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Original Research Paper

Bayesian and Classical Estimation of Strength-Stress Reliability for Gompertz Distribution Based on Upper Record Values

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Abstract. In this paper, we consider the problem of estimating stress-strength reliability $R = \Pr(X > Y)$ for Gompertz lifetime models having the same shape parameters but different location parameters under a set of upper record values. We obtain the maximum likelihood estimator (MLE), the approximate Bayes estimator and the exact confidence intervals of stress-strength reliability when the shape parameter is known. Also, when the shape parameter is unknown, the MLE, the asymptotic confidence interval and some bootstrap confidence intervals of stress-strength reliability are studied. Furthermore, a Bayesian approach is proposed for estimating the parameters and then the corresponding credible interval are achieved using Gibbs sampling technique

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via OpenBUGS software. Monte Carlo simulations are performed to compare the performance of different proposed estimation methods. Finally, analysis of a real dataset is performed.

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1 Introduction

Increasing the reliability of any system is an important issue in many fields of engineering. The stress-strength problem originates from reliability engineering. The problem arises naturally in mechanical reliability of a system or equipment. In the reliability engineering literature, the stress-strength model is an evaluation of reliability of a system in terms of random variates X representing an external stress experienced by the system and Y representing the inherent strength of the system available to overcome the stress. The system fails if at any time the external stress is larger than its inherent strength. Assessment of $R = \Pr(Y > X)$ where X and Y are taken to be non-negative independent continuous random variables is a common statistical approach of the Stress-Strength testing. Birnbaum and McCarty [5] were the first who introduced this model under the context of reliability engineering. The book by Kotz et al. [12] is an excellent reference which provides a comprehensive account of this topic on the theory and applications of the stress-strength relationships in industrial, economic and engineering systems under classical and Bayesian point of views. When the X and Y are independent and follow the generalized exponential, three-parameter Weibull distributions, and inverse Pareto, the estimation of R was studied by Kundu and Gupta [13], Kundu and Raqab [14] and Guo and Gui [11], respectively. Rao et al. [18] estimate the multicomponent stress-strength reliability of a system when strength and stress variates are drawn from an exponentiated Weibull distribution with different shape parameters and common shape and scale parameters respectively. Bi and Gui [4] consider the problem of estimating stress-strength reliability for inverse Weibull and lifetime models having the same shape but different scale parameters.

Since record values arise in numerous real-life situations involving data relating to seismology, hydrology, weather, sporting and athletic events, economics, life tests, industrial stress testing, meteorological analysis, oil mining surveys, stock market, and other similar situations, they are extensively used in statistical modeling. In industrial stress-strength testing, the measurements made chronologically and only values which are larger or smaller than all previous ones are observed. That is, if at a stage, an observation recorded has a value which exceeded the value of the previous observations is called a record value. According to the model of record values that is examined in Chandler [6], there are many situations in lifetime testing in which a failure time of a product is recorded if it surpasses all preceding failure times. These recorded failure times are the upper record value sequence. Wen-Chuan Lee et al. [15] evaluate the lifetime performance index based on the Bayesian estimation for the Rayleigh lifetime products with the upper record values. Nadar et al. [17] used the maximum likelihood and Bayesian approaches to estimate stress-strength reliability based on a set of upper record values from Kumaraswamy distribution. The problem of estimating $\Pr(Y > X)$ based on upper record values is considered by Tarvirdizade and Ahmadpour [21] when X and Y are independent random variables from a two-parameter bathtub-shaped lifetime distribution with the same shape but different scale parameters.

The Gompertz distribution, due to Benjamin Gompertz, was introduced in 1825 to describe human mortality and establish actuarial tables. This distribution plays an important role in modelling survival times and customarily used as life time distribution in demography, biology, actuarial, and medical science. In reliability and survival studies, many equipment life time are characterized by an increasing hazard rate and having Gompertz distribution. The problem of stress-strength reliability model is studied by Surinder and Mayank [20] when Stress and Strength follows Gompertz and Power function distribution, respectively.

The cumulative distribution function (cdf) and the probability density function (pdf) of Gompertz distribution with location parameter α and shape parameter β, respectively, is given by

$$F(x) = 1 - e^{-\frac{\alpha}{\beta}(e^{\beta x}-1)}, \qquad x > 0, \quad \alpha, \beta > 0, \tag{1}$$

and

$$f(x) = \alpha e^{\beta x} e^{-\frac{\alpha}{\beta}(e^{\beta x}-1)}, \qquad x > 0, \quad \alpha, \beta > 0. \tag{2}$$

Based on (1) and (2), the corresponding failure rate function of this distribution is given by

$$h_F(x) = \frac{f(x)}{1-F(x)} = \alpha e^{\beta x}, \qquad x > 0, \quad \alpha, \beta > 0. \tag{3}$$

The rest of the paper is organized as follows. In Section 2, the likelihood inference for R is discussed and its asymptotic confidence interval and some bootstrap confidence intervals are obtained. In addition, Bayesian inference is considered in Section 3. In Section 4, Monte Carlo simulation is used to compare the performance of different types of estimators presented in this paper. In Section5, a real data analysis is presented to illustrate the proposed methods. Finally, some stochastic comparisons are made based on stress-strength of inactivity times of two Gompertz random variables in Section 6.

2 Likelihood inference

Let X and Y be independent random variables from the Gompertz lifetime distribution in (2) with the parameters α_1 , β and α_2 , β respectively. Let $R = Pr(X > Y)$ be the stress-strength reliability. then,

$$R = \int_0^\infty F_Y(x).f_X(x) = \frac{\alpha_2}{\alpha_1 + \alpha_2}.$$

We are interested in estimating the quantity R based on two sets of upper record values on both variables. Let $\underset{\sim}{r} = (r_1, \cdots, r_n)$ be a set of upper records from distribution of X with pdf f and cdf F and let $\underset{\sim}{s} = (s_1, \cdots, s_m)$ be an independent set of upper records from distribution of Y with pdf g and cdf G. The likelihood functions are given by (Ahsanullah [2]),

$$\begin{aligned} L(\alpha_1, \beta|\underset{\sim}{r}) &= f(r_n) \prod_{i=1}^{n} \left(\tfrac{f(r_i)}{1-F(r_i)}\right), \quad 0 < r_1 < \cdots < r_n < \infty \\ L(\alpha_2, \beta|\underset{\sim}{s}) &= g(s_m) \prod_{i=1}^{m} \left(\tfrac{g(s_i)}{1-G(s_i)}\right), \quad 0 < s_1 < \cdots < s_m < \infty. \end{aligned} \tag{4}$$

Substituting f , F , g and G in the likelihood functions and using (4), we obtain

$$\begin{aligned} L(\alpha_1, \beta | \underset{\sim}{r}) &= \alpha_1^n e^{\beta \sum_{i=1}^n r_i} e^{-\frac{\alpha_1}{\beta}(e^{\beta r_n}-1)}, \\ L(\alpha_2, \beta | \underset{\sim}{s}) &= \alpha_2^m e^{\beta \sum_{i=1}^m s_i} e^{-\frac{\alpha_2}{\beta}(e^{\beta s_m}-1)}. \end{aligned} \tag{5}$$

