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Original Research Paper

On Quasi Bi-Slant Submersions from Kenmotsu Manifolds onto any Riemannian Manifolds

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Abstract. The paper deals with the notion of quasi bi-slant submersions from almost contact metric manifolds onto Riemannian manifolds. These submersions are generalization of hemi-slant submersions and semi-slant submersions. We study such submersions from Kenmotsu manifolds onto Riemannian manifolds and discuss some examples of it. In this paper, we also study the geometry of leaves of distributions which are involved in the definition of the submersion. Further, we obtain the conditions for such submersions to be integrable and totally geodesic.

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1 Introduction

Differential geometry is the most popular branch of mathematics and physics since ancient days. There are several topics in differential geometry that have very important applications in both, mathematics and physics. Immersions and submersions are one of them. The properties of slant submersions became interesting subject in complex geometry and also in contact geometry.

The theory of Riemannian submersions was initiated by O'Neill [17] and Gray [9] in 1966 and 1967, respectively. After some time of this theory, an almost complex type of Riemannian submersions was studied by Watson [26] in 1976. He also defined almost Hermitian submersions between almost Hermitian manifolds in which the Riemannian submersion is an almost complex map. The phenomenon of almost Hermitian submersion to different kinds of sub-classes of almost contact manifolds was extended by D. Chinea [6] in 1985. In 2013, B. Şahin introduced the semi-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds [19] as a generalization of holomorphic submersions and anti-invariant submersions in [23]. Further, the notion of slant submersions from almost Hermitian manifolds onto arbitrary Riemannian manifolds was also defined and studied by B. Şahin [20]. The notion of semi-slant submersions from an almost Hermitian manifold onto a Riemannian manifold were defined and studied by K. S. Park and R. Prasad [15]. In 2015, the hemi-slant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds was introduced in [25]. As a generalization of hemi-slant submersions, C. Sayar et al. defined the notion of bi-slant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds in [24]. Recently, R. Prasad et al. in [16] defined quasi bi-slant submersions as natural generalization of slant, semi-slant, hemi-slant, bi-slant, quasi hemi-slant and show that the geometry of this kind of submersions is different from previous notions. The different kinds of Riemannian submersions between Riemannian manifolds endowed with different structures were studied by several geometers ([1], [2], [3], [4], [8], [10], [11], [12], [13], [14], [18]). Recent developments in the theory of submersions can be found in the book [22]. Taking into account these all previos notions, we are motivated to fill a gap in the literature by giving the notion of quasi bi-slant submersions from Kenmotsu manifolds onto Riemannian manifolds in which the fibers consist of one invariant distribution, two slant distributions and one Reep vector field. In this paper, as a special case of the above notion and a generalization of invariant, anti-invariant, semi-invariant, slant, semi-slant, hemi-slant, bi-slant, quasi hemi-slant Riemannian submersions, we introduce quasi bi-slant submersion from Kenmotsu manifolds and investigate the geometry of base space, the total space and the fibers.

The paper is organized as follows: In the second section, we present some basic information related to quasi bi-slant Riemannian submersion needed throughout this paper. In the third section, we obtain some results on quasi bi-slant Riemannian submersions from Kenmotsu manifold onto Riemannian manifold and provide some examples of such submersions. We also study the geometry of leaves of distribution involved in the above submersion. Finally, we obtain certain conditions for such submersions to be integrable and totally geodesic.

2 Preliminaries

Let M be an almost contact metric manifold [7]. So there exist on M, a $(1,1)$ tensor field ϕ, a vector field ξ, a $1-$form η and a Riemannian metric g_M such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi \circ \xi = 0, \quad \eta \circ \phi = 0, \tag{1}$$

$$g_M(X,\xi) = \eta(X), \eta(\xi) = 1 \tag{2}$$

and

$$g_M(\phi X, \phi Y) = g_M(X,Y) - \eta(X)\eta(Y),$$
$$g_M(\phi X, Y) = -g_M(X, \phi Y), \tag{3}$$

for any vector fields X and Y on M and I is the identity tensor field [27]. An almost contact metric manifold M equipped with an almost contact metric structure (ϕ, ξ, η, g_M) is denoted by $(M, \phi, \xi, \eta, g_M)$.

An almost contact metric manifold M is called a Kenmotsu manifold if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{4}$$

for any vector fields X and Y on M, where ∇ is the Riemannian connection of the Riemannian metric g_M. If $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold, then the following equation holds:

$$\nabla_X \xi = X - \eta(X)\xi. \tag{5}$$

Now, we recall following definitions:

Definition 2.1. [21] *Let F be a Riemannian submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then we say that F is an invariant Riemannian submersion if the vertical distribution is invariant with respect to the complex structure J, i.e.,*

$$J(\ker F_*) = \ker F_*.$$

Definition 2.2. [23] *Let M be an almost Hermitian manifold with Hermitian metric g_M and almost complex structure J and N be a Riemannian manifold with Riemannian metric g_N. Suppose that there exists a Riemannian submersion $F : (M, g_M, J) \to (N, g_N)$ such that $J(\ker F_*) \subseteq (\ker F_*)^\perp$. Then we say that F is an anti-invariant Riemannian submersion.*

Definition 2.3. [19] *Let F be a Riemannian submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then we say that F is a semi-invariant Riemannian submersion if there is a distribution $\mathfrak{D}_1 \subseteq \ker F_*$ such that*

$$\ker F_* = \mathfrak{D}_1 \oplus \mathfrak{D}_2,$$

and

$$J\mathfrak{D}_1 = \mathfrak{D}_1, J\mathfrak{D}_2 \subseteq (\ker F_*)^\perp,$$

where $\mathfrak{D}_2$ is orthogonal complementary to $\mathfrak{D}_1$ in $\ker F_*$.

Let μ denotes the complementary orthogonal subbundle to $J(\ker F_*)$ in $(\ker F_*)^\perp$.

