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Original Research Paper

Specific Complete Measure in the Structure of a Utility Function

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Abstract. In this article, an outer measure is constructed on a pseudo-ordered set $(X, \succsim)$ and then it will be shown that it is in fact a measure defined on the whole power set of X . Applying this, a measurable utility function θ is defined which represents the relation $\succsim$ on X. Also, we discuss the continuity and upper semi-continuity of θ in certain points of X. Finally, the results are used to improve some of the theorems in economics.

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1 Introduction

Suppose that $(X, \succsim)$ is a pseudo-ordered space; that is, $\succsim$ is a binary relation on X with two following properties:

(i) Completeness: For all $x, y \in X$, either $x \succsim y$ or $y \succsim x$;

(ii) Transitivity: For all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.
The relation $x \succ y$ means $x \succsim y$, but not $y \succsim x$. Also $x \sim y$ means $x \succsim y$ and $y \succsim x$. For any $x_0 \in X$, consider the subsets $W(x_0) = \{x \in X; x_0 \succ x\}$, $B(x_0) = \{x \in X; x \succ x_0\}$ and $I(x_0) = \{x \in X; x \sim x_0\}$

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of X. The topology on X generated by the collections $\{W(y); y \in X\}$ and $\{B(y); y \in X\}$ is called the order topology. For each $x, y \in X$ the open interval (x, y) is defined by $(x, y) = \{z \in X; x \prec z \prec y\}$. The other intervals $[x, y]$, $(x, y]$ and $[x, y)$ are defined similarly. In the following, we will assume that X has a minimum element and it is unbounded from above.
A real-valued function $u : X \longrightarrow \mathbb{R}$ is called a utility function representing the relation $\succsim$, if it is an strictly increasing function with respect to the relation $\succsim$; that is, for all $x, y \in X$,

$$x \sim y \Rightarrow u(x) = u(y) \ and \ x \succ y \Rightarrow u(x) > u(y).$$

A function $u : X \longrightarrow \mathbb{R}$ is said to be upper semi-continuous if $\{x \in X; u(x) < r\}$ is open for each $r \in \mathbb{R}$. The relation $\succsim$ on X is called order separable if there is a countable set $D \subseteq X$ such that for all $x, y \in X$,

$$x \succ y \Rightarrow (\exists\ d_1, d_2 \in D; x \succsim d_1 \succ d_2 \succsim y).$$

For every $D \subseteq X$, the topology on X generated by the family $\{W(y); y \in D\}$ is called D-lower order topology. Also the topology generated by $\{W(y); y \in D\}$ and $\{B(y); y \in D\}$ is called D-order topology.

In the study of utility theory in economics, one of the topics discussed is the existence of a continuous utility function on a pseudo-ordered space $(X, \succsim)$; although, in many cases the weaker condition upper semi-continuity for the utility function works; See [4], [2] and [7]. recently, Voorneveld and Weibull constructed a utility function u based on specific outer measure μ^*; they proved the upper semi-continuity of u on D-lower order topology of X, [8]. Also, in [5], the combination of measure and utility theory have been used to represent an order relation.

In this paper, to improve the results of [8], an outer measure ν is constructed and it is shown that ν is in fact a complete measure on all power set of X. In the last section, as an application of this measure function in economics, we define a utility function θ based on ν and discuss about the continuity and upper semi-continuity of this function on specific subsets of X. Although, the structure of the function ν in this paper is very similar to the function μ^* in [8], it can help to use the properties of complete measure to achieve new results about the utility function θ. This is while, the outer measure μ^* is not accurate for measurement many subsets of X such as $B(x)$ and (x, y).

2 Introducing a Complete Measure

Suppose $\succsim$ is an order separable relation on X. The set D in the definition of order separability is countable, so we can give each $d \in D$ a positive weight, $\omega(d)$, such that weights have a finite sum. Put $\varepsilon_{t(d)} = \omega(d)$ in which $t : D \longrightarrow \mathbb{N}$ is an injection. Without loss of generality we assume that $\sum_{d\in D} \omega(d) = \sum_{d\in D} \varepsilon_{t(d)} = \sum_{i=1}^{\infty} \varepsilon_i = 1$; see [8].