Thus, the joint log-likelihood function of the observed records r and s is given by

$$\begin{aligned} \ell(\alpha_1, \alpha_2, \beta | \underset{\sim}{r}, \underset{\sim}{s}) =& n \ln \alpha_1 + m \ln \alpha_2 + \beta (\sum_{i=1}^n r_i + \sum_{i=1}^m s_i) \\ & - \frac{\alpha_1 (e^{\beta r_n} - 1)}{\beta} - \frac{\alpha_2 (e^{\beta s_m} - 1)}{\beta}, \end{aligned}$$

and subsequently the likelihood equations are found to be

$$\frac{\partial \ell}{\partial \alpha_1} = \frac{n}{\alpha_1} - \frac{1}{\beta}(e^{\beta r_n} - 1) = 0, \tag{6}$$

$$\frac{\partial \ell}{\partial \alpha_2} = \frac{m}{\alpha_2} - \frac{1}{\beta}(e^{\beta s_m} - 1) = 0, \tag{7}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} =& \sum_{i=1}^n r_i + \sum_{i=1}^m s_i + \frac{\alpha_1}{\beta^2}(e^{\beta r_n} - 1) + \frac{\alpha_2}{\beta^2}(e^{\beta s_m} - 1) \\ & - \frac{1}{\beta}\left(\alpha_1 r_n e^{\beta r_n} + \alpha_2 s_m e^{\beta s_m}\right) = 0. \end{aligned} \tag{8}$$

Now, we consider likelihood inference for R in the following two cases:

2.1 When β is known

Under the assumption that the shape parameter β is known, the MLEs of α_1 and α_2, respectively, are obtained from (6) and (7) as

$$\hat{\alpha}_1 = \frac{n\beta}{e^{\beta r_n} - 1}, \quad \hat{\alpha}_2 = \frac{m\beta}{e^{\beta s_m} - 1}. \tag{9}$$

Therefore the MLE of R is given by

$$\hat{R} = \frac{\hat{\alpha}_2}{\hat{\alpha}_1 + \hat{\alpha}_2}.$$

To study the distribution of $\hat{R}$ we need the distributions of $\hat{\alpha}_1$ and $\hat{\alpha}_2$. Consider first $\hat{\alpha}_1 = \frac{n\beta}{e^{\beta r_n}-1}$. The pdf of R_n is given by (see [2]):

$$\begin{aligned} f_{R_n}(r_n) &= \frac{1}{(n-1)!} f(r_n)\big[-\ln(1-F(r_n))\big]^{n-1} \\ &= \frac{1}{(n-1)!} \alpha_1^n e^{\beta r_n} e^{-\frac{\alpha_1}{\beta}(e^{\beta r_n}-1)} \Big[\frac{e^{\beta r_n}-1}{\beta}\Big]^{n-1}, \qquad r_n > 0. \end{aligned}$$

Consequently, the pdf of $Z_1 = \hat{\alpha}_1$ is given by:

$$\begin{aligned} f_{Z_1}(z_1) &= \frac{n}{z_1^2 + n\beta z_1} f_{R_n}\Big(\frac{1}{\beta}\ln\big(1 + \frac{n\beta}{z_1}\big)\Big) \\ &= \frac{(n\alpha_1)^n}{(n-1)! z_1^{n+1}} e^{-\frac{n\alpha_1}{z_1}}, \end{aligned}$$

which is an inverted gamma distribution, i.e., $Z_1 \sim I\Gamma(n, n\alpha_1)$. Similarly, for $Z_2 = \hat{\alpha}_2$, we can obtain $Z_2 \sim I\Gamma(m, m\alpha_2)$. So we can find the pdf of $\hat{R} = \frac{\hat{\alpha}_2}{\hat{\alpha}_1+\hat{\alpha}_2} = \frac{Z_2}{Z_1+Z_2} = \frac{1}{1+\frac{Z_1}{Z_2}}$. We have $\frac{n\alpha_1}{Z_1} \sim \Gamma(n,1)$ and $\frac{m\alpha_2}{Z_2} \sim \Gamma(m,1)$ according to properties of the inverted gamma distribution and its relation with the gamma distribution. Therefore $\frac{2n\alpha_1}{Z_1} \sim \chi^2_{(2n)}$ and $\frac{2m\alpha_2}{Z_2} \sim \chi^2_{(2m)}$. Note that, by the independence of two random quantities we have

$$\frac{(2m\alpha_2/2mZ_2)}{(2n\alpha_1/2nZ_1)} = \frac{\alpha_2 Z_1}{\alpha_1 Z_2} = \frac{R}{(1-R)}\frac{\hat{\alpha}_1}{\hat{\alpha}_2} \sim F_{(2m,2n)}.$$

This fact can be used to construct the following $(1-\gamma)\%$ confidence interval for R,

$$\left(\left(1 + \frac{\hat{\alpha}_1}{\hat{\alpha}_2 F_{(\frac{\gamma}{2},2n,2m)}}\right)^{-1}, \left(1 + \frac{\hat{\alpha}_1}{\hat{\alpha}_2 F_{(1-\frac{\gamma}{2},2n,2m)}}\right)^{-1}\right). \qquad (10)$$

2.2 When β is unknown

Here, likelihood inference for R when all of the parameters α_1, α_2 and β are unknown, is discussed. Based on (6) and (7), we obtain

$$\hat{\alpha}_1 = \frac{n\hat{\beta}}{e^{\hat{\beta}r_n} - 1}, \quad \hat{\alpha}_2 = \frac{m\hat{\beta}}{e^{\hat{\beta}s_m} - 1}. \tag{11}$$

By solving likelihood equations, (6), (7) and (8), the MLE of parameter β , i.e $\hat{\beta}$, is obtained by substituting (9) in non-linear equation (8) as follows

$$\sum_{i=1}^{n} r_i + \sum_{i=1}^{m} s_i + \frac{n+m}{\beta} - \frac{nr_n e^{\beta r_n}}{e^{\beta r_n} - 1} - \frac{ms_m e^{\beta s_m}}{e^{\beta s_m} - 1} = 0. \tag{12}$$

Therefore, $\hat{\beta}$ can be obtained as a solution of the non-linear equation of the form $h(\beta) = \beta$ where

$$h(\beta) = -(n+m)\left[\sum_{i=1}^{n} r_i + \sum_{i=1}^{m} s_i - \frac{nr_n e^{\beta r_n}}{e^{\beta r_n} - 1} - \frac{ms_m e^{\beta s_m}}{e^{\beta s_m} - 1}\right]^{-1}.$$

Since $\hat{\beta}$ is a fixed point solution of this non-linear equation, therefore, it can be obtained by using a simple iterative procedure as $h(\beta_j) = \beta_{j+1}$, where β_j is the jth iteration of $\hat{\beta}$. The iteration procedure should be stopped when $|\beta_j - \beta_{j+1}|$ is sufficiently small. Once we obtain $\hat{\beta}$, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ can be deduced from (15) and therefore, the MLE of R is $\hat{R} = \frac{\hat{\alpha}_2}{\hat{\alpha}_1 + \hat{\alpha}_2}$. It is clear that the study of the distribution of R ˆ is very complicated and difficult, therefore, it is not possible to obtain exact confidence interval of R. In this case, some confidence intervals based on the asymptotic distribution of $\hat{R}$ and the bootstrap method are suggested as follows.

