backslash A)$. Then there exists some $r > 0$ such that $B_J(x, r) \subset int(X \backslash A) \subset X \backslash A$. So $B_J(x, r) \cap A = \emptyset$ and we have $x \in X \backslash A'$ (using Theorem 4.13). It implies that $x \in (X \backslash A) \cap (X \backslash A') \subset X \backslash \overline{A}$. Therefore $int(X \backslash A) \subset X \backslash \overline{A}$, which shows that $int(X \backslash A) = X \backslash \overline{A}$. □

Theorem 4.24. *Let (X, J) be an S^{JS}-metric space and $A(\neq \emptyset) \subset X$ be a nowhere dense set in X. Then for any open set $U \neq \emptyset$ there exists an open set $V(\neq \emptyset) \subset U$ such that $V \cap A = \emptyset$.*

Proof. Since $int(\overline{A}) = \emptyset$ then $\overline{A} \neq X$. So $int(X \backslash A) = X \backslash \overline{A} \neq \emptyset$. Let U be a nonempty open set in X. Then $U \cap int(X \backslash A) \neq \emptyset$ because if $U \cap int(X \backslash A) = \emptyset$ then $U \cap (X \backslash \overline{A}) = \emptyset$ implying that $U \subset \overline{A}$, a contradiction. Let $V = U \cap int(X \backslash A)$. Then V is open and $V \subset int(X \backslash A) \subset X \backslash A$. Therefore $V \cap A = \emptyset$. □

Definition 4.25. An S^{JS}-metric space (X, J) is said to have property (c) if every nonempty subset of X has a closure.

Conjecture: A complete S^{JS}-metric space (X, J) with property (c) is not expressable as a countable union of nowhere dense sets.

5 Conclusion

In this paper we initiated the study of S^{JS}- metric and topological spaces and proved several classical theorems. In future we plan to further investigate topological properties of these spaces. Proving Baire's Category Theorem in S^{JS}- metric spaces is still an open challenging problem. We also expect applications of these spaces in approximation theory, variational problems, fixed point theory, and optimization theory.

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