Suppose that ν be a real-valued function on power set of X, 2^X, such that $\nu(\emptyset) = 0$, $\nu(X) = 1$ and for every nonempty proper subset A of X,

$$\nu(A) := \inf\{\sum_i \sum_{d^{'} \in D\cap A; d^{'} \precsim d_i} \omega(d^{'}); W_i = W(d_i) \in \mathrm{F} \text{ and } A \subseteq \bigcup_{i\in\mathbb{N}} W_i\}.$$

Without loss of generality we consider the set D consists of the minimum of X which will be denoted by $\tilde{d}$. In this case we assume that $\omega(\tilde{d}) = 0$ and hence $\nu(W(\tilde{d})) = 0$.

Proposition 2.1. *The function ν defines an outer measure on X.*

Proof. Suppose that $A \subseteq B \subseteq X$. Then $A \subseteq \cup W_i$ for each sequence $\{W_i\}$ in F with $B \subseteq \cup W_i$. Thus

$$\nu(A) \leq \sum_i \sum_{d^{'} \in D\cap A; d^{'} \precsim d_i} \omega(d^{'}) \leq \sum_i \sum_{d^{'} \in D\cap B; d^{'} \precsim d_i} \omega(d^{'})$$

for every such sequence $\{W_i\}$ and so $\nu(A) \leq \nu(B)$.
Now let $\{A_t\}$ be a sequence of subsets of X and for each $t \in \mathbb{N}$ let

$$\nu(A_t) = \inf\{\sum_i \sum_{d^{'} \in D\cap A_t; d^{'} \precsim d_i^t} \omega(d^{'}); W_i^t = W(d_i^t) \in \mathrm{F} \textit{ and } A_t \subseteq \bigcup_{i\in\mathbb{N}} W_i^t\}.$$

Then for every $\varepsilon > 0$ there exist a sequence $\{W_i^t\}_{i\in\mathbb{N}}$ such that

$$\sum_i \sum_{d^{'} \in D\cap A_t; d^{'} \precsim d_i^t} \omega(d^{'}) \leq \nu(A_t) + \frac{\varepsilon}{2^t}.$$

Therefore

$$\begin{aligned} \sum_i \sum_{d^{'} \in D\cap(\cup_t A_t); d^{'} \precsim d_i^t} \omega(d^{'}) &\leq \sum_i \sum_t \sum_{d^{'} \in D\cap A_t; d^{'} \precsim d_i^t} \omega(d^{'}) \\ &\leq \sum_t \nu(A_t) + \varepsilon. \end{aligned}$$

Since $\bigcup_{t\in\mathbb{N}} A_t \subseteq \bigcup_{i,t\in\mathbb{N}} W_i^t$,

$$\nu(\bigcup_{t\in\mathbb{N}} A_t) \leq \sum_t \nu(A_t) + \varepsilon \quad \forall\varepsilon > 0$$

and this completes the proof. $\square$

Recall that a set $A \subseteq X$ is called ν- measurable if for each $E \subseteq X$,

$$\nu(E) = \nu(E \cap A) + \nu(E \cap A^c).$$

By the Caratheodory's theorem, [1, Theorem 1.11], the collection $\mathcal{M}$ of ν-measurable sets is a σ-algebra and the restriction of ν to $\mathcal{M}$ is a complete measure.The elements of $\mathcal{M}$ are called measurable sets.