2.3 Asymptotic confidence interval

In this subsection, the approximate confidence interval of R is obtained based on the asymptotic distribution of $\hat{R}$ which depends on calculating the Fisher information matrix. Since the expected information matrix is very complicated and requires numerical integration, the observed information matrix is used. The 3×3 observed information matrix I is given by

$$I(\alpha_1,\alpha_2,\beta)=\begin{bmatrix} I_{11} & I_{12} & I_{13}\\ I_{21} & I_{22} & I_{23}\\ I_{31} & I_{32} & I_{33}\end{bmatrix},$$

where

$$\begin{aligned}
I_{11} &= -\frac{\partial^2 \ell}{\partial \alpha_1^2}=\frac{n}{\alpha_1^2}, \qquad I_{12}=I_{21}=-\frac{\partial^2\ell}{\partial\alpha_1\alpha_2}=0, \qquad I_{22}=-\frac{\partial^2\ell}{\partial\alpha_2^2}=\frac{m}{\alpha_2^2},\\
I_{13} =&I_{31}=-\frac{\partial^2\ell}{\partial\alpha_1\partial\beta}=-\frac{1}{\beta^2}\left(e^{\beta r_n}-1\right)+\frac{r_n}{\beta}e^{\beta r_n},\\
I_{23} =&I_{32}=-\frac{\partial^2\ell}{\partial\alpha_2\partial\beta}=-\frac{1}{\beta^2}\left(e^{\beta s_m}-1\right)+\frac{s_m}{\beta}e^{\beta s_m},\\
I_{33} &= -\frac{\partial^2\ell}{\partial\beta^2}=-\frac{2\alpha_1}{\beta^3}(e^{\beta r_n}-1)-\frac{2\alpha_2}{\beta^3}(e^{\beta s_m}-1)\\
&\quad+\frac{2}{\beta^2}(\alpha_1 r_n e^{\beta r_n}+\alpha_2 s_m e^{\beta s_m})-\frac{1}{\beta}(\alpha_1 r_n^2 e^{\beta r_n}+\alpha_2 s_m^2 e^{\beta s_m}).
\end{aligned}$$

As $n\to\infty$ and $m\to\infty$, by the asymptotic properties of the MLE, $\hat{R}$ is asymptotically normal with mean R and asymptotic variance

$$\sigma_R^2=\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial R}{\partial\lambda_i}\frac{\partial R}{\partial\lambda_j}I_{ij}^{-1},$$

where $\lambda_1=\alpha_1$, $\lambda_2=\alpha_2$, $\lambda_3=\beta$ and I_{ij}^{-1} is the (i,j)th element of the inverse of the $I(\alpha_1,\alpha_2,\beta)$ (see Rao [18]). Since

$$\frac{\partial R}{\partial\lambda_1}=\frac{-\alpha_2}{(\alpha_1+\alpha_2)^2}, \qquad \frac{\partial R}{\partial\lambda_2}=\frac{\alpha_1}{(\alpha_1+\alpha_2)^2}, \qquad \frac{\partial R}{\partial\lambda_3}=0,$$

therefore,

$$\sigma_R=\sqrt{\frac{\alpha_2^2 I_{11}^{-1}+\alpha_1^2 I_{22}^{-1}-2\alpha_1\alpha_2 I_{12}^{-1}}{(\alpha_1+\alpha_2)^4}}.$$

Now, the asymptotic $(1-\gamma)\%$ confidence interval of R is given by

$$\left(\hat{R}-z_{1-\gamma/2}\hat{\sigma}_R \quad , \quad \hat{R}+z_{1-\gamma/2}\hat{\sigma}_R\right),$$

where $\hat{\sigma}_R$ is obtained by replacing α_1, α_2, and β involved in σ_R by their corresponding MLEs and z_γ is the γ quantile of the standard normal distribution.

2.4 Bootstrap confidence intervals

Some confidence intervals for R based on the parametric bootstrap methods are obtained here. The following method is proposed to generate parametric bootstrap samples of R, as suggested by Efron and Tibshirani [9].

Step1. Compute $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\beta}$ and $\hat{R}$ based on the original two samples of upper records $\underset{\sim}{r}$ and $\underset{\sim}{s}$.

Step2. Generate independent bootstrap upper record samples $\underset{\sim}{r}^* = (r_1^*, r_2^*, \cdots, r_n^*)$ and $\underset{\sim}{s}^* = (s_1^*, s_2^*, \cdots, s_m^*)$ from the Gompertz lifetime distribution with the parameters $\hat{\alpha}_1$, $\hat{\beta}$ and $\hat{\alpha}_2$, $\hat{\beta}$ respectively. Using these data, we compute the bootstrap estimators $\hat{\alpha}_1^*$, $\hat{\alpha}_2^*$, $\hat{\beta}^*$ and $\hat{R}^*$.

Step3. Repeat step 2, B times to obtained a set of bootstrap samples of R, say $\hat{R}_1^*, \cdots, \hat{R}_B^*$.

Now, we obtain the following three types of bootstrap confidence interval using the above bootstrap samples of R:

(I) Standard normal interval:
The simplest $100(1-\gamma)\%$ bootstrap interval is the standard normal interval as

$$(\hat{R} - z_{1-\alpha/2}\hat{se}_{boot}, \hat{R} + z_{1-\alpha/2}\hat{se}_{boot}),$$

where $\hat{se}_{boot}$ is the bootstrap estimate of the standard error based on $\hat{R}_1^*, \cdots, \hat{R}_B^*$.

(II) Percentile bootstrap (Boot-p) interval:
Let $G(x) = Pr(\hat{R}^* \leqslant x)$ be the cdf of $\hat{R}^*$. Define $\hat{R}_{Boot}(x) = G^{-1}(x)$ for a given x. Then the $100(1-\gamma)\%$ bootstrap percentile interval for R is defined by

$$\left(\hat{R}_{Boot}\left(\frac{\gamma}{2}\right), \hat{R}_{Boot}\left(1 - \frac{\gamma}{2}\right)\right),$$

that is, just use the $\gamma/2$ and $1-\gamma/2$ quantiles of the bootstrap sample $\hat{R}_1^*, \cdots, \hat{R}_B^*$.

(III) Student's t bootstrap (Boot-t) interval:
Let

$$T_b^* = \frac{\hat{R}_b^* - \hat{R}}{\hat{se}_b^*}, \qquad b = 1, 2, \ldots, B,$$

where $\hat{se}_b^*$ is an estimate of the standard error of $\hat{R}^*{}_b$ and can be replaced by its asymptotic standard error. Then $100(1-\gamma)\%$ bootstrap student's t interval is given by

$$(\hat{R} - t^*_{1-\gamma/2}\hat{se}_{boot}, \hat{R} - t^*_{\gamma/2}\hat{se}_{boot}),$$

where t^*_γ is the γ quantile of $T_1^*, \ldots, T_B^*$.

3 Bayesian Inference

In this section, we discuss Bayesian methods for making inferences about R based on upper record values. Again, two cases are considered separately to draw inference on R, namely when the shape parameter β is known and unknown.

