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Integral Closure of a Filtration Relative to an Injective Module

F. Dorostkar
University of Guilan

Abstract. In this paper we will introduce the integral closure of a filtration relative to an injective module.

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1 Introduction

Throughout this paper R denotes a commutative Noetherian ring with identity. Further $\mathbf{N}$ and $\mathbf{N}_0$ will denote the set of natural integers and non-negative integers respectively. Also $\mathbf{Z}$ will denote the set of integer numbers. Further E is an injective $R-$module.

The ideas of reduction and integral closure of an ideal in a commutative Noetherian ring R (with identity) were introduced by Northcott and Rees in [3]. It is appropriate for us to recall these definitions.

Let I and J be ideals of a commutative Noetherian ring R. The ideal I is a reduction of the ideal J if $I \subseteq J$ and there exists an integer $n \in \mathbf{N}$ such that $IJ^n = J^{n+1}$. Also an element x of R is said to be integrally

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dependent on I if there exist a positive integer n and elements $c_k \in I^k$, $k = 1, ..., n$, such that

$$x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n = 0.$$

We know from [3], $x \in R$ is integrally dependent on I if and only if I is a reduction of the ideal $I + Rx$. Further, we know that the set of all elements of R which are integrally dependent on I is an ideal of R. This ideal is called the integral closure of I and is denoted by I^-.

Now let E be an injective $R-$module. In [2], H. Ansari Toroghy and R. Y. Sharp introduced integral closure of an ideal I of a commutative ring R relative to an injective $R-$module E.

Let I and J be ideals of R. The ideal I is said to be a reduction of the ideal J relative to E, if $I \subseteq J$ and there exists an integer $n \in \mathbf{N}$ such that $(0 :_E IJ^n) = (0 :_E J^{n+1})$. Also an element x of R is said to be integrally dependent on I relative to an injective $R-$module E, if there exists a positive integer n such that

$$(0 :_E \sum_{i=1}^{n} x^{n-i} I^i) \subseteq (0 :_E x^n).$$

We know from [2], an element x of R is integrally dependent on I relative to an injective $R-$module E, if and only if I is a reduction of the ideal $I + Rx$ relative to E. Moreover in [2], it is shown that the set of all elements of R which are integrally dependent on I relative to E is an ideal of R. This is denoted by $I^{*(E)}$ and is called the integral closure of I relative to E.

Here, we give some definitions and notations which will be helpful for us in the rest of the paper.

A filtration $\mathcal{F} = \{I_n\}_{n \geq 0}$ on R is a descending sequence of ideals I_n of R such that $I_0 = R$ and $I_n I_m \subseteq I_{n+m}$ for all $n, m \in \mathbf{N}_0$. Let $\mathcal{F} = \{I_n\}_{n \geq 0}$ and $\mathcal{G} = \{J_n\}_{n \geq 0}$ be two filtrations. We say $\mathcal{F} \subseteq \mathcal{G}$ if $I_n \subseteq J_n$ for every n. Also the filtration $\{I_n J_n\}_{n \geq 0}$ is denoted by $\mathcal{F}\mathcal{G}$.

The integral closure of a filtration $\mathcal{F} = \{I_n\}_{n \geq 0}$ is defined in [4]. For every $n \geq 0$, let J_n be the set of all $x \in R$ such that x satisfies an equation

$$x^k + a_1 x^{k-1} + \cdots + a_{k-1} x + a_k = 0$$

for a positive integer k and $a_i \in I_{ni}$. Then $\mathcal{F}^- = \{J_n\}_{n\geq 0}$ is a filtration such that $\mathcal{F} \subseteq \mathcal{F}^-$. The filtration $\mathcal{F}^- = \{J_n\}_{n\geq 0}$ is called the integral closure of the filtration $\mathcal{F} = \{I_n\}_{n\geq 0}$.

In this paper we will introduce the integral closure of a filtration relative to an injective module and study some related topics.

2 Auxiliary results

In this section we define the concepts of reduction and integral closure of a filtration relative to injective modules and prove some of their properties. We begin to remind some definitions.