Then, we have

$$(\ker F_*)^\perp = J\mathfrak{D}_2 \oplus \mu.$$

Obviously μ is an invariant subbundle of $(\ker F_*)^\perp$ with respect to the complex structure J.

Definition 2.4. [20] *Let F be a Riemannian submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). If for any non-zero vector $X \in (\ker F_*)_p$, $p \in M$, the angle $\theta(X)$ between JX and the space $(\ker F_*)_p$ is constant, i.e., it is independent of the choice of the point $p \in M$ and the tangent vector X in $\ker F_*$, then we say that F is a slant submersion. In this case, the angle θ is called the slant angle of the submersion. If the slant angle is $0 < \theta < \frac{\pi}{2}$, then the submersion is called a proper slant submersion.*

Definition 2.5. [15] *Let (M, g_M, J) be an almost Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, g_M, J) \to (N, g_N)$ is called a semi-slant submersion if there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that*

$$\ker F_* = \mathcal{D} \oplus \mathcal{D}_1, J(\mathcal{D}) = \mathcal{D},$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_1)_p$ is constant for non-zero $X \in (\mathcal{D}_1)_p$ and $p \in M$, where $\mathcal{D}_1$ is the orthogonal complement of $\mathcal{D}$ in $\ker F_*$.

We call the angle θ, a semi-slant angle.

Definition 2.6. [25] *Let M be an almost Hermitian manifold with Hermitian metric g_M and almost complex structure J, and N be a Riemannian manifold with Riemannian metric g_N. A Riemannian submersion $F : (M, g_M, J) \to (N, g_N)$ is called a hemi-slant submersion if the vertical distribution $\ker F_*$ of F admits two orthogonal complementary distributions D^θ and $D^\perp$ such that D^θ is slant with angle θ and $D^\perp$ is anti-invariant, i.e, we have*

$$\ker F_* = D^\theta \oplus D^\perp.$$

In this case, the angle θ is called the hemi-slant angle of the submersion.

Definition 2.7. [24] *Let (M, g, J) be a Kaehler manifold and (N, g_N) be a Riemannian manifold. A Riemannian submersion $\pi : (M, g, J) \to (N, g_N)$ is called a bi-slant submersion, if there are two slant distributions $\mathcal{D}^{\theta_1} \subset ker\pi_*$ and $\mathcal{D}^{\theta_2} \subset ker\pi_*$ such that*

$$ker\pi_* = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2},$$

where, $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ has slant angles θ_1 and θ_2, respectively.

Define O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_E L = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}L + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}L, \tag{6}$$

$$\mathcal{T}_E L = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}L + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}L, \tag{7}$$

for any vector fields E, L on M, where ∇ is the Levi-Civita connection of g_M. It is easy to see that $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions.

From equations (6) and (7), we have

$$\nabla_X Y = \mathcal{T}_X Y + \mathcal{V}\nabla_X Y, \tag{8}$$

$$\nabla_X U = \mathcal{T}_X U + \mathcal{H}\nabla_X U, \tag{9}$$

$$\nabla_U X = \mathcal{A}_U X + \mathcal{V}\nabla_U X, \tag{10}$$

$$\nabla_U V = \mathcal{H}\nabla_U V + \mathcal{A}_U V, \tag{11}$$

for $X, Y \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^{\perp}$, where $\mathcal{H}\nabla_X V = \mathcal{A}_V X$, if V is basic. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution .

It is seen that for $q \in M$, $X \in \mathcal{V}_q$ and $U \in \mathcal{H}_q$ the linear operators

$$\mathcal{A}_U,\ \mathcal{T}_X : T_qM \to T_qM$$

are skew-symmetric, that is

$$g(\mathcal{A}_U E, L) = -g(E, \mathcal{A}_U L) \text{ and } g(\mathcal{T}_X E, L) = -g(E, \mathcal{T}_X L)$$

for each $E, L \in T_qM$. Since $\mathcal{T}_V$ is skew-symmetric, we observe that F has totally geodesic fibers if and only if $\mathcal{T} \equiv 0$.

Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold and $F : M \to N$ is smooth map. Then the second fundamental form of F is given by

$$(\nabla F_*)(V, W) = \nabla^F_V F_* W - F_*(\nabla_V W), \quad \text{for } V, W \in \Gamma(T_pM), \tag{12}$$

where we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N and ∇^F is the pullback connection.

We recall that a differentiable map F between two Riemannian manifolds is totally geodesic if

$$(\nabla F_*)(V,W) = 0, \text{ for all } V, W \in \Gamma(TM). \tag{13}$$

A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

Now, we can easily prove the following lemma as in [5].

Lemma 2.8. *Let F be a Riemannian submersion from a Riemannian manifold (M, g_M) onto an other Riemannian manifold (N, g_N), then we have*

(i) $(\nabla F_*)(U,V) = 0$,

(ii) $(\nabla F_*)(X,Y) = -F_*(\mathcal{T}_X Y) = -F_*(\nabla_X Y)$,

(iii) $(\nabla F_*)(U,X) = -F_*(\nabla_U X) = -F_*(\mathcal{A}_U X)$,

where U and V are horizontal vector fields and X and Y are vertical vector fields.

3 Quasi Bi-Slant Submersions

In this section, we introduce the notion of a quasi bi-slant submersion from Kenmotsu manifolds onto Riemannian manifold and give non-trivial examples of this kind of submersions and investigate the geometry of leaves of distributions which are involved in the submersion.