3.1 When β is known

Under the assumption that the shape parameter β is known, the likelihood functions of α_1 and α_2 in (5) suggest that the conjugate family of prior distributions for α_1 and α_2 is the Gamma family of probability distributions as

$$\pi(\alpha_i) = \frac{b_i^{a_i}}{\Gamma(a_i)} \alpha_i^{a_i-1} e^{-b_i\alpha_i}, \qquad \alpha_i > 0, a_i > 0, b_i > 0, i = 1, 2, \tag{13}$$

where a_i and b_i are the parameters of the prior distributions of α_i for $i = 1, 2$. Using these prior distributions and the likelihood functions in (5), the posterior distributions of α_1 and α_2 are obtained to be

$$\begin{aligned} (\alpha_1|\underset{\sim}{r}) &\sim \Gamma\left(n + a_1, b_1 + \tfrac{1}{\beta}(e^{\beta r_n} - 1)\right), \\ (\alpha_2|\underset{\sim}{s}) &\sim \Gamma\left(m + a_2, b_2 + \tfrac{1}{\beta}(e^{\beta s_m} - 1)\right). \end{aligned} \tag{14}$$

Since the priors α_1 and α_2 are independent, then, using standard transformation techniques and after some manipulations, the posterior pdf of R will be

$$f_R(r) = C\frac{(1-r)^{n+a_1-1}r^{m+a_2-1}}{(1-r)\left(b_1+\frac{e^{\beta r_n}-1}{\beta}\right)+r\left(b_2+\frac{e^{\beta s_m}-1}{\beta}\right)},$$

where $0 < r < 1$ and

$$C = \frac{\Gamma(n+m+a_1+a_2)}{\Gamma(n+a_1)\Gamma(m+a_2)}\left(b_1+\frac{e^{\beta r_n}-1}{\beta}\right)^{n+a_1}\left(b_2+\frac{e^{\beta s_m}-1}{\beta}\right)^{m+a_2}.$$

Under squared error loss function, the Bayes estimate of R is the expected value of R. This expected value contains an integral which is not obtainable in a simple closed form. Alternatively, using the approximate method of Lindley [16], it can be seen that the approximate Bayes estimate of R, say $\tilde{R}_B$, under squared error loss function is

$$\tilde{R}_B = \tilde{R}\left(1-\frac{\tilde{R}(1-\tilde{R})}{n+a_1-1}+\frac{(1-\tilde{R})^2}{m+a_2-1}\right),$$

where $\tilde{R} = \frac{\tilde{\alpha}_2}{\tilde{\alpha}_1+\tilde{\alpha}_2}$ and

$$\tilde{\alpha}_1 = \frac{n+a_1-1}{b_1+\frac{e^{\beta r_n}-1}{\beta}}, \qquad \tilde{\alpha}_2 = \frac{m+a_2-1}{b_2+\frac{e^{\beta s_m}-1}{\beta}},$$

are the mode of the posterior densities α_1 and α_2, respectively. Furthermore, it follows from (14) that

$$2\left(b_1+\frac{e^{\beta r_n}-1}{\beta}\right)(\alpha_1|\underset{\sim}{r}) \sim \chi^2_{2(n+a_1)},$$
$$2\left(b_2+\frac{e^{\beta s_m}-1}{\beta}\right)(\alpha_2|\underset{\sim}{s}) \sim \chi^2_{2(m+a_2)}.$$

Then, the posterior distribution of R, i.e. $\pi(R|\underset{\sim}{r},\underset{\sim}{s})$, is equal to that of $(1+AW)^{-1}$, where

$$W \sim F_{2(n+a_1),2(m+a_2)}, \quad A = \frac{(n+a_1)\left(b_2+(e^{\beta s_m}-1)/\beta\right)}{(m+a_2)\left(b_1+(e^{\beta r_m}-1)/\beta\right)}.$$

Therefore, a Bayesian $100(1-\gamma)\%$ confidence interval for R is given by

$$\left((AF_{\gamma/2,2(n+a_1),2(m+a_2)}+1)^{-1},(AF_{1-\gamma/2,2(n+a_1),2(m+a_2)}+1)^{-1}\right). \quad (15)$$

3.2 When β is Unknown

In this subsection, the Bayes estimation of R under assumption that all of the parameters α_1, α_2 and β are unknown, is obtained. It is assume that α_1 and α_2 have conjugate priors $\Gamma(a_1,b_1)$ and $\Gamma(a_2,b_2)$ as mentioned in (13), respectively. Also, we consider a prior $\Gamma(a_3,b_3)$ for the shape parameter β. Furthermore, it is assumed that α_1, α_2 and β are independent. Therefore,

$$\pi(\alpha_1,\alpha_2,\beta|\underset{\sim}{r},\underset{\sim}{s}) \propto \pi(\alpha_1)\pi(\alpha_2)\pi(\beta)L(\alpha_1,\alpha_2,\beta|\underset{\sim}{r},\underset{\sim}{s}), \quad (16)$$

where the right hand of (16) by using (5) and (13) is given as

$$\frac{b_1^{a_1}\alpha_1^{n+a_1-1}e^{-\alpha_1\left(b_1+\frac{1}{\beta}(e^{\beta r_n}-1)\right)}}{\Gamma(a_1)} \times \frac{b_2^{a_2}\alpha_2^{m+a_2-1}e^{-\alpha_2\left(b_2+\frac{1}{\beta}(e^{\beta s_m}-1)\right)}}{\Gamma(a_2)}$$

$$\times\frac{b_3^{a_3}\beta^{a_3-1}e^{-\beta\left(b_3-\sum_{i=1}^{n}r_i-\sum_{i=1}^{m}s_i\right)}}{\Gamma(a_3)}.$$

We use the Gibbs sampling technique via OpenBUGS software which uses the posterior distributions of each parameter conditional on all others to obtain the bayes estimation and the credible interval of R (see [10]). The unknown parameter's posterior PDF's are,

$$\pi(\alpha_1|\alpha_2,\beta,\underset{\sim}{r},\underset{\sim}{s}) \propto \alpha_1^{n+a_1-1}e^{-\alpha_1\left(b_1+\frac{1}{\beta}(e^{\beta r_n}-1)\right)},$$

$$\pi(\alpha_2|\alpha_1,\beta,\underset{\sim}{r},\underset{\sim}{s}) \propto \alpha_2^{m+a_2-1}e^{-\alpha_2\left(b_2+\frac{1}{\beta}(e^{\beta s_m}-1)\right)},$$

and

$$\pi(\beta|\alpha_1, \alpha_2, \underset{\sim}{r}, \underset{\sim}{s}) \propto \beta^{a_3-1} e^{-\frac{\alpha_1}{\beta}\left(e^{\beta r_n}-1\right)-\frac{\alpha_2}{\beta}\left(e^{\beta s_m}-1\right)-\beta\left(b_3-\sum_{i=1}^{n} r_i-\sum_{i=1}^{m} s_i\right)}.$$

Therefore, the algorithm of Gibbs Sampling is described as follows:

Step1. Start with $\beta^{(0)} = \hat{\beta}$ as an initial guess and set t=1.

Step2. Generate $\alpha_1^{(t)}$, $\alpha_2^{(t)}$ and $\beta^{(t)}$.

Step3. Compute $R^{(t)} = \alpha_2^{(t)}/(\alpha_1^{(t)} + \alpha_2^{(t)})$.

Step4. Set $t = t + 1$.

Step5. Repeat Steps 2 to 4, N times.

Based on N and $R^{(t)}$ values, using the method introduced by Chen and Shao [7], we can construct the $100(1-\gamma)\%$ highest posterior density (HPD) credible interval for R as,

$$\left(R_{[\frac{\gamma}{2}N]}, R_{[(1-\frac{\gamma}{2})N]}\right), \tag{17}$$

where $R_{[\frac{\gamma}{2}N]}$ and $R_{[(1-\frac{\gamma}{2})N]}$ are the $[\frac{\gamma}{2}N]$-th and $[(1-\frac{\gamma}{2})N]$-th smallest integers of $\left\{R^{(t)}, t = M+1, M+2, \cdots, N\right\}$, respectively.

