Journal of Mathematical Extension
Vol. 15, No. 4, (2021) (9)1-11
URL: https://doi.org/10.30495/JME.2021.1576
ISSN: 1735-8299
Original Research Paper

A Conjecture on a Symmetric Diagonal Diophantine Equation of Degree Six

A. Jalali
Urmia University

A. S. Janfada*
Urmia University

H. Shabani-Solt
Urmia University

Abstract. In this paper, we conjecture that the symmetric diagonal Diophantine equation $x^6 + ky^3 + k'z^3 = u^6 + kv^3 + k'w^3$ has infinitely many nontrivial solutions for all rational numbers k and k'. This conjecture is proved for certain cases.

AMS Subject Classification: Primary: 11D45; Secondary: 11H52.

Keywords and Phrases: Counting solutions of Diophantine equations, elliptic curves.

Received: May 2020; Accepted: December 2020

*Corresponding Author

1 Introduction

The Diophantine equation of diagonal type

$$\sum_{i=1}^{k} A_i x_i^{m_i} = 0,\ \ A_i, m_i \in \mathbb{Z}, m_i > 0$$

is one of the historical classical equations. A number of the mathematicians have worked on this equation. Of well-known results among them are the generalized Fermat's theorem [1], Euler conjecture [5], Waring problem [3], and equal sums of like powers [4, 6].

By a symmetric equation, it is meant an equation of the form

$$f(x_1, x_2, \dots, x_n) = f(y_1, y_2, \dots, y_n),$$

where f is a symmetric polynomial of n unknowns with integer coefficients. In this article, we concentrate on the solutions of the symmetric diagonal Diophantine equations of the form

$$\sum_{i=1}^{k} A_i x_i^{m_i} = \sum_{i=1}^{k} A_i y_i^{m_i},$$

and consider the equations

$$Ax^6 + By^3 + Cz^3 = Au^6 + Bv^3 + Cw^3,\ \ A, B, C \in \mathbb{Z},\ \ A \neq 0.$$

For more details about symmetric diagonal equations we cite [2]. By dividing both sides of the above equation by A we get

$$x^6 + ky^3 + k'z^3 = u^6 + kv^3 + k'w^3,\ \ k, k' \in \mathbb{Q}.$$

In the case $k' = 0$, the rational points of the symmetric diagonal equation $x^6+ky^3 = u^6+kv^3,\ k \in \mathbb{Q}$ are studied in [7] in which, using the elliptic curve method, it is conjectured that this equation has infinitely many nontrivial rational solutions. The same strategy is used in [9] to claim the existence of infinitely many nontrivial rational solutions of the equation $x^5 + ky^3 = u^5 + kv^3$.

The aim of this article is to claim the following conjecture. We approach to this conjecture in two different ways in Section 3.

Conjecture 1.1. *For all rational numbers k and k', the symmetric diagonal Diophantine equation*

$$x^6 + ky^3 + k'z^3 = u^6 + kv^3 + k'w^3$$

has infinitely many nontrivial solutions.

In this article, all rank computations are implemented by the 'mwrank' software. Moreover, we assume that all solutions of the equation $x^6+ky^3+k'z^3 = u^6+kv^3+k'w^3$ are integral and nontrivial. Note that any rational solution leads to an integral solution. By a trivial solution of the equation $ax^m + by^n + cz^k = au^m + bv^n + cw^k$ we mean either of the following cases.

(i) $(x, y, z) = (u, v, w)$;

(ii) $x = r^{nk}, y = s^{mk}, z = t^{mn}, u = t^{nk}, v = r^{mk}, w = s^{mn}$, for some rationals r, s, t, when $a = b = c$.