Definition 3.1. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion*

$$F : (M, \phi, \xi, \eta, g_M) \to (N, g_N),$$

is called a quasi bi-slant submersion if there exist four mutually orthogonal distributions D, D_1, D_2 and $<\xi>$ such that

(i) $\ker F_* = D \oplus D_1 \oplus D_2 \oplus <\xi>$,

(ii) $J(D) = D$ *i.e., D is invariant,*

(iii) $J(D_1) \perp D_2$ *and* $J(D_2) \perp D_1$,

(iv) for any non-zero vector field $X \in (D_1)_p$, $p \in M$, the angle θ_1 between JX and $(D_1)_p$ is constant and independent of the choice of point p and X in $(D_1)_p$,

(v) for any non-zero vector field $X \in (D_2)_q$, $q \in M$, the angle θ_2 between JX and $(D_2)_q$ is constant and independent of the choice of point q and X in $(D_2)_q$,

These angles θ_1 and θ_2 are called slant angles of the submersion.

We easily observe that

(a) If $\dim D \neq 0$, $\dim D_1 = 0$ and $\dim D_2 = 0$, then F is an invariant submersion.

(b) If $\dim D \neq 0, \dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 = 0$, then F is proper semi-slant submersion.

(c) If $\dim D = 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 = 0$, then F is slant submersion with slant angle θ_1.

(d) If $\dim D = 0, \dim D_1 = 0$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then F is slant submersion with slant angle θ_2.

(e) If $\dim D = 0, \dim D_1 \neq 0$, $\theta_1 = \frac{\pi}{2}$ and $\dim D_2 = 0$, then F is an anti-invariant submersion.

(f) If $\dim D \neq 0, \dim D_1 \neq 0$, $\theta_1 = \frac{\pi}{2}$ and $\dim D_2 = 0$, then F is an semi-invariant submersion.

(g) If $\dim D = 0, \dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $\theta_2 = \frac{\pi}{2}$, then F is a hemi-slant submersion.

(h) If $\dim D = 0, \dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then F is a bi-slant submersion.

(i) If $\dim D \neq 0, \dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $\theta_2 = \frac{\pi}{2}$, then we may call F is an quasi-hemi-slant submersion.

(j) If $\dim D \neq 0, \dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then F is proper quasi bi-slant submersion.

(k) If $\dim D \neq 0, \dim D_1 \neq 0$, $\dim D_2 \neq 0$ and $\theta_1 = \theta_2 = \theta$, then F is semi-slant submersion with semi-slant angle θ.

Now, we will give non-trivial examples in order to guarantee the existence of quasi bi-slant submersions from a Kenmotsu manifold onto a Riemannian manifold and demonstrate that the method presented in this paper is effective.

Example 3.2. Let (x_i, y_i, z) be cartesian coordinates on $\mathbb{R}^{2n+1}$ for $i =$

$1, 2, 3, ..., n$. An almost contact metric structure (ϕ, ξ, η, g) is defined as follows:

$$\phi(a_1\frac{\partial}{\partial x_1}+a_2\frac{\partial}{\partial x_2}+........+a_n\frac{\partial}{\partial x_n}+b_1\frac{\partial}{\partial y_1}+b_2\frac{\partial}{\partial y_2}+........+b_n\frac{\partial}{\partial y_n}+c\frac{\partial}{\partial z})$$

$$= (-b_1\frac{\partial}{\partial x_1} + a_1\frac{\partial}{\partial y_1} - b_2\frac{\partial}{\partial x_2} + a_2\frac{\partial}{\partial y_2} - .... - b_n\frac{\partial}{\partial x_n} + a_n\frac{\partial}{\partial y_n}),$$

where $\xi = \frac{\partial}{\partial z}$ and a_i, b_i, c are C^∞ real valued functions in R^{2n+1}.

Let $\eta = dz$, g is Euclidean metric and

$$\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, ..., \frac{\partial}{\partial y_n}, \frac{\partial}{\partial z}\}$$

is orthonormal base field of vectors on $\mathbb{R}^{2n+1}$. We can easily show that (ϕ, ξ, η, g) is Kenmotsu structure on $\mathbb{R}^{2n+1}$. Hence, it is Kenmotsu manifold.

Define a map $\psi : \mathbb{R}^{15} \longrightarrow \mathbb{R}^6$ by

$\psi(x_1, ..., x_7, y_1, ..., y_7, z) \mapsto (x_2 \sin\theta_1 - y_3 \cos\theta_1, y_2, x_4 \cos\theta_2 - y_5 \sin\theta_2, x_5, x_7, y_7)$,

which is a quasi bi-slant submersion such that

$$X_1 = \frac{\partial}{\partial x_1},\ X_2 = \frac{\partial}{\partial y_1},\ X_3 = \frac{\partial}{\partial x_2}\cos\theta_1 + \frac{\partial}{\partial y_3}\sin\theta_1$$

$$X_4 = \frac{\partial}{\partial x_3},\ X_5 = \frac{\partial}{\partial x_4}\sin\theta_2 + \frac{\partial}{\partial y_5}\cos\theta_2,\ X_6 = \frac{\partial}{\partial y_4},$$

$$X_7 = \frac{\partial}{\partial x_6},\ X_8 = \frac{\partial}{\partial y_6},\ X_9 = \xi = \frac{\partial}{\partial z}.$$

$$(\ker\psi_*) = (D \oplus D_1 \oplus D_2 \oplus <\xi>),$$

where

$$D = <X_1, X_2, X_7, X_8>,$$

$$D_1 =< X_3, X_4 >,$$

$$D_2 =< X_5, X_6 >,$$

$$< \xi >=< X_9 >,$$

and

$$(\ker \psi_*)^\perp =< \frac{\partial}{\partial x_2}\sin\theta_1 - \frac{\partial}{\partial y_3}\cos\theta_1, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial x_4}\cos\theta_2 - \frac{\partial}{\partial y_5}\sin\theta_2,$$

$$\frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial y_7} >,$$

with bi-slant angles θ_1 and θ_2.