4 Simulation

In this section, we conduct the Monte Carlo simulation study to assess on the finite sample behavior of the different estimators of R. All results are obtained from 5000 Monte Carlo replications.

In each replication, the upper set of records $\underset{\sim}{r} = (r_1, \cdots, r_n)$ is drawn from $\Gamma(\alpha_1, \beta)$ and an independent set of upper records $\underset{\sim}{s} = (s_1, \cdots, s_m)$ is drawn from $\Gamma(\alpha_2, \beta)$ such that $\Gamma(\alpha, \beta)$ is Gompertz distribution with parameters α and β.

The true value of parameter β in data generating processes is $\beta = 0.1$, and $\alpha_1 = (0.1, 0.3, 0.5, 0.7, 0.9)$ and $\alpha_2 = (0.9, 0.7, 0.5, 0.3, 0.1)$ are different values for the parameters α_1 and α_2. Tables 1, 2 and 3 represent the empirical means and confidence intervals of the corresponding estimators of R for sample sizes n = 3, 5, 10, 15 and 20.

Table 1: Average estimates of R (AVR).

$R(\alpha_1, \alpha_2)$	(n,m)	MLE (Knownβ)	MLE (Unknown β)	BE (Knownβ)	BE (Unknown β)
0.9(0.9,0.1)	(3,3)	0.8455	0.9563	0.8455	0.8159
	(5,5)	0.8940	0.9594	0.894	0.8535
	(10,10)	0.9262	0.9375	0.9162	0.8828
	(15,15)	0.9159	0.9167	0.9012	0.8899
	(20,20)	0.9112	0.9034	0.9009	0.9108
0.7(0.7,0.3)	(3,3)	0.6567	0.8937	0.6567	0.6579
	(5,5)	0.6848	0.8616	0.6848	0.6810
	(10,10)	0.7273	0.8466	0.7081	0.6956
	(15,15)	0.7187	0.8006	0.7076	0.7042
	(20,20)	0.7123	0.7846	0.7023	0.7094
0.5(0.5,0.5)	(3,3)	0.5076	0.4823	0.5076	0.5284
	(5,5)	0.4952	0.4739	0.5052	0.5170
	(10,10)	0.5045	0.5073	0.5041	0.5096
	(15,15)	0.5008	0.5037	0.5082	0.5077
	(20,20)	0.5005	0.5030	0.5003	0.5034
0.3(0.3,0.7)	(3,3)	0.3364	0.3464	0.3355	0.3248
	(5,5)	0.3105	0.3281	0.3164	0.3151
	(10,10)	0.2972	0.3154	0.3052	0.3106
	(15,15)	0.2958	0.3117	0.3078	0.3059
	(20,20)	0.2838	0.3089	0.3038	0.3015
0.1(0.1,0.9)	(3,3)	0.1533	0.1345	0.1533	0.2364
	(5,5)	0.1119	0.1147	0.1119	0.2081
	(10,10)	0.1031	0.1169	0.1030	0.1507
	(15,15)	0.0963	0.1095	0.0965	0.1172
	(20,20)	0.0852	0.1082	0.0853	0.1013

Table 1 represents the MLEs and Bayes estimators (BEs) of parameter R for different values of α_1, α_2 and sample sizes. The prevailing view is that by increasing the size of sample, the estimation of R towards to correct values as expected also, it is notable that in most cases, when the value of parameter β is known, the results of ML and Bayes estimators are close together.

Table 2: MLE, Asymptotic and Bayes confidence intervals of R.

$R(\alpha_1, \alpha_2)$	(n,m)	MLE (Known β)	MLE (Unknown β)	Asy. (Known β)	Bayes (known β)	Bayes (Unknown β)
0.9(0.9,0.1)	(3,3)	(0.512 , 0.968)	(0.828 , 0.992)	(0.656 , 1.035)	(0.512 , 0.968)	(0.496 , 0.923)
	(5,5)	(0.701 , 0.969)	(0.882 , 0.988)	(0.786 , 1.002)	(0.700 , 0.969)	(0.510 , 0.968)
	(10,10)	(0.808 , 0.954)	(0.865 , 0.973)	(0.855 , 0.978)	(0.818 , 0.964)	(0.532 , 0.932)
	(15,15)	(0.830 , 0.940)	(0.851 , 0.953)	(0.853 , 0.949)	(0.830 , 0.940)	(0.553 , 0.948)
	(20,20)	(0.863 , 0.930)	(0.797 , 0.974)	(0.877 , 0.935)	(0.863 , 0.930)	(0.597 , 0.928)
0.7(0.7,0.3)	(3,3)	(0.253 , 0.916)	(0.693 , 0.977)	(0.308 , 1.006)	(0.325 , 0.916)	(0.348 , 0.893)
	(5,5)	(0.372 , 0.889)	(0.688 , 0.959)	(0.424 , 0.945)	(0.416 , 0.889)	(0.406 , 0.883)
	(10,10)	(0.496 , 0.856)	(0.769 , 0.949)	(0.531 , 0.884)	(0.496 , 0.856)	(0.473 , 0.929)
	(15,15)	(0.540 , 0.834)	(0.750 , 0.927)	(0.565 , 0.853)	(0.540 , 0.834)	(0.518 , 0.822)
	(20,20)	(0.581 , 0.830)	(0.744 , 0.918)	(0.601 , 0.844)	(0.581 , 0.830)	(0.624 , 0.760)
0.5(0.5,0.5)	(3,3)	(0.159 , 0.849)	(0.191 , 0.777)	(0.128 , 0.887)	(0.159 , 0.849)	(0.157 , 0.931)
	(5,5)	(0.211 , 0.782)	(0.215 , 0.742)	(0.189 , 0.801)	(0.211 , 0.782)	(0.217 , 0.872)
	(10,10)	(0.290 , 0.711)	(0.296 , 0.710)	(0.282 , 0.719)	(0.290 , 0.711)	(0.264 , 0.882)
	(15,15)	(0.329 , 0.678)	(0.333 , 0.679)	(0.326 , 0.683)	(0.329 , 0.678)	(0.247 , 0.814)
	(20,20)	(0.356 , 0.659)	(0.351 , 0.655)	(0.354 , 0.662)	(0.356 , 0.659)	(0.355 , 0.649)
0.3(0.3,0.7)	(3,3)	(0.077 , 0.699)	(0.041 , 0.549)	(0.004 , 0.625)	(0.077 , 0.699)	(0.042 , 0.856)
	(5,5)	(0.121 , 0.549)	(0.119 , 0.526)	(0.067 , 0.605)	(0.121 , 0.649)	(0.059 , 0.789)
	(10,10)	(0.147 , 0.504)	(0.126 , 0.507)	(0.119 , 0.471)	(0.147 , 0.504)	(0.119 , 0.742)
	(15,15)	(0.163 , 0.455)	(0.141 , 0.459)	(0.145 , 0.431)	(0.163 , 0.455)	(0.183 , 0.694)
	(20,20)	(0.165 , 0.382)	(0.238 , 0.423)	(0.187 , 0.261)	(0.199 , 0.382)	(0.218 , 0.607)
0.1(0.1,0.9)	(3,3)	(0.031 , 0.484)	(0.006 , 0.144)	(0.009 , 0.341)	(0.031 , 0.484)	(0.038 , 0.677)
	(5,5)	(0.033 , 0.310)	(0.019 , 0.188)	(0.009 , 0.225)	(0.033 , 0.225)	(0.016 , 0.540)
	(10,10)	(0.036 , 0.182)	(0.024 , 0.126)	(0.022 , 0.144)	(0.036 , 0.182)	(0.014 , 0.453)
	(15,15)	(0.038 , 0.146)	(0.033 , 0.126)	(0.050 , 0.133)	(0.038 , 0.146)	(0.018 , 0.391)
	(20,20)	(0.007 , 0.137)	(0.093 , 0.169)	(0.064 , 0.125)	(0.041 , 0.137)	(0.022 , 0.280)

Tables 2 and 3 give the confidence intervals of R with various methods that explained in above. It can be seen that generally by increasing the sample size, confidence intervals get shorter. The results of simulation study shows that the confidence intervals of ML and Bayes estimations with known β are more accurate than similar estimators with unknown β. By and large, the bootstrap methods had worst confidence intervals in comparison of other methods.