Proposition 2.2. *Every subset of X is a ν-measurable set.*

Proof. Suppose that E is an arbitrary subset of X. Then for each $A \subseteq X$ and for $\varepsilon > 0$, there exist a sequence $\{W_i\} \subseteq$ F such that

$$\nu(A) + \varepsilon \geq \sum_i \sum_{d' \in D\cap A; d' \precsim d_i} \omega(d')$$

for which $A \subseteq \cup W_i$ and $W_i = W(d_i)$ for all $i \in \mathbb{N}$.
Put $A_1 = A \cap E$ and $A_2 = A \cap E^c$. Then $A_1, A_2 \subseteq \cup W_i$ and so

$$\begin{aligned} \nu(A) + \varepsilon &\geq \sum_i \sum_{d' \in D\cap A\cap E; d' \precsim d_i} \omega(d') + \sum_i \sum_{d' \in D\cap A\cap E^c; d' \precsim d_i} \omega(d') \\ &= \sum_i \sum_{d' \in D\cap A_1; d' \precsim d_i} \omega(d') + \sum_i \sum_{d' \in D\cap A_2; d' \precsim d_i} \omega(d') \\ &\geq \nu(A_1) + \nu(A_2). \end{aligned}$$

Since this hold for every $\varepsilon > 0$, thus

$$\nu(A) \geq \nu(A \cap E) + \nu(A \cap E^c) \quad \forall A \subseteq X$$

Therefore E is ν-measurable. $\square$

Corollary 2.3. *ν is a complete measure on 2^X.*

3 Applications in Economics

In terms of economics, the set $(X, \succsim)$ with the conditions mentioned in previous sections can be considered as a consumption set. In any model of consumer choice, a consumption set is the set of all alternatives that the consumer can conceive. In this case, X is considered as a closed convex subset of $\mathbb{R}_+^n$ which contains 0. The pseudo-ordering relation $\succsim$ is called the preference relation and the sets $W(x_0)$, $B(x_0)$ and $I(x_0)$ are called the sets of elements 'worse than x_0 ', 'preferred to x_0' and 'indifference to x_0', respectively. For more information about these economic terms one can refer to [3].

In this section we suppose that $\succsim$ is an order separable relation on X. Define the function θ from X into $\mathbb{R}$ in the form

$$\theta(x) := \nu(W(x)) \quad \forall x \in X$$

and put $I(D) = \{x \in X; x \sim d \text{ for some } d \in D\}$. In the following, we examine some of the properties of θ as a utility function .

Theorem 3.1. *The mapping θ is a measurable utility function represents $\succsim$ on X and is continuous on $X \setminus I(D)$ in the D-order topology of X.*

Proof. Note that for each $d \in D$, $\theta(d) = \sum_{d' \in D \cap W(d)} \omega(d')$. Hence for each $x, y \in X$ if $x \sim y$, then $W(x) = W(y)$ and so $\theta(x) = \theta(y)$; if $x \prec y$, there are $d_1, d_2 \in D$ with $x \precsim d_1 \prec d_2 \precsim y$ and so

$$\theta(x) = \nu(W(x)) \leq \nu(W(d_1)) < \nu(W(d_2)) \leq \nu(W(y)) = \theta(y);$$

that is, θ represents $\succsim$ on X. Now for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that

$$\forall m, n \in \mathbb{N} \qquad m > n \geq k \Rightarrow \sum_{i=n+1}^{m} \varepsilon_i < \varepsilon.$$

Suppose $x \in X \setminus I(D)$. By order separability of $\succsim$, the sets $W(x) \cap D$ and $B(x) \cap D$ are infinite and for each $d_1 \in W(x) \cap D$ and $d_2 \in B(x) \cap D$ there exist $d_3, d_4 \in D$ such that $d_1 \prec d_3 \prec x \prec d_4 \prec d_2$. Since there are only finitely many $d' \in D$ with $t(d') < k$ we can choose $d_1 \in W(x) \cap D$ and $d_2 \in B(x) \cap D$ such that $t(d') \geq k$ for each $d' \in D$ with $d_1 \precsim d' \prec d_2$.