2 Preliminary results

Let K be a field and C be the algebraic curve defined over K by

$$v^2 = au^4 + bu^3 + cu^2 + du + e, \ a \neq 0, \tag{1}$$

Consider the K-rational affine point $(u, v) = (p, q)$ on C. We may assume $p = 0$ by changing u to $u+p$, if necessary. Then $e = q^2$ and the equation (1) turns to

$$v^2 = au^4 + bu^3 + cu^2 + du + q^2, \ a \neq 0. \tag{2}$$

Let $q = 0$. If $d = 0$, the curve (2) will have a singularity at $(u, v) = (0, 0)$. Therefore, assume $d \neq 0$. Dividing both side of (2) by u^4 we get

$$\left(\frac{v}{u^2}\right)^2 = d\left(\frac{1}{u}\right)^3 + c\left(\frac{1}{u}\right)^2 + b\left(\frac{1}{u}\right) + a.$$

Put $X = 1/u$ and $Y = 1/u^2$ Then, we obtain the elliptic curve $Y^2 = dX^3 + cX^2 + bX + a$ in the Weierstrass form. The harder case is when $q \neq 0$ in which case we have the following result [8, Theorem 2.17].

Theorem 2.1. *Let K be a field of characteristic not 2 and C be the algebraic curve defined over K by*

$$C : v^2 = au^4 + bu^3 + cu^2 + du + q^2,\ q \neq 0.$$

Suppose C has a K-rational point (p, q). Let

$$X = \frac{2q(v + q) + du}{u^2},\ Y = \frac{4q^2(v + q) + 2q(du + cu^2) - (d^2u^2/2q)}{u^3}.$$

Define

$$a_1 = d/q,\ a_2 = c - (d^2/4q^2),\ a_3 = 2qb,\ a_4 = -4q^2a,\ a_6 = a_2a_4.$$

Then the curve C is in one to one corresponding with the elliptic curve

$$E : Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6.$$

The inverse transformation is

$$u = \frac{2q(X + c) - (d^2/2q)}{Y},\ v = -q + \frac{u(uX - d)}{2q}.$$

The point $(u, v) = (0, q)$ on C corresponds to the point $(X, Y) = \infty$ on E and $(u, v) = (0, -q)$ on C corresponds to $(X, Y) = (-a_2, a_1a_2 - a_3)$ on E.

In [7, Theorem 1.1], the authors showed that for a large number of values of k the equation

$$x^6 + ky^3 = u^6 + kv^3,\ k \in \mathbb{Q}. \tag{3}$$

has infinitely many nontrivial solutions; that is, $(x,y) \neq (u,v)$. Then they exhibited more collection of rational numbers k for which the equation is satisfied and due to these observations, they conjectured that for any rational number k, the equation (3) has infinitely many nontrivial solutions. Since part of the basic demonstration in the current paper is hanging on this result, we restate it with a bit difference and improve its proof by resolving a mistake.

Theorem 2.2. *For each integer k with $1 \leq k \leq 100$, the Diophantine equation*

$$x^6 + ky^3 = z^6 + kw^3$$

has infinitely many nontrivial solutions.

Following the proof of Theorem 1.1 in [7], the parametric system of changing variables

$$x = u + \frac{4}{k}s^2,\ y = v - \frac{u}{2},\ z = u - \frac{4}{k}s^2,\ w = u + y = v + \frac{u}{2}$$

leads to the quartic equation

$$v^2 = \big(\frac{4}{k}s\big)^2 u^4 + \big(\frac{10}{3}\big(\frac{4}{k}\big)^4 s^6 - \frac{1}{12}\big)u^2 + \big(\frac{4}{k}\big)^6 s^{10}.$$

Then, using Theorem 2.1 we get the elliptic curve

$$E_{k,s} : Y^2 = X^3 + a_2X^2 + a_4X + a_6, \tag{4}$$

over $\mathbb{Q}(k,s)$, where

$$a_2 = \frac{10}{3}\big(\frac{4}{k}\big)^4 s^6 - \frac{1}{12},\ a_4 = -4\big(\frac{4}{k}\big)^8 s^{12},$$
$$a_6 = -\frac{40}{3}\big(\frac{4}{k}\big)^{12} s^{18} + \frac{1}{3}\big(\frac{4}{k}\big)^8 s^{12}.$$