Table 3: Bootstrap confidence intervals of R.

R(α_1, α_2)	(n,m)	Boot-s	Boot-p	Boot-t
0.9(0.9,0.1)	(3,3)	(0.872 , 1.020)	(0.809 , 1.112)	(0.801 , 1.297)
	(5,5)	(0.807 , 1.089)	(0.833 , 1.065)	(0.835 , 1.144)
	(10,10)	(0.874 , 1.037)	(0.881 , 0.998)	(0.747 , 1.081)
	(15,15)	(0.794 , 1.048)	(0.821 , 0.996)	(0.841 , 1.037)
	(20,20)	(0.850 , 1.016)	(0.865 , 0.986)	(0.847 , 0.972)
0.7(0.7,0.3)	(3,3)	(0.594 , 0.813)	(0.598 , 0.811)	(0.608 , 0.909)
	(5,5)	(0.579 , 0.798)	(0.607 , 0.795)	(0.632 , 0.893)
	(10,10)	(0.574 , 0.823)	(0.594 , 0.872)	(0.627 , 0.841)
	(15,15)	(0.635 , 0.849)	(0.625 , 0.940)	(0.686 , 0.796)
	(20,20)	(0.673 , 1.053)	(0.616 , 0.877)	(0.644 , 0.749)
0.5(0.5,0.5)	(3,3)	(0.480 , 0.529)	(0.489 , 0.524)	(0.448 , 0.567)
	(5,5)	(0.488 , 0.519)	(0.492 , 0.515)	(0.392 , 0.612)
	(10,10)	(0.471 , 0.524)	(0.478 , 0.515)	(0.425 , 0.674)
	(15,15)	(0.459 , 0.550)	(0.474 , 0.536)	(0.400 , 1.598)
	(20,20)	(0.344 , 0.656)	(0.390 , 0.610)	(0.245 , 1.261)
0.3(0.3,0.7)	(3,3)	(0.046 , 0.513)	(0.460 , 0.499)	(0.094 , 0.424)
	(5,5)	(0.044 , 0.514)	(0.444 , 0.498)	(0.014 , 0.426)
	(10,10)	(0.015 , 0.524)	(0.209 , 0.460)	(0.157 , 0.458)
	(15,15)	(0.004 , 0.471)	(0.063 , 0.381)	(0.168 , 0.397)
	(20,20)	(0.060 , 0.339)	(0.022 , 0.347)	(0.181 , 0.343)
0.1(0.1,0.9)	(3,3)	(0.004 , 0.522)	(0.091 , 0.493)	(-0.056 , 0.495)
	(5,5)	(0.020 , 0.517)	(0.082 , 0.461)	(-0.044 , 0.323)
	(10,10)	(0.001 , 0.349)	(0.038 , 0.309)	(0.004 , 0.306)
	(15,15)	(0.000 , 0.242)	(0.004 , 0.246)	(0.001 , 0.265)
	(20,20)	(0.005 , 0.199)	(0.006 , 0.193)	(0.005 , 0.277)

Figure 1 represent the AMSE of the parameter R for fixed value of $\beta = 0.1$ and different sample sizes. In all figures, black and green lines show AMSEs of MLE and Bayes estimators with known β and red and blue lines show AMSEs of MLE and Bayes estimators with unknown β respectively. We note that, as the sample size increases, the mean squared errors decrease in all the cases analyzed, as expected. Moreover, similar to results of Table 1, the AMSE of ML and Bayes estimators are close together. The simulations were carried out in 'R' and 'OpenBugs' statistical softwares.

5 Data Analysis

As an example, we analyze the real dataset to illustrate the methods of inference discussed in this article. We consider two datasets of the amount of annual rainfall (in inches) recorded at the Los Angeles Civic

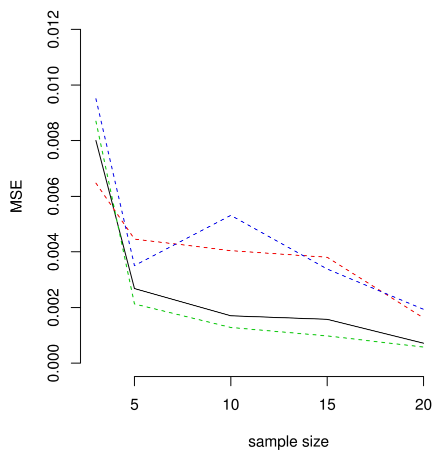

Figure 1: The mean square error of parameter R when the true value is 0.1.

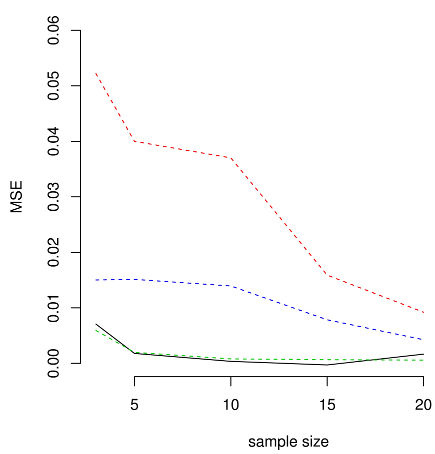

Figure 2: The mean square error of parameter R when the true value is 0.3.

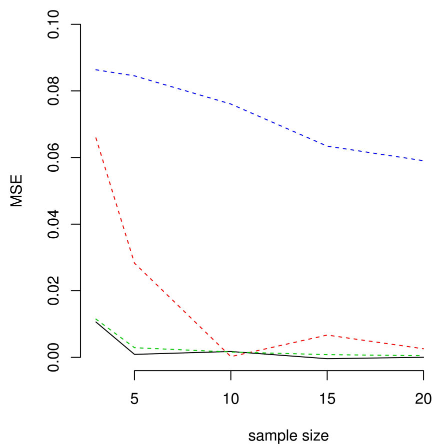

Figure 3: The mean square error of parameter R when the true value is 0.5.

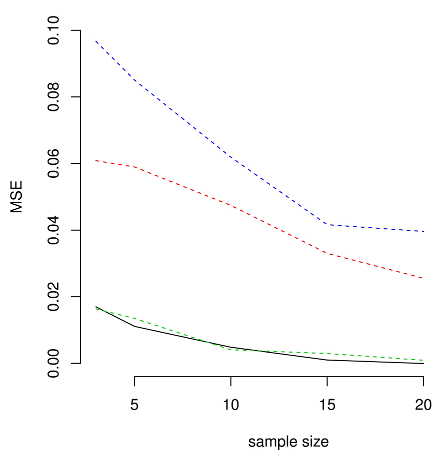

Figure 4: The mean square error of parameter R when the true value is 0.7.