Put $V = B(d_1) \cap W(d_2) = (d_1, d_2)$. Trivially V is an open set in D-order topology contains x. Let $y \in V$; then

$$\begin{aligned} |\theta(y) - \theta(x)| &< \theta(d_2) - \theta(d_1) = \nu(W(d_2)) - \nu(W(d_1)) \\ &= \sum_{d' \in D; d_1 \precsim d' \prec d_2} \omega(d') = \sum_{d' \in D; d_1 \precsim d' \prec d_2} \varepsilon_{t(d')} < \varepsilon. \end{aligned}$$

□

By the above theorem, θ is upper semi-continuous on $X \setminus I(D)$ in the D-order topology. In the next theorem, we discuss the upper semi-continuity of θ on the weaker topology, D-lower order topology.

Theorem 3.2. *The function θ is upper semi-continuous in the D-lower order topology in*

(i) any point x of $I(D)$ for which there exists $d' \in B(x) \cap D$ with $(x, d') = \emptyset$;

(ii) any point of $X \setminus I(D)$.

Proof. Suppose that $r \in \mathbb{R}$ and $\{x \in X; \theta(x) < r\}$ is a non trivial subset of X. If $x \in I(D)$ and there is a $d' \in B(x) \cap D$ with $(x, d') = \emptyset$, then

$$\{y \in X; y \precsim x\} = \{y \in X; y \prec d'\} = W(d')$$

is a neighborhood of x in D-lower order topology and for each $y \in W(d')$, $\theta(y) \leq \theta(x) < r$ and this proves (i).

Now let $x \in X \setminus I(D)$. Then there is $k \in \mathbb{N}$ such that for all $n \geq k$, $\sum_{i=n+1}^{\infty} \varepsilon_i < r - \theta(x)$. similar to the proof of the last theorem, there exist $d_1 \in W(x) \cap D$ and $d_2 \in B(x) \cap D$ such that $t(d') \geq k$ for all $d' \in D$ with $d_1 \precsim d' \prec d_2$. For each $y \in W(d_2)$,

$$\begin{aligned} \theta(y) < \theta(d_2) &= \sum_{d' \in D \cap W(d_2)} \omega(d') \\ &= \sum_{d' \in D; d' \prec d_1} \omega(d') + \sum_{d' \in D; d_1 \precsim d' \prec d_2} \omega(d') \\ &< \theta(d_1) + (r - \theta(x)) < \theta(x) + (r - \theta(x)) = r; \end{aligned}$$

that is, $W(d_2)$ is a neighborhood of x which is a subset of $\{x \in X; \theta(x) < r\}$; this completes the proof of (ii). □

The use of each of theorems 3.1 and 3.2, depends on the conditions of $(X, \succsim)$. For example, if $\succsim$ is upper semi-continuous, it is order separable [6].

We hope that applying θ as a measure function will lead to more desirable results about the utility function. The properties such as monotonicity, subadditivity, continuity from below, etc. can be applied to the function ν and examined what new results would be obtained for the utility function θ. For example, If $\{x_n\}$ is an increasing sequence in X (i.e., $x_1 \precsim x_2 \precsim x_3 \precsim ...$), then $\{W(x_n)\}$ is an increasing sequence of measurable sets in X and by the property "continuity from below" of measure, $\lim_n \theta(x_n) = \lim_n \nu(W(x_n)) = \nu(\bigcup_n W(x_n))$. In this case, if $\{x_n\}$ is in $X \setminus I(D)$ and converges to some element x_0 of this space, in D-order topology, then by theorem 3.1, $\theta(x_0) = \lim_n \theta(x_n) = \nu(\bigcup_n W(x_n))$. Also, by the monotonicity property of measure, since $\nu(X) = 1$, for every $x \in X$,

$$\theta(x) = \nu(W(x)) = \nu(X) - \nu(\bar{B}(x)) = 1 - \nu(\bar{B}(x)),$$

where, $\bar{B}(x) = B(x) \cup I(x)$ is called the set of elements "at least as good as x". In other words, the utility function θ can be obtained by measuring the set $\bar{B}(x)$, too.

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