Example 3.3. Define a map

$$\phi : \mathbb{R}^{13} \to \mathbb{R}^6$$

$$\phi(x_1, ..., x_6, y_1, ..., y_6, z) \mapsto (\frac{x_1 - x_2}{\sqrt{2}}, y_1, \frac{\sqrt{3}x_4 - x_5}{2}, y_5, x_6, y_6),$$

which is a quasi bi-slant submersion such that

$$X_1 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}), X_2 = \frac{\partial}{\partial y_2}, X_3 = \frac{\partial}{\partial x_3}, X_4 = \frac{\partial}{\partial y_3},$$

$$X_5 = \frac{1}{2}(\frac{\partial}{\partial x_4} + \sqrt{3}\frac{\partial}{\partial x_5}), X_6 = \frac{\partial}{\partial y_4},$$

$$X_7 = \xi = \frac{\partial}{\partial z}.$$

$$(\ker \phi_*) = (D \oplus D_1 \oplus D_2 \oplus < \xi >),$$

where

$$D =< X_3, X_4 >,$$

$$D_1 =< X_1, X_2 >,$$

$$D_2 =< X_5, X_6 >,$$

$$< \xi >=< X_7 >,$$

and

$$(\ker \phi_*)^{\perp} =< \frac{\partial}{\partial y_1}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}), \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}), \frac{\partial}{\partial y_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial y_6} >,$$

with bi-slant angles $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{\pi}{3}$.

Remark 3.4. *In this paper, we assume that all horizontal vector fields are basic vector fields.*

Let F be quasi bi-slant submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have

$$TM = \ker F_* \oplus (\ker F_*)^{\perp}. \tag{14}$$

Now, for any vector field $X \in \Gamma(\ker F_*)$, we put

$$X = PX + QX + RX + \eta(X)\xi, \tag{15}$$

where P, Q and R are projection morphisms of $\ker F_*$ onto D, D_1 and D_2, respectively.

For $X \in (\Gamma \ker F_*)$, we set

$$\phi X = \psi X + \omega X, \tag{16}$$

where $\psi X \in (\Gamma \ker f_*)$ and $\omega X \in \Gamma(\omega D_1 \oplus \omega D_2)$.

From equations (15) and (16), we have

$$\begin{aligned} \phi X &= \phi(PX) + \phi(QX) + \phi(RX), \\ &= \psi(PX) + \omega(PX) + \psi(QX) + \omega(QX) + \psi(RX) + \omega(RX). \end{aligned}$$

Since $\phi D = D$, we get $\omega PX = 0$.

Hence above equation reduces to

$$\phi X = \psi(PX) + \psi QX + \omega QX + \psi RX + \omega RX.$$

Thus we have the following decomposition

$$\phi(\ker F_*) = D \oplus (\psi D_1 \oplus \psi D_2) \oplus (\omega D_1 \oplus \omega D_2),$$

where $\oplus$ denotes orthogonal direct sum.

Further, let $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. Then

$$g_M(X, Y) = 0.$$

From definition 3.1(iii), we have

$$g_M(\phi X, Y) = g_M(X, \phi Y) = 0.$$

Now, consider

$$\begin{aligned} g_M(\psi X, Y) &= g_M(\phi X - \omega X, Y), \\ &= g_M(\phi X, Y), \\ &= 0. \end{aligned}$$

Similarly, we have

$$g_M(X, \psi Y) = 0.$$

Let $Z \in \Gamma(D)$ and $X \in \Gamma(D_1)$. Then we have

$$\begin{aligned} g_M(\psi X, Z) &= g_M(\phi X - \omega X, Z), \\ &= g_M(\phi X, Z), \\ &= -g(X, \phi Z), \\ &= 0, \end{aligned}$$

as D is invariant i.e., $\phi Z \in \Gamma(D)$.

Similarly, for $Z \in \Gamma(D)$ and $Y \in \Gamma(D_2)$, we obtain

$$g_M(\psi Y, Z) = 0,$$

From above equations, we have

$$g_M(\psi X, \psi Y) = 0,$$

and

$$g_M(\omega X, \omega Y) = 0,$$

for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$.

So, we can write

$$\psi D_1 \cap \psi D_2 = \{0\}, \omega D_1 \cap \omega D_2 = \{0\}.$$

If $\theta_2 = \frac{\pi}{2}$, then $\psi R = 0$ and D_2 is anti-invariant, i.e., $\phi(D_2) \subseteq (\ker F_*)^\perp$. In this case we denote D_2 by $D^\perp$.

We also have

$$\phi(\ker F_*) = D \oplus \psi D_1 \oplus \omega D_1 \oplus J D^\perp.$$

Since $\omega D_1 \subseteq (\ker F_*)^\perp$, $\omega D_2 \subseteq (\ker F_*)^\perp$. So we can write

$$(\ker F_*)^\perp = \omega D_1 \oplus \omega D_2 \oplus \mathcal{V},$$

where $\mathcal{V}$ is orthogonal complement of $(\omega D_1 \oplus \omega D_2)$ in $(\ker F_*)^\perp$.

Also for any non-zero vector field $Z \in \Gamma(\ker F_*)^\perp$, we have

$$\phi Z = BZ + CZ, \tag{17}$$

where $BZ \in \Gamma(\ker F_*)$ and $CZ \in \Gamma(\mathcal{V})$.

Lemma 3.5. *Let F be a quasi bi-slant submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\psi^2 X + B\omega X = -X + \eta(X)\xi, \omega\psi X + C\omega X = 0,$$

$$\omega BZ + C^2 Z = -Z, \psi BZ + BCZ = 0,$$

for all $X \in \Gamma(\ker F_*)$ and $Z \in \Gamma(\ker F_*)^\perp$.

Proof. Using equations (1), (16) and (17), we have Lemma 3.5. □

Lemma 3.6. *Let F be a quasi bi-slant submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

(*i*) $\psi^2 X = -(\cos^2\theta_1)X$

(*ii*) $g_M(\psi X, \psi Y) = \cos^2\theta_1 g_M(X, Y)$,

(*iii*) $g_M(\omega X, \omega Y) = \sin^2\theta_1 g_M(X, Y)$,

for all $X, Y \in \Gamma(D_1)$.