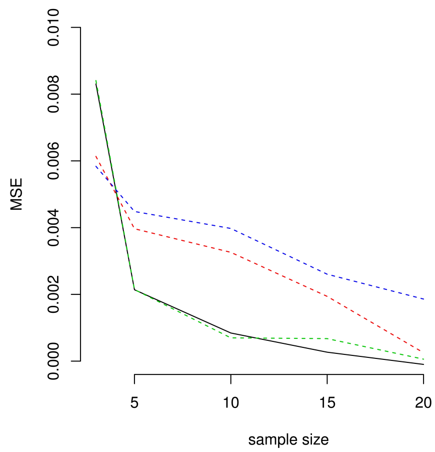

Figure 5: The mean square error of parameter R when the true value is 0.9.

Center for two 30-year periods as follows:

Dataset 1 (from 1960 to 1989): 4.85, 18.79, 8.38, 7.93, 13.69, 20.44, 22.00, 16.58, 27.47, 7.77, 12.32, 7.17, 21.26, 14.92, 14.35, 7.22, 12.31, 33.44, 19.67, 26.98, 8.98, 10.71, 31.25, 10.43, 12.82, 17.86, 7.66, 12.48, 8.08, 7.35,

Dataset 2 (from 1990 to 2019): 11.47, 21.00, 27.36, 8.11, 24.35, 12.46, 12.40, 31.01, 9.09, 11.57, 17.94, 4.42, 16.49, 9.24, 37.25, 13.19, 3.21, 13.53, 9.08, 16.36, 20.20, 8.69, 5.85, 6.08, 8.52, 9.65, 19.00, 4.79, 18.82, 14.86.

The Gompertz distribution given in (1) is fitted to the two datasets separately, and its performances is examined. The estimated location and shape parameters (based on the ML method), Kolmogorov-Smirnov (K-S) distances and corresponding p-values are presented in Table 4. Obviously, the Gompertz distributions with equal shape parameters fit

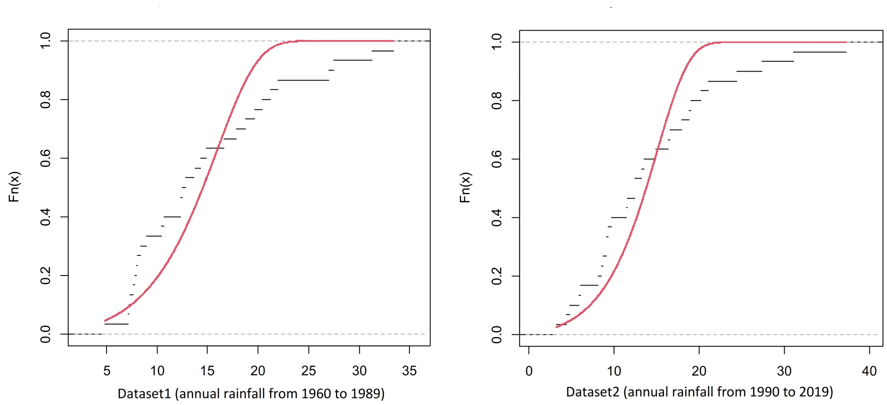

Figure 6: The empirical distribution function (dashed) and fitted distribution function for Datasets 1 and 2.

reasonably well to the both datasets. It is clear that we cannot reject the null hypothesis that the two shape parameters are equal. Figures 6 confirms this result

Table 4: Location parameter, shape parameter, K-S distances and p-values of the fitted Gompertz distribution to Datasets 1 and 2.

Dataset	Location Parameter	Shape Parameter	K-S	p-value
1	0.2459	0.0049	0.1843	0.2297
2	0.2659	0.0049	0.2014	0.1525

For the above data, we observe that the upper record values are as follows:

$\underset{\sim}{r}$: 4.85, 18.79, 20.44, 22.00, 27.47, 33.44,

$\underset{\sim}{s}$: 11.47, 21.00, 27.36, 31.01, 37.25.

Based on these record values, we consider the following two cases:

Case I : Under the assumption that the shape parameter is known, we take $\beta = 0.0049$ from Table 4. The MLEs of α_1 and α_2, are obtained from (9) as, 0.1651 and 0.1223, respectively. The MLE of R is $\hat{R} = 0.4255$ and the corresponding 95% confidence interval based on (10) is equal to (0.1802, 0.7284). Since we have no prior information, to obtain the approximate Bayes estimator of R, we use very small non-negative values of the hyper-parameters, i.e. $a_1 = a_2 = b_1 = b_2 = 0.0001$, as suggested by Congdom [8]. Therefore, we can conduct $\tilde{R}_B$ as 0.4310 and corresponding Bayesian 95% confidence interval based on (15) can be obtained as (0.2057, 0.7596).

Case II : Under the assumption that the shape parameter is known, the MLEs of β, α_1 and α_2 are 0.0054, 0.1637 and 0.1211 from (11) and (12), respectively. Therefore, the MLE of R is obtained as $\hat{R} = 0.4253$. The asymptotic 95% confidence interval of R (12) is obtained as (0.1352, 0.6903). Based on 1000 bootstrap samples, the 95% standard normal, percentile bootstrap and Student's t bootstrap confidence intervals constructed in subsection (2.4) are obtained as, (0.1143, 0.7028), (0.1167, 0.7139) and (0.0908, 0.6962), respectively. Moreover, based on $N = 10000$ samples and using hyper-parameters $a_i = b_i = 0.0001$ for $i = 1, 2, 3$, the approximate Bayes estimator of R becomes 0.4301 and the corresponding 95% HPD credible interval from (17) is equal to (0.1731, 0.6924).

6 Stochastic comparisons

In this section, strength-stress reliability in terms of sequences of records instead of the original variables is firstly derived. Then, a preservation property of a stochastic ordering which constructed by strength-stress reliability, is established for two random variables with Gompertz distributions. We assume that R_n and S_n are the nth upper records based on sequences of i.i.d random lifetimes $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ from distributions F known as Gompertz(α_1, β) and G known as Gompertz(α_2, β), respectively. From [3], recall that R_n and S_n have pdf's

$$f_{R_n}(r) = \frac{(\Lambda_F(r))^n}{n!} f(r), \;\; r > 0 \text{ and } \;\; f_{S_n}(s) = \frac{(\Lambda_G(s))^n}{n!} g(s), \;\; s > 0$$

where $\Lambda_F(r) = \int_0^r h_F(x)dx$ and $\Lambda_G(s) = \int_0^s h_G(y)dy$ are cumulative hazards of F and G, respectively, given by

$$\Lambda_F(r) = -\ln(\bar{F}(r)) \;\; \text{and} \;\; \Lambda_G(s) = -\ln(\bar{G}(s)).$$

To extend the quantity R evaluated before, to a more developed setting based on record values, $R^{'} = P(R_n > S_n)$ as the probability that R_n exceeds S_n could be obtained. In spirit of the Gompertz distribution, since for $\alpha_1 = \alpha_2$ it holds that $P(R_n > S_n) = 1/2$ and because $R^{'} = 1-P(S_n > R_n)$, thus without loss of generality, we assume that $\alpha_2 < \alpha_1$ as the other case will be derived by symmetry.