Table 1 shows the positive-rank elliptic curves (4) for $1 \leq k \leq 100$ and appropriate values of s. The computation for higher integer values of k is similar and not complicated. Even for rational values

k,s	rank r
1,1; 2,1; 3,1; 5,1/4; 6,1; 7,1; 9,1/2; 10,1/2; 11,1; 12,1; 13,1/2; 14,1; 15,1; 16,1/2; 18,1; 19,1; 20,2; 21,3/2; 22,2; 25,1; 28,1; 29,1/2; 30,1; 32,1; 35,5; 37,1; 38,1; 41,1/4; 43,2; 49,14; 53,1; 54,1; 55,5; 56,1; 58,3; 62,1; 64,1; 65,4; 69,3; 71,1; 73,1/4; 75,1; 79,5; 82,4; 83,1; 85,2; 90,3; 92,2; 93,3; 94,2; 95,1; 98,3; 100,2; **1/2,1; 2/3,2; 2/7,5; 1/16,6; 4/7,8; 5/3,10**	$r = 1$
4,4; 17,1/2; 23,3/4; 24,1/3; 26,1; 27,1/3; 31,2; 33,3; 34,3; 36,1; 39,3; 40,6; 42,1; 44,2; 45,1; 46,7; 47,7; 48,5; 50,4; 51,3; 52,4; 57,7; 59,1; 60,7; 61,2; 63,5; 67,4; 68,4; 70,1; 72,5; 74,6; 76,13; 77,4; 80,3; 81,3; 84,3; 86,3; 87,2; 88,7; 89,3; 91,6; 96,8; 97,4; 99,1;**3/7,3; 2/9,4; 3/4,9**	$r = 2$
8,14; 66,3; 78,13	$r = 3$
3/5,7	$r = 4$

Table 1: Positive ranks for the elliptic curve $E_{k,s}$.

of k there are positive-rank elliptic curves. Some of these elliptic curves are denoted in Table 1 in bold fonts.

Of course, one can transform the elliptic curve (4) into the following short Weierstrass form and then attempt to find the positive-rank elliptic curves for that. The transformation can be done either by software (such as maple, sage, etc.) or using the manipulation formulas (see for example [8, Section 2.1]).

$$E_{A,B} : Y^2 = X^3 + AX + B,$$

where

$$\begin{aligned} A &= -\frac{218103808\, s^{12} - 20480\, k^4 s^6 + k^8}{432 k^8}, \\ B &= -\frac{(8192\, s^6 + k^4)\,(-28672\, s^6 + k^4)\,(-10240\, s^6 + k^4)}{23328\, k^{12}}. \end{aligned}$$

3 Settling Conjecture 1.1

In this section, we settle Conjecture 1.1. The next result shows that the sufficient condition for the equation

$$x^6 + ky^3 + k'z^3 = u^6 + kv^3 + k'w^3, \;\; k, k' \in \mathbb{Q}.$$

to have infinitely many solutions is that it has the same property for the case $k' = 0$.

Proposition 3.1. *If the the equation*

$$X^6 + \ell Y^3 = U^6 + \ell V^3 \tag{5}$$

has infinitely many solutions for any nonzero rational number ℓ, then the equation

$$x^6 + ky^3 + k'z^3 = u^6 + kv^3 + k'w^3$$

has also infinitely many solutions for nonzero rational numbers k, k'.

Proof. Let k and k' be rational numbers. The hypothesis guarantees the existence of infinitely many solutions of the equation (5) for $\ell = kt^3 + k't'^3$, where t and t' are rational numbers. That is to say that,

$$X^6 + k(tY)^3 + k'(t'Y)^3 = U^6 + k(tV)^3 + k'(t'V)^3$$

has infinitely many solutions. The result is now follows. □

Now, Conjecture 1.1 is settled by Theorem 2.2.

Corollary 3.2. *Let k, k', t, t' be any rational number such that $kt^3 + k't'^3$ is an integer with $1 \leq kt^3 + k't'^3 \leq 100$. Then, the equation*

$$x^6 + ky^3 + k'z^3 = u^6 + kv^3 + k'w^3$$

has infinitely many solutions.

As an alternative way, we settle Conjecture 1.1 directly for some values of k and k' by a computational process.