Proof. (*i*) Let F be a quasi bi-slant submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with the quasi bi-slant angle θ_1.

Then for a non-vanishing vector field $X \in \Gamma(D_1)$, we have

(A) $$\cos\theta_1 = \frac{|\psi X|}{|JX|}$$

and $$\cos\theta_1 = \frac{g_M(JX, \psi X)}{|JX||\psi X|}.$$

By using equation (3.3), we have

$$\cos\theta_1 = \frac{g_M(\psi X, \psi X)}{|JX||\psi X|}.$$

(B) $$\cos\theta_1 = -\frac{g_M(X, \psi^2 X)}{|JX||\psi X|},$$

from equations (A) and (B), we get

$$\psi^2 X = -(\cos^2\theta_1)X, \text{ for } X \in \Gamma(D_1).$$

(*ii*) For all $X, Y \in \Gamma(D_1)$, using equation (16) and Lemma 3.6(*i*), we have

$$\begin{aligned} g_M(\psi X, \psi Y) &= g_M(\psi X + \omega X, \psi Y), \\ &= -g_M(X, \psi^2 Y), \\ &= \cos^2\theta_1 g_M(X, Y). \end{aligned}$$

(*iii*) Using equation (16) and Lemma 3.6(*i*), (*ii*), we have Lemma 3.6(*iii*). □

In a similar way as in above, we obtain the following Lemma:

Lemma 3.7. *Let F be a quasi bi-slant submersion from an contact metric manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

(*i*) $\psi^2 Z = -(\cos^2\theta_2)Z$

(*ii*) $g_M(\psi Z, \psi W) = \cos^2\theta_2 g_M(Z, W)$,

(*iii*) $g_M(\omega Z, \omega W) = \sin^2\theta_2 g_M(Z, W)$,

for all $Z, W \in \Gamma(D_2)$.

Lemma 3.8. *Let F be a quasi bi-slant submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\mathcal{V}\nabla_X \psi Y + \mathcal{T}_X \omega Y - g_M(\psi X, Y)\xi + \eta(Y)\psi X = \psi \mathcal{V}\nabla_X Y + B\mathcal{T}_X Y, \quad (18)$$

$$\mathcal{T}_X \psi Y + \mathcal{H}\nabla_X \omega Y + \eta(Y)\omega X = \omega \mathcal{V}\nabla_X Y + C\mathcal{T}_X Y, \quad (19)$$

$$\mathcal{V}\nabla_U BV + \mathcal{A}_U CV - g_M(CU, V)\xi = \psi \mathcal{A}_U V + B\mathcal{H}\nabla_U V, \quad (20)$$

$$\mathcal{A}_U BV + \mathcal{H}\nabla_U CV = \omega \mathcal{A}_U V + C\mathcal{H}\nabla_U V, \quad (21)$$

$$\mathcal{V}\nabla_X BU + \mathcal{T}_X CU - g_M(\omega X, U)\xi = \psi \mathcal{T}_X U + B\mathcal{H}\nabla_X U, \quad (22)$$

$$\mathcal{T}_X BU + \mathcal{H}\nabla_X CU = \omega \mathcal{T}_X U + C\mathcal{H}\nabla_X U, \quad (23)$$

$$\mathcal{V}\nabla_V \psi X + \mathcal{A}_V \omega X - g_M(BV, X)\xi + \eta(X)BV = B\mathcal{A}_V X + \psi \mathcal{V}\nabla_V X, \quad (24)$$

$$\mathcal{A}_V \psi X + \mathcal{H}\nabla_V \omega X + \eta(X)CV = C\mathcal{A}_V X + \omega \mathcal{V}\nabla_V X, \quad (25)$$

for any $X, Y \in \Gamma(\ker F_*)$ and $U, V \in \Gamma(\ker F_*)^\perp$.

Proof. Using equations $(5)-(7), (16)$ and (17), we can easily obtain all assertions. □

Now, we define

$$(\nabla_X \psi)Y = \mathcal{V}\nabla_X \psi Y - \psi \mathcal{V}\nabla_X Y, \quad (26)$$

$$(\nabla_X \omega)Y = \mathcal{H}\nabla_X \omega Y - \omega \mathcal{V}\nabla_X Y, \quad (27)$$

$$(\nabla_U C)V = \mathcal{H}\nabla_U CV - C\mathcal{H}\nabla_U V, \quad (28)$$

$$(\nabla_U B)V = \mathcal{V}\nabla_U BV - B\mathcal{H}\nabla_U V, \quad (29)$$

for any $X, Y \in \Gamma(\ker F_*)$ and $U, V \in \Gamma(\ker F_*)^\perp$.

Lemma 3.9. *Let F be a quasi bi-slant submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$(\nabla_X \phi)Y = B\mathcal{T}_X Y - \mathcal{T}_X \omega Y + g_M(\psi X, Y)\xi - \eta(Y)\psi X,$$

$$(\nabla_X \omega)Y = C\mathcal{T}_X Y - \mathcal{T}_X \psi Y - \eta(Y)\omega X,$$

$$(\nabla_U C)V = \omega \mathcal{A}_U V - \mathcal{A}_U BV,$$

$$(\nabla_U B)V = \psi \mathcal{A}_U V - \mathcal{A}_U CV + g_M(CU, V)\xi,$$

for any vectors $X, Y \in \Gamma(\ker F_*)$ and $U, V \in \Gamma(\ker F_*)^\perp$.

Proof. Using equations $(18)-(21)$ and $(24)-(27)$ we get all equations of Lemma 3.9. □

The proofs of above Lemmas follow from straightforward computations, so we omit them.