$$
\begin{aligned}
R^{'} &= \int_0^{\infty} F_{S_n}(r) f_{R_n}(r)\ dr \\
&= \int_0^{\infty}\int_0^{r} \frac{(\Lambda_G(s))^n}{n!} g(s) \frac{(\Lambda_F(r))^n}{n!} f(r)\ ds\ dr \\
&= \int_0^{\infty}\int_0^{\Lambda_G(r)} \frac{(y)^n e^{-y}}{n!} \frac{(\Lambda_F(r))^n}{n!} f(r)\ dy\ dr \\
&= \frac{(\alpha_1/\beta)^{n+1}}{(n!)^2} \int_0^{\infty} U^n(x) B(U(x)) e^{-(\alpha_1/\beta)U(x)}\ dU(x),
\end{aligned}
$$

where $B(u) = \int_0^{u\alpha_2/\beta} y^n e^{-y}\ dy$ and $U(x) = e^{\beta x} - 1$. By making the change of variable $v = U(x)$ and using the expansion of the incomplete gamma function,

$$
\begin{aligned}
R^{'} &= \frac{(\alpha_1/\beta)^{n+1}}{(n!)^2} \int_0^{\infty} v^n B(v) e^{-(\alpha_1/\beta)v}\ dv \\
&= \frac{(\alpha_1/\beta)^{n+1}}{(n!)^2} \int_0^{\infty} v^n (v\alpha_2/\beta)^{n+1} e^{-(\alpha_1/\beta)v} \sum_{k=0}^{\infty} \frac{(-v\alpha_2/\beta)^k}{k!(k+n+1)}\ dv \\
&= \frac{(\alpha_1\alpha_2)^{n+1}}{(n!)^2\beta^{2n+2}} \int_0^{\infty} \sum_{k=0}^{\infty} \frac{(-\alpha_2/\beta)^k}{k!(k+n+1)} v^{k+2n+1} e^{-(\alpha_1/\beta)v}\ dv \\
&= \frac{(\alpha_1\alpha_2)^{n+1}}{(n!)^2\beta^{2n+2}} \sum_{k=0}^{\infty} \frac{(-\alpha_2/\beta)^k}{k!(k+n+1)} (\alpha_1/\beta)^{-(k+2n+2)} \Gamma(k+2n+2) \\
&= \frac{(\alpha_2/\alpha_1)^{n+1}}{(n!)^2} \sum_{k=0}^{\infty} \frac{(-\alpha_2/\alpha_1)^k (k+2n+1)!}{k!(k+n+1)}.
\end{aligned}
$$

In view of (3), if X follows Gompertz(α_1, β) and Y follows Gompertz(α_2, β) then X and Y have proportional hazards as $h_G(x) = \theta h_F(x)$, in which $\theta = \alpha_2/\alpha_1$. The concept of inactivity time of an equipment at time of observation of its failure has attracted the attention of many researchers. The conditional random variables $X_{(t)} = (t - X \mid X \leq t)$ and $Y_{(t)} = (t - Y \mid Y \leq t)$ are called the inactivity times associated with X and Y, respectively. In hydrology the Gompertz distribution is applied to extreme events such as annual maximum one-day rainfalls and river discharges. Thus, in this case to evaluate water shortage, it may be useful to investigate the deficiency of water in days in which the the amount of water falling in rain is below a (necessary) threshold. The inactivity time is, therefore, useful for such an analysis in terms of inactivity time probability order ([1]).

Definition 6.1. *Let X and Y be non-negative continuous random variables denoting the lifetimes of two systems. The lifetime Y is said to be greater than X in inactivity probability (denoted by $X \leq_{ipr} Y$) if for all $t > 0$, $\widetilde{R}(t) \geq 1/2$.*

In the following theorem, we demonstrate that the ipr order passes from the maximum Gompertz variables into the variables themselves. Let $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ be two random (i.i.d) samples from Gompertz(α_1, β) and Gompertz(α_2, β), respectively, and let $X_{(n)}$ and $Y_{(n)}$ be the maximum order statistics in the specified samples, respectively. We can see that the pdf's of $X_{(n)}$ and $Y_{(n)}$ are, respectively, obtained as

$$f_{(n)}(\tau) = n\alpha_1 e^{\beta\tau} e^{-\frac{\alpha_1}{\beta}(e^{\beta\tau}-1)}(1 - e^{-\frac{\alpha_1}{\beta}(e^{\beta\tau}-1)})^{n-1}, \quad \tau > 0$$

and

$$g_{(n)}(\tau) = n\alpha_2 e^{\beta\tau} e^{-\frac{\alpha_2}{\beta}(e^{\beta\tau}-1)}(1 - e^{-\frac{\alpha_2}{\beta}(e^{\beta\tau}-1)})^{n-1}, \quad \tau > 0.$$

The cdf's of $X_{(n)}$ and $Y_{(n)}$ are, respectively, derived as

$$F_{(n)}(\tau) = (1 - e^{-\frac{\alpha_1}{\beta}(e^{\beta\tau}-1)})^n, \ \tau > 0$$

and

$$G_{(n)}(\tau) = (1 - e^{-\frac{\alpha_2}{\beta}(e^{\beta\tau}-1)})^n, \ \tau > 0.$$

Theorem 6.2. *In the Gompertz distribution, $X_{(n)} \leq_{ipr} Y_{(n)}$ implies $X \leq_{ipr} Y$.*

Proof. It holds that $X_{(n)} \leq_{ipr} Y_{(n)}$, if and only if,

$$\int_0^x [g_{(n)}(\tau)F_{(n)}(\tau) - f_{(n)}(\tau)G_{(n)}(\tau)]\ d\tau \geq 0, \quad \text{for all } x > 0,$$

which is equivalent to the condition that for all $x > 0$,

$$\int_0^x [\alpha_2 e^{\beta\tau} e^{-\frac{\alpha_2}{\beta}(e^{\beta\tau}-1)}(v_2(\tau))^{n-1}(v_1(\tau))^n - \alpha_1 e^{\beta\tau} e^{-\frac{\alpha_1}{\beta}(e^{\beta\tau}-1)}(v_1(\tau))^{n-1}(v_2(\tau))^n]\ d\tau \geq 0.$$

where $v_i(\tau) = 1 - e^{-\frac{\alpha_i}{\beta}(e^{\beta\tau}-1)}$ for $i = 1, 2$. Let us denote, for all $x > 0$,

$$W(x) = \int_0^x [v_1(\tau)v_2(\tau)]^{n-1}(g(\tau)F(\tau) - f(\tau)G(\tau))\ d\tau.$$

Then, from what resulted above, one concludes that $\int_0^T dW(x) \geq 0$, for all $T > 0$. For each fixed $T > 0$, we further have

$$\int_0^T [g(\tau)F(\tau) - f(\tau)G(\tau)]\ d\tau \overset{sign}{=} \int_0^\infty h(\tau)\ dW(\tau),$$

where

$$h(\tau) = \begin{cases} \frac{1}{[v_1(\tau)v_2(\tau)]^{n-1}}, & \tau \leq T \\ 0, & \tau > T, \end{cases}$$

which is non-negative and non-increasing in τ, for all $\tau > 0$. The result now follows from Lemma 7.1(b) in [19]. □

7 Conclusion

The paper has achieved two goals. The first is the problem of estimation of stress-strength reliability in Gompertz distribution via classical and bayesian inference. Data analysis has been carried out by simulation and a real dataset. The second goal is to make some stochastic comparisons based on stress-strength of inactivity times of two Gompertz random variables.

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