Proposition 3.3. *For any positive integers k, k' with $1 \leq k \leq 50$, $1 \leq k' \leq 20$, the equation*

$$x^6 + ky^3 + k'z^3 = u^6 + kv^3 + k'w^3, \;\; k, k' \in \mathbb{Q}. \tag{6}$$

has infinitely many solutions.

k,k',s	rank r
1,4,1; 1,5,1; 1,7,1; 1,8,1; 1,9,4; 1,15,1; 2,1,1; 2,2,1; 2,3,1; 2,6,2; 2,7,2; 2,20,1/4; 3,1,2; 3,4,3; 3,9,1; 3,11,1; 3,14,4; 3,17,1; 3,20,1; 4,6,2; 4,7,2; 4,10,3; 4,14,1; 4,16,2; 4,19,3; 4,20,3; 5,3,1; 5,4,1; 5,5,2; 5,11,1/4; 5,12,2; 5,13,1; 5,16,1; 5,17,1; 6,4,1; 6,5,1; 6,9,1; 6,13,1/5; 6,14,2; 6,18,11; 7,3,1; 7,4,1; 7,13,1; 7,15,1; 7,18,1; 8,2,1; 8,3,1; 8,13,2; 8,16,1; 9,1,1; 9,2,1; 9,6,1; 9,7,1; 9,8,1/2; 9,9,1; 9,10,1/4; 9,11,1; 9,12,1; 9,15,1; 9,19,1; 10,4,1; 10,7,1; 10,15,1; 10,20,1; 11,14,1; 11,15,13; 11,20,2; 12,3,1; 12,9,2; 12,10,2; 12,14,7; 12,19,2; 13,3,3; 13,5,2; 13,8,1; 13,11,2/3; 13,12,1; 13,20,2; 14,1,4; 14,7,1/2; 14,11,6; 14,15,1; 14,18,1; 14,20,1; 15,1,2; 15,2,7; 15,6,1; 15,9,2; 15,19,1; 16,1,1; 16,8,1; 16,9,1; 16,12,1; 16,13,2; 16,14,1; 16,16,1; 16,18,2; 17,10,1; 17,12,2; 17,13,5; 17,14,7/12; 18,9,3; 18,11,2; 18,15,3/5; 18,16,1; 18,19,1; 19,1,1; 19,6,10; 19,15,6/5; 19,17,1/17; 20,2,3; 20,11,12; 20,13,1; 20,15,2; 20,19,2; 21,3,2; 21,8,4; 21,10,1; 21,11,3; 21,16,1; 21,20,1; 22,5,2; 22,11,1; 22,16,2; 23,5,3; 23,6,2; 23,8,2; 23,12,1/5; 23,14,2; 23,18,2; 23,19,11; 24,1,2; 24,3,2; 24,12,2; 24,13,1; 24,17,1; 24,18,1; 25,6,2; 25,7,1; 25,9,1; 25,12,2; 25,14,2; 25,17,1/2; 25,19,1; 26,7,7; 26,8,2; 26,9,1; 26,12,1; 26,15,1/5; 26,18,1; 26,19,11/2; 26,20,1; 27,1,1; 27,2,2; 27,4,5; 27,9,7; 27,10,3/2; 27,13,1; 27,16,1; 27,17,1/4; 27,19,37/76; 28,5,2; 28,10,4; 28,11,1; 28,13,2; 28,17,5; 29,6,1; 29,8,8; 29,12,2; 29,14,1/5; 29,15,1; 29,18,2; 29,20,2; 30,1,1; 30,2,1; 30,3,1; 30,8,1; 30,14,1; 30,15,2; 30,16,7; 31,12,1; 31,13,1; 31,16,1; 31,17,3; 31,18,2; 32,3,6; 32,10,2; 32,14,1; 32,16,1; 32,17,1; 32,19,1/19; 33,1,34; 33,2,1; 33,4,1; 33,5,1/5; 33,7,2; 33,14,1/2; 33,16,1/2; 34,6,1; 34,11,20; 34,12,1/2; 34,13,2; 34,17,1; 34,18,1; 35,4,1; 35,8,2; 35,9,3; 35,10,2; 35,17,1; 36,2,6; 36,4,1; 36,5,2; 36,13,2; 36,16,2; 36,20,1/2; 37,5,2; 37,9,1; 37,16,2; 37,18,1; 38,3,3; 38,12,1; 38,14,2; 38,19,2; 39,3,7; 39,4,1; 39,7,1; 39,11,1; 39,14,2; 39,16,1; 39,17, 11/8; 39,20,2; 40,5,4; 40,6,3; 40,7,1; 40,13,1; 