If the tensors ϕ and ω are parallel with respect to the linear connection ∇ on M respectively, then

$$B\mathcal{T}_X Y = \mathcal{T}_X \omega Y - g_M(\psi X, Y)\xi + \eta(Y)\psi X,$$

and

$$C\mathcal{T}_X Y = \mathcal{T}_X \psi Y + \eta(Y)\omega X,$$

for any $X, Y \in \Gamma(TM)$.

Theorem 3.10. *Let F be a proper quasi bi-slant submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, the slant distribution D is integrable if and only if*

$$g_M(\mathcal{T}_Y \phi X - \mathcal{T}_X \phi Y, \omega QZ + \omega RZ) = g_M(\mathcal{V}\nabla_X \phi Y - \mathcal{V}\nabla_Y \phi X, \psi QZ + \psi RZ),$$

for $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D_1 \oplus D_2 \oplus < \xi >)$.

Proof. For $X, Y \in \Gamma(D)$, and $Z \in \Gamma(D_1 \oplus D_2 < \xi >)$, using equations $(1)-(5), (7), (15)$ and (16), we have

$$\begin{aligned}
& g_M([X, Y], Z) \\
= \ & g_M(\nabla_X \phi Y, \phi Z) + \eta(Z)\eta(\nabla_X Y) - g_M(\nabla_Y \phi X, \phi Z) - \eta(Z)\eta(\nabla_Y X), \\
= \ & g_M(\nabla_X \phi Y, \phi Z) - g_M(\nabla_Y \phi X, \phi Z), \\
= \ & g_M(\mathcal{T}_X \phi Y - \mathcal{T}_Y \phi X, \omega QZ + \omega RZ) - g_M(\mathcal{V}\nabla_X \phi Y - \mathcal{V}\nabla_Y \phi X, \\
& \psi QZ + \psi RZ),
\end{aligned}$$

which completes the proof. □

Theorem 3.11. *Let F be a proper quasi bi-slant submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, the slant distribution D_1 is integrable if and only if*

$$\begin{aligned}
& g_M(\mathcal{T}_Z \omega\psi W - \mathcal{T}_W \omega\psi Z, U) \\
= \ & g_M(\mathcal{T}_Z \omega W - \mathcal{T}_W \omega Z, \phi PU + \psi RU) + g_M(\mathcal{H}\nabla_Z \omega W - \mathcal{H}\nabla_W \omega Z, \\
& \omega RU),
\end{aligned}$$

for all $Z, W \in \Gamma(D_1)$ and $U \in \Gamma(D \oplus D_2 \oplus < \xi >)$.

Proof. For all $Z, W \in \Gamma(D_1)$ and $U \in \Gamma(D \oplus D_2 \oplus < \xi >)$, we have

$$g_M([Z,W],U) = g_M(\nabla_Z W, U) - g_M(\nabla_W Z, U).$$

Using equations $(1), (2), (8), (15), (16)$ and Lemma 3.6, we have

$$\begin{aligned} & g_M([Z,W],U) \\ = \; & g_M(\nabla_Z \phi W, \phi U) - g_M(\nabla_W \phi Z, \phi U), \\ = \; & g_M(\nabla_Z \psi W, \phi U) + g_M(\nabla_Z \omega W, \phi U) - g_M(\nabla_W \psi Z, \phi U) \\ & -g_M(\nabla_Z \omega W, \phi U), \\ = \; & \cos^2\theta_1 g_M(\nabla_Z W, U) - \cos^2\theta_1 g_M(\nabla_W Z, U) - g_M(\mathcal{T}_Z \omega\psi W \\ & -\mathcal{T}_W \omega\psi Z, U) + g_M(\mathcal{H}\nabla_Z \omega W + \mathcal{T}_Z \omega W, \phi PU \\ & +\psi RU + \omega RU) - g_M(\mathcal{H}\nabla_W \omega Z + \mathcal{T}_W \omega Z, \\ & \phi PU + \psi RU + \omega RU). \end{aligned}$$

Now, we have

$$\begin{aligned} & \sin^2\theta_1 g_M([Z,W],U) \\ = \; & g_M(\mathcal{T}_Z \omega W - \mathcal{T}_W \omega Z, \phi PU + \psi RU) + g_M(\mathcal{H}\nabla_Z \omega W - \mathcal{H}\nabla_W \omega Z, \\ & \omega RU) - g_M(\mathcal{T}_Z \omega\psi W - \mathcal{T}_W \omega\psi Z, U), \end{aligned}$$

which completes the proof. □

Theorem 3.12. *Let F be a proper quasi bi-slant submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, the slant distribution D_2 is integrable if and only if*

$$\begin{aligned} & g_M(\mathcal{T}_X \omega\psi Y - \mathcal{T}_Y \omega\psi X, V) \\ = \; & g_M(\mathcal{H}\nabla_X \omega Y - \mathcal{H}\nabla_Y \omega X, \omega QV) + g_M(\mathcal{T}_X \omega Y - \mathcal{T}_Y \omega X, \phi PV + \\ & \psi QV), \end{aligned}$$

for all $X, Y \in \Gamma(D_2)$ and $V \in \Gamma(D \oplus D_1 \oplus < \xi >)$.