41,2,1; 41,4,1; 41,5,4; 41,9,2; 41,9,2; 41,13,1; 42,1,4; 42,3,2; 42,7,1; 42,11,2; 42,15,7; 42,18,1; 42,20,2; 43,6,1; 43,8,3; 43,15,1; 43,16,3/8; 43,18,7; 44,3,5; 44,12,1/12; 44,13,2; 44,15,1/12; 44,16,2; 44,19,4/19; 45,4,1; 45,9,1; 45,11,2; 45,12,2; 45,13,1; 45,14,4; 45,15,5; 45,18,2; 45,20,1; 46,7,2/3; 46,10,2; 46,11,2; 46,16,2; 46,17,4/15; 47,3,1; 47,6,3; 47,8,3/2; 47,15,2; 47,16,8; 48,1,1; 48,5,2; 48,6,5; 48,9,5; 48,12,1/36; 48,13,2; 48,20,1; 49,2,4; 49,4,2; 49,8,2; 49,12,2; 49,13,9/17; 49,16,5; 49,17,7; 49,18,2; 50,2,2; 50,8,7; 50,11,1/5; 50,16,9/8; 50,18,3	$r = 1$
1,10,1; 1,16,16; 1,18,6; 3,2,1/4; 3,10,7; 3,15,2; 3,16,3; 4,5,6; 4,13,38; 4,15,3; 4,17,12; 6,8,6; 6,16,7; 7,2,10; 7,16,2; 8,14,3/5; 9,18,10; 9,20,1/17; 10,5,7; 10,14,1/5; 10,16,4; 10,18,5; 11,10,1; 11,12,1/18; 11,13,6; 12,5,8; 12,12,7; 12,15,4; 12,18,1; 13,2,1; 13,15,4; 13,18,3; 14,10,7; 14,12,9; 14,16,1/6; 15,7,3/4; 15,10,2; 15,12,9; 15,14,12; 16,15,1; 16,19,10; 17,6,8; 17,9,7; 18,13,11/5; 18,17,10; 19,13,7; 19,20,4; 20,4,8; 20,8,10; 21,4,3; 21,9,5; 21,12,5/2; 21,14,28/3; 22,13,1/2; 22,14,4; 23,2,7; 23,13,2; 23,15,3; 24,11,1/2; 24,19,3; 25,8,4; 25,10,10; 25,13,2; 25,16,1/3; 26,10,5; 27,8,1; 28,2,1; 28,8,16; 28,12,7; 28,14,12; 29,2,3; 29,3,2; 29,9,8; 29,13,7; 30,5,10; 30,11,7/33; 30,17,4; 31,1,7; 31,2,7; 31,8,3; 31,9,6; 31,14,2/5; 31,20,1/18; 32,8,3; 32,11,11; 33,3,5; 33,18,1/12; 35,11,7; 35,19,7; 36,3,4; 36,10,8; 36,12,11; 36,14,6; 36,17,8/9; 37,4,11; 37,6,3; 37,10,3; 37,12,3; 37,13,4; 38,6,10; 38,8,2; 38,11,1/6; 38,15,7; 38,16,17; 38,17,2; 38,20,9; 39,15,4; 39,18,9; 39,19,3; 40,8,31; 40,11,4; 40,12,9; 40,19,3/2; 40,20,6; 41,10,8; 41,16,16; 41,17,5; 41,18,1/15; 42,4,9; 42,6,9; 43,3,8; 43,7,3; 43,10,4/5; 43,12,2; 43,14,25/4; 43,19,4; 44,11,16; 44,14,2; 44,17,3; 44,20,5; 45,19,1/19; 46,8,4; 46,14,1/4; 46,15,17/8; 47,4,8; 47,13,16; 48,11,13/11; 48,17,4; 48,18,15; 49,6,5; 49,10,1; 49,11,7/8; 50,7,20; 50,17,2/17; 50,19,4/5	$1 \leq r \leq 2$
1,12,1; 1,17,2; 1,20,19; 2,4,2; 2,5,5; 2,9,6; 3,19,5; 4,11,2; 5,6,1; 5,10,4; 5,19,6; 6,11,3; 6,17,1/2; 7,5,8; 8,10,7; 8,17,1/22; 9,13,7; 10,19,7/18; 11,17,2; 12,17,2; 13,10,3/4; 13,19,7; 14,14,4; 14,17,2; 15,8,10; 15,13,2; 15,18,12; 16,6,4; 16,17,13; 18,3,8; 18,18,1; 18,20,2; 19,3,5; 19,8,10; 19,11,1; 20,7,4; 20,9,4; 20,16,4; 21,17,11/8; 21,18,1; 22,1,4; 22,2,10; 22,7,4; 22,9,41; 22,10,3; 22,17,10; 22,19,25/19; 22,20,11; 23,7,1/17; 23,10,5; 23,11,7; 24,2,6; 24,8,10; 24,16,6; 25,11,13; 25,20,5; 26,3,7; 26,5,3/10; 26,11,6; 27,3,2/3; 27,5,2; 27,6,37; 27,14,5/4; 27,18,2; 27,20,5; 29,11,21/11; 29,17,1; 30,4,5; 30,19,1; 30,20,2; 31,6,8; 31,10,4; 31,19,1; 32,15,1; 33,11,8; 33,17,4; 34,1,3; 34,7,1; 34,8,3; 34,14,3; 34,15,2; 35,18,1/8; 36,7,6; 36,11,7; 36,18,3; 37,19,15; 38,2,9; 39,6,1; 39,8,6; 40,2,5; 40,9,30; 40,10,8; 40,14,2; 40,17,4/17; 40,18,1; 41,3,10; 41,11,5; 41,14,1/4; 42,19,7; 43,13,8; 43,20,14; 44,6,4; 45,5,4; 45,7,10; 45,8,26; 45,16,1; 46,4,10; 46,5,2; 46,9,14; 46,19,1; 47,5,3; 47,7,8; 47,11,3/5; 47,17,15; 47,18,9; 49,5,10; 49,20,1/2; 50,1,10; 50,3,6; 50,14,9/2	$1 \leq r \leq 3$
13,14,17; 21,19,3; 23,17,1/25; 30,13,1/19; 32,4,12; 33,19,15; 45,17,14/31; 48,10,9	$1 \leq r \leq 4$
1,1,7; 1,2,2; 1,3,3; 1,6,1; 1,13,1/3; 1,14,1/34; 1,19,2; 2,8,2; 2,10,2; 2,13,1; 2,14,1/4; 2,15,1; 2,16,1; 2,17,1/4; 2,18,2; 3,3,2; 3,5,1; 3,6,5; 3,7,1; 3,8,2; 3,18,1; 4,1,5; 4,3,2; 4,4,2; 4,8,3; 4,9,2; 4,12,2; 4,18,2; 5,1,2; 5,7,1/8; 5,8,1/3; 5,9,8; 5,14,1; 5,18,1/3; 6,1,10; 6,6,2; 6,7,5/3; 6,10,1; 6,15,2; 6,19,4; 6,20,7; 7,6,1; 7,7,1; 7,9,2; 7,10,4; 7,11,2; 7,14,1; 7,17,11; 7,20,4; 8,4,1; 8,5,7; 8,6,6; 8,7,7; 8,8,7; 8,9,5; 8,15,2; 8,18,1; 8,19,2; 8,20,3; 9,3,7; 9,4,1; 9,5,2; 9,14,1; 9,16,1/10; 9,17,1; 10,1,1; 10,2,2; 10,3,2; 10,6,6; 10,8,1; 10,9,2; 10,10,5; 10,12,1; 10,13,2; 10,17,1; 11,1,7; 11,2,2; 11,3,3; 11,4,5; 11,5,2; 11,6,2; 11,7,1; 11,8,7; 11,9,1; 11,11,17; 11,16,2; 11,18,2; 12,1,2; 12,2,2; 12,4,5; 12,6,1; 12,7,2; 12,8,1; 12,11,7; 12,13,1; 12,16,1; 13,1,2; 13,4,3; 13,7,1; 13,9,5; 13,13,1; 13,16,2; 14,2,2; 14,3,3; 14,6,6; 14,8,7; 14,9,1; 14,13,2; 15,3,1; 15,4,1; 15,5,2; 15,11,2; 15,16,3; 15,17,3; 15,20,1/16; 16,3,2; 16,4,9; 16,5,1; 16,7,2; 16,11,1; 16,20,2; 17,1,4; 17,2,2; 17,3,5; 17,4,4; 17,5,1; 17,11,1; 17,15,7; 17,16,2; 17,17,2; 17,18,2; 17,19,5; 17,20,2; 18,1,2; 18,2,4; 18,4,1; 18,5,1/10; 18,6,5; 18,7,5; 18,8,1; 18,10,1; 18,12,5; 18,14,2; 19,4,7; 19,5,2; 19,7,1; 19,10,2; 19,14,2; 19,16,1/5; 19,18,7; 19,19,9; 20,1,2; 20,3,4; 20,5,2; 20,6,3; 20,10,2; 20,12,3; 20,18,4; 20,20,1; 21,1,7; 21,2,7; 21,5,1; 21,7,1; 21,13,1; 21,15,2; 22,3,1; 22,4,9; 22,6,2; 22,8,4; 22,12,2; 