Proof. For all $X, Y \in \Gamma(D_2)$ and $V \in \Gamma(D \oplus D_1 < \xi >)$, using equations $(1) - (5)$ and (16), we have

$$\begin{aligned} g_M([X,Y],V) \;=\; & g_M(\nabla_X \psi Y, \phi V) + g_M(\nabla_X \omega Y, \phi V) \\ & -g_M(\nabla_Y \psi X, \phi V) - g_M(\nabla_Y \omega X, \phi V). \end{aligned}$$

From equations (8), (15) and Lemma 3.7, we have

$$\begin{aligned} & g_M([X,Y],V) \\ = \; & \cos^2\theta_2 g_M([X,Y],V) + g_M(\mathcal{H}\nabla_X\omega Y - \mathcal{H}\nabla_Y\omega X, \omega QV) \\ & + g_M(\mathcal{T}_X\omega Y - \mathcal{T}_Y\omega X, \phi PV + \psi QV) - g_M(\mathcal{T}_X\omega\psi Y - \mathcal{T}_Y\omega\psi X, V). \end{aligned}$$

Now, we have

$$\begin{aligned} & \sin^2\theta_2 g_M([X,Y],V) \\ = \; & g_M(\mathcal{T}_X\omega Y - \mathcal{T}_Y\omega X, \phi PV + \psi QV) - g_M(\mathcal{T}_X\omega\psi Y - \mathcal{T}_Y\omega\psi X, V) \\ & + g_M(\mathcal{H}\nabla_X\omega Y - \mathcal{H}\nabla_Y\omega X, \omega QV), \end{aligned}$$

the proof follows from the above equations. □

Theorem 3.13. *Let F be a proper quasi bi-slant submersion from a Kenmotsu manifold (M,ϕ,ξ,η,g_M) onto a Riemannian manifold (N,g_N). Then the vertical distribution $(\ker F_*)$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned} & g_M(\mathcal{T}_U PV + \cos^2\theta_1 \mathcal{T}_U QV + \cos^2\theta_2 \mathcal{T}_U RV, X) \\ = \; & g_M(\mathcal{H}\nabla_U\omega\psi PV + \mathcal{H}\nabla_U\omega\psi QV + \mathcal{H}\nabla_U\omega\psi RV, X) \\ & + g_M(\mathcal{T}_U\omega V, BX) + g_M(\mathcal{H}\nabla_U\omega V, CX), \end{aligned}$$

for all $U,V \in \Gamma(\ker F_*)$ and $X \in \Gamma(\ker F_*)^\perp$.

Proof. For all $U,V \in \Gamma(\ker F_*)$ and $X \in \Gamma(\ker F_*)^\perp$, using equations $(1)-(5)$, we have

$$\begin{aligned} & g_M(\nabla_U V, X) \\ = \; & g_M(\nabla_U\phi PV, \phi X) + g_M(\nabla_U\phi QV, \phi X) + g_M(\nabla_U\phi RV, \phi X). \end{aligned}$$

Now, using equations (7), (8), (15), (16), (17) and Lemmas 3.6 and 3.7, we have

$$\begin{aligned} & g_M(\nabla_U V, X) \\ = \; & g_M(\mathcal{T}_U PV, X) + \cos^2\theta_1 g_M(\mathcal{T}_U QV, X) + \cos^2\theta_2 g_M(\mathcal{T}_U RV, X) \\ & - g_M(\mathcal{H}\nabla_U\omega\psi PV + \mathcal{H}\nabla_U\omega\psi QV + \mathcal{H}\nabla_U\omega\psi RV, X) \\ & + g_M(\nabla_U\omega PV + \nabla_U\omega QV + \nabla_U\omega RV, \phi X). \end{aligned}$$

Now, since $\omega PV + \omega QV + \omega RV = \omega V$ and $\omega PV = 0$, we have

$$\begin{aligned} & g_M(\nabla_U V, X) \\ = \; & g_M(\mathcal{T}_U PV + \cos^2\theta_1 \mathcal{T}_U QV + \cos^2\theta_2 \mathcal{T}_U RV, X) \\ & - g_M(\mathcal{H}\nabla_U \omega\psi PV + \mathcal{H}\nabla_U \omega\psi QV + \mathcal{H}\nabla_U \omega\psi RV, X) \\ & + g_M(\mathcal{T}_U \omega V, BX) + g_M(\mathcal{H}\nabla_U \omega V, CX), \end{aligned}$$

which completes the proof. □

Theorem 3.14. *Let F be a proper quasi bi-slant submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, the horizontal distribution $(\ker F_*)^\perp$ is not totally geodesic foliation on M.*

Proof. Let $X, Y \in \Gamma(\ker F_*)^\perp$, using equations (2) and (5), we have

$$\begin{aligned} g_M(\nabla_X Y, \xi) &= -g_M(Y, \nabla_X \xi) \\ &= -g_M(Y, X), \end{aligned}$$

since $g_M(Y, X) \neq 0$, so $g_M(\nabla_X Y, \xi) \neq 0$. Hence, $(\ker F_*)^\perp$ is not totally geodesic foliation on M. □

Proposition 3.15. *Let F be a proper quasi bi-slant submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the distribution D is not totally geodesic foliation on M.*

Proof. For all $X, Y \in \Gamma(D)$, using equations $(1), (2), (3)$ and (5), we have

$$g_M(\nabla_X Y, \xi) = -g_M(X, Y),$$

which is $g_M(\nabla_X Y, \xi) \neq 0$, so D is not totally geodesic foliation. □

Theorem 3.16. *Let F be a proper quasi bi-slant submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the distribution $D \oplus < \xi >$ defines a totally geodesic foliation if and only if*

$$g_M(\mathcal{T}_X \phi PY, \omega QZ + \omega RZ) = -g_M(\mathcal{V}\nabla_X \phi PY, \psi QZ + \psi RZ),$$

and

$$g_M(\mathcal{T}_X\phi PY, CV) = -g_M(\mathcal{V}\nabla_X\phi PY, BV),$$

for all $X, Y \in \Gamma(D\oplus <\xi>), Z \in \Gamma(D_1 \oplus D_2)$ *and* $V \in \Gamma(\ker F_*)^\perp$.