22,15,1; 22,18,4; 23,3,1; 23,4,1/8; 23,9,2; 23,16,2; 24,4,4; 24,5,3; 24,7,4; 24,9,7; 24,19,2; 24,14,2; 24,15,4; 24,20,2; 25,1,1; 25,2,7; 25,3,5; 25,4,1; 25,5,2; 25,15,2; 25,18,1/12; 26,1,2; 26,2,7; 26,4,1; 26,6,2; 26,13,2; 26,14,3; 26,16,1; 26,17,11/17; 27,7,1; 27,11,1/3; 27,15,3; 28,1,10; 28,3,2; 28,4,2; 28,6,3; 28,9,2; 28,15,2; 28,18,6; 28,19,2; 29,1,3; 29,5,4; 29,7,5; 29,10,2; 29,16,1; 29,19,1; 30,6,2; 30,7,2; 30,18,2; 31,3,2; 31,4,4; 31,5,1; 31,7,1; 31,11,2; 31,15,4; 32,1,2; 32,2,2; 32,6,4; 32,9,5; 32,12,3; 32,13,2; 32,18,1/6; 32,20,2; 33,6,2; 33,8,2; 33,9,3; 33,10,1/4; 33,12,9; 33,13,7; 33,15,1; 33,20,1/2; 34,2,4; 34,5,7; 34,9,2; 34,10,1; 34,16,1; 34,19,2; 34,20,1/2; 35,1,2; 35,2,2; 35,3,1; 35,5,5; 35,6,2; 35,7,1; 35,12,1; 35,13,2; 35,14,3; 35,15,1; 35,16,2; 35,20,1; 36,6,2; 36,8,1; 36,15,2; 36,19,2; 37,1,3; 37,8,2; 37,11,1; 37,14,1/7; 37,20,1/2; 38,1,3; 38,4,1; 38,5,2; 38,7,2; 38,9,1/18; 38,10,2; 38,13,2; 38,18,1; 39,1,7; 39,5,5; 39,9,3; 39,10,4; 40,1,2; 40,3,10; 40,4,5; 40,15,2; 40,16,2; 41,1,3; 41,6,3; 41,12,1; 41,15,2; 41,19,2; 41,20,2; 42,2,4; 42,5,2; 42,8,1; 42,9,2; 42,10,2; 42,12,6; 42,17,1; 43,4,1; 43,5,1; 43,9,1; 43,11,7; 43,17,1/4; 44,1,10; 44,4,3; 44,5,4; 44,8,2; 44,9,2; 44,10,1; 44,18,1; 42,13,2; 42,14,4; 45,1,4; 45,2,1; 45,3,1; 45,6,2; 45,10,1; 46,2,10; 46,3,3; 46,12,7; 46,13,43/39; 46,20,8; 47,1,11; 47,9,5; 47,10,2; 47,12,2; 47,14,1/2; 47,20,1; 48,2,10; 48,3,5; 48,4,3; 48,7,10; 48,8,8; 48,14,1; 48,15,2; 48,19,2; 49,1,10; 49,33,3; 49,7,1; 49,9,2; 49,14,8; 49,15,3; 50,4,5; 50,5,2; 50,6,9; 50,9,1; 50,10,1; 50,12,2; 50,20,2	$r = 2$
7,5,4; 13,17,1; 14,19,7; 17,8,14; 23,20,2; 28,16,5; 28,20,7; 32,5,4; 37,3,15; 37,15,21/5; 41,7,8; 41,8,4	$2 \leq r \leq 3$
2,19,1/22; 3,12,6; 4,2,10; 5,2,6; 6,3,10; 7,19,1/3; 20,14,3; 24,6,4; 27,12,7; 32,7,7; 42,16,1/24; 48,16,1/24	$2 \leq r \leq 4$
1,11,7; 2,11,7; 2,12,2; 3,13,4; 5,15,2; 6,2,2; 6,12,5; 7,1,2; 7,12,3; 8,1,8; 8,11,2; 8,12,9; 10,11,10; 11,19,3/2; 12,20,2; 13,6,2; 14,4,3; 14,5,2; 15,15,1/3; 16,2,7; 17,7,1; 19,2,2; 19,12,2; 20,17,2; 21,6,8; 23,1,1; 28,7,10; 29,4,4; 30,12,2; 34,3,2; 34,4,10; 36,1,2; 36,9,9; 37,2,2; 37,17,7; 39,2,7; 39,12,3; 39,13,2; 43,1,7; 43,2,6; 44,2,2; 44,7,4; 46,1,2; 46,6,4; 46,18,2; 47,2,22; 49,19,3/19; 50,13,4; 50,15,4	$r = 3$
16,10,7; 37,7,7	$r = 4$