Proof. For all $X, Y \in \Gamma(D\oplus <\xi>), Z \in \Gamma(D_1 \oplus D_2)$ and $V \in \Gamma(\ker F_*)^\perp$, using equations $(1)-(5), (15)$ and (16), we have

$$\begin{aligned} g_M(\nabla_X Y, Z) &= g_M(\nabla_X\phi Y, \phi Z), \\ &= g_M(\nabla_X\phi PY, \phi QZ + \phi RZ), \\ &= g_M(\mathcal{T}_X\phi PY, \omega QZ + \omega RZ) + g_M(\mathcal{V}\nabla_X\phi PY, \\ &\quad \psi QZ + \psi RZ). \end{aligned}$$

Now, again using equations $(1)-(5), (7)$, and $(15)-(17)$, we have

$$\begin{aligned} g_M(\nabla_X Y, V) &= g_M(\nabla_X\phi Y, \phi V), \\ &= g_M(\nabla_X\phi PY, BV + CV), \\ &= g_M(\mathcal{V}\nabla_X\phi PY, BV) + g_M(\mathcal{T}_X\phi PY, CV), \end{aligned}$$

which completes the proof. □

Proposition 3.17. *Let F be a proper quasi bi-slant submersion from a Kenmotsu manifold* $(M, \phi, \xi, \eta, g_M)$ *onto a Riemannian manifold* (N, g_N). *Then the distribution* D_i *is not totally geodesic foliation on* M, *for* $i = 1, 2$.

Proof. For all $Z, W \in \Gamma(D_i)$, using equations $(1)-(3)$ and (5), we have

$$g_M(\nabla_Z W, \xi) = -g_M(Z, W),$$

which is $g_M(\nabla_Z W, \xi) \neq 0$, so D_i is not totally geodesic foliation on M, for $i = 1, 2$. □

Theorem 3.18. *Let F be a proper quasi bi-slant submersion from a Kenmotsu manifold* $(M, \phi, \xi, \eta, g_M)$ *onto a Riemannian manifold* (N, g_N). *Then the distribution* $D_1\oplus <\xi>$ *defines a totally geodesic foliation if and only if*

$$g_M(\mathcal{T}_Z\omega\psi W, X) = g_M(\mathcal{T}_Z\omega QW, \phi PX + \psi RX) + g_M(\mathcal{H}\nabla_Z\omega QW, \omega RX),$$

and

$$g_M(\mathcal{H}\nabla_Z\omega\psi W, V) = g_M(\mathcal{H}\nabla_Z\omega W, CV) + g_M(\mathcal{T}_Z\omega W, BV),$$

for all $Z, W \in \Gamma(D_1 \oplus <\xi>), X \in \Gamma(D \oplus D_2)$ *and* $V \in \Gamma(\ker F_*)^\perp$.

Proof. For all $Z, W \in \Gamma(D_1 \oplus <\xi>), X \in \Gamma(D \oplus D_2)$ and $V \in \Gamma(\ker F_*)^\perp$, using equations $(1)-(5), (8), (15), (16)$ and Lemma 3.6, we have

$$\begin{aligned}
& g_M(\nabla_Z W, X) \\
= \;& g_M(\nabla_Z \phi W, \phi X) \\
= \;& g_M(\nabla_Z \psi W, \phi X) + g_M(\nabla_Z \omega W, \phi X), \\
= \;& \cos^2\theta_1 g_M(\nabla_Z W, X) - g_M(\mathcal{T}_Z\omega\psi W, X) \\
& + g_M(\mathcal{T}_Z\omega Q W, \phi P X + \psi R X) + g_M(\mathcal{H}\nabla_Z\omega Q W, \omega R X).
\end{aligned}$$

Now, we have

$$\begin{aligned}
& \sin^2\theta_1 g_M(\nabla_Z W, X) \\
= \;& -g_M(\mathcal{T}_Z\omega\psi W, X) + g_M(\mathcal{T}_Z\omega Q W, \phi P X + \psi R X) \\
& + g_M(\mathcal{H}\nabla_Z\omega Q W, \omega R X)
\end{aligned}$$

Next, from equations $(1)-(5), (9), (16), (17)$ and Lemma 3.6, we have

$$\begin{aligned}
g_M(\nabla_Z W, V) &= g_M(\nabla_Z \phi W, \phi V), \\
&= g_M(\nabla_Z \psi W, \phi V) + g_M(\nabla_Z \omega W, \phi V), \\
&= \cos^2\theta_1 g_M(\nabla_Z W, V) - g_M(\mathcal{H}\nabla_Z\omega\psi W, V) \\
&\quad + g_M(\mathcal{H}\nabla_Z\omega W, CV) + g_M(\mathcal{T}_Z\omega W, BV).
\end{aligned}$$

Now, we have

$$\begin{aligned}
& \sin^2\theta_1 g_M(\nabla_Z W, V) \\
= \;& -g_M(\mathcal{H}\nabla_Z\omega\psi W, V) + g_M(\mathcal{H}\nabla_Z\omega W, CV) + g_M(\mathcal{T}_Z\omega W, BV),
\end{aligned}$$

which completes the proof. □

Similar to Theorem 3.18, we can prove the following theorem:

Theorem 3.19. *Let F be a proper quasi bi-slant submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the distribution $D_2 \oplus < \xi >$ defines a totally geodesic foliation if and only if*

$$g_M(\mathcal{T}_X \omega\psi Y, Z) = g_M(\mathcal{T}_X \omega QY, \phi PZ + \phi RZ) + g_M(\mathcal{H}\nabla_X \omega QY, \omega RZ),$$

and

$$g_M(\mathcal{H}\nabla_X \omega\psi Y, V) = g_M(\mathcal{H}\nabla_X \omega Y, CV) + g_M(\mathcal{T}_X \omega Y, BV),$$

for all $X, Y \in \Gamma(D_2 \oplus < \xi >), Z \in \Gamma(D \oplus D_1)$ and $V \in \Gamma(\ker F_*)^{\perp}$.

Using Theorem 3.14 we can give the following theorem:

Theorem 3.20. *Let F be a proper quasi bi-slant submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the map F is not a totally geodesic map.*

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