Table 2: Positive ranks for the elliptic curve $E_{(k,k',s)}$.

Proof. Intersect the variety (6) with the hyperplanes

$$x = tg + 1, \; y = 1 - g, \; z = h - g,$$
$$u = tg - 1, \; v = 1 + g, \; w = h + g,$$

where $g \neq 0$. Then we get the following quartic equation

$$h^2 = \frac{2t^5}{k'} g^4 + \frac{1}{3k'}(20t^3 - k - k')g^2 + \frac{2t - k}{k'}.$$

Put $\frac{2t-k}{k'} = s^2$. Then, by Theorem 2.1, we obtain the following elliptic curve over $\mathbb{Q}(k, k', s)$.

$$E_{(k,k',s)} : Y^2 = X^3 + a_2 X^2 + a_4 X + a_6,$$

where

$$a_2 = \frac{1}{6k'} \left(5(s^2k' + k)^3 - 2k - 2k'\right), \; a_4 = \frac{-s^2}{4k'}(s^2k'+k)^5, \; a_6 = a_2 a_4.$$

As depicted in Table 2, for each pair of the 1000 values of k, k' in Proposition 3.3, a positive-rank elliptic curve $E_{(k,k',s)}$ is found for some s. This settles once more Conjecture 1.1. $\square$

Acknowledgements
The authors would like to thank the helpful comments and suggestion of the referee(s) which leads to improve the paper.

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Azar Jalali
Department of Mathematics
PhD student
Urmia University, Urmia, Iran
E-mail: azarjalali1360@gmail.com

Ali S. Janfada
Department of Mathematics
Associate Professor of Mathematics
Urmia University, Urmia, Iran
E-mail: asjanfada@gmail.com; a.sjanfada@urmia.ac.ir

Hassan Shabani-Solt
Department of Mathematics
Assistant Professor of Mathematics
Urmia University, Urmia, Iran
E-mail: h.shabani.solt@gmail.com