Definition 2.1. (See [5, 2.1.3].) Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ and $\mathcal{G} = \{J_n\}_{n\geq 0}$ be filtrations on R. $\mathcal{F}$ is said to be a reduction of $\mathcal{G}$ if $\mathcal{F} \subseteq \mathcal{G}$ and there exists a positive integer d such that

$$J_n = \sum_{i=0}^{d} I_{n-i}J_i \quad for\ every\ n \geq 1.$$

Here, and throughout this paper, $I_i = R$ if $i \leq 0$.

Definition 2.2. (See [5, 2.1.4].) Let R be a Noetherian ring. A filtration $\mathcal{F} = \{I_n\}_{n\geq 0}$ on R is Noetherian in case there exists a positive integer d such that

$$I_n = \sum_{i=0}^{d} I_{n-i}I_i \quad for\ every\ n \geq 1.$$

Definition 2.3. Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ and $\mathcal{G} = \{J_n\}_{n\geq 0}$ be filtrations on R. Then $\mathcal{F}$ is said to be a reduction of $\mathcal{G}$ relative to an injective $R-$module E if $\mathcal{F} \subseteq \mathcal{G}$ and there exists a positive integer d such that

$$(0 :_E J_n) = (0 :_E \sum_{i=0}^{d} I_{n-i}J_i) \quad for\ every\ n \geq 1.$$

Remark 2.4. Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ and $\mathcal{G} = \{J_n\}_{n\geq 0}$ be filtrations on R. Let $\mathcal{F}$ be a reduction of $\mathcal{G}$ relative to an injective $R-$module E. Then there exists a positive integer d such that

$$(0:_E J_n) = (0:_E \sum_{i=0}^{d} I_{n-i}J_i) \quad for\ every\ n \geq 1.$$

Let $d < d'$. Since $\sum\limits_{i=d+1}^{d'} I_{n-i}J_i \subseteq J_n$, we have

$$(0:_E J_n) = (0:_E \sum_{i=0}^{d} I_{n-i}J_i) \cap (0:_E \sum_{i=d+1}^{d'} I_{n-i}J_i) = (0:_E \sum_{i=0}^{d'} I_{n-i}J_i).$$

Theorem 2.5. (See [2, 1.3]). *Let $\mathcal{F} = \{I_n\}_{n\geq 0}, \mathcal{G} = \{J_n\}_{n\geq 0}, \mathcal{H} = \{H_n\}_{n\geq 0}$, and $\mathcal{K} = \{K_n\}_{n\geq 0}$ be filtrations on R and let E be an injective $R-$module.*

(a) If $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H}$ and $\mathcal{F}$ is a reduction of $\mathcal{H}$ relative to E then $\mathcal{G}$ is a reduction of $\mathcal{H}$ relative to E.

(b) If $\mathcal{F}$ is a reduction of $\mathcal{G}$ relative to E and $\mathcal{G}$ is a reduction of $\mathcal{H}$ relative to E then $\mathcal{F}$ is a reduction of $\mathcal{H}$ relative to E.

(c) If $\mathcal{F}$ is a reduction of $\mathcal{G}$ relative to E and $\mathcal{H}$ is a reduction of $\mathcal{K}$ relative to E then $\mathcal{FH}$ is a reduction of $\mathcal{GK}$ relative to E.

Proof. (a) and (b) are clear.

(c) Since $\mathcal{F}$ is a reduction of $\mathcal{G}$ relative to E and $\mathcal{H}$ is a reduction of $\mathcal{K}$ relative to E then there are two positive integers d, d' such that for every $n \geq 1$,

$$(0:_E J_n) = (0:_E \sum_{i=0}^{d} I_{n-i}J_i)$$

and

$$(0:_E K_n) = (0:_E \sum_{t=0}^{d'} H_{n-t}K_t).$$

By Remark 2.4, we can assume $d = d'$. Then for every $n \geq 1$, we have

$$(0:_E J_nK_n) = ((0:_E J_n):_E K_n) = ((0:_E \sum_{i=0}^{d} I_{n-i}J_i):_E K_n)$$

$$= ((0 :_E K_n) :_E \sum_{i=0}^{d} I_{n-i}J_i)$$

$$= ((0 :_E (\sum_{i=0}^{d} I_{n-i}J_i)(\sum_{t=0}^{d} H_{n-t}K_t)).$$

It is easy to see that $(\sum\limits_{i=0}^{d} I_{n-i}J_i)(\sum\limits_{t=0}^{d} H_{n-t}K_t) = \sum\limits_{i=0}^{d} I_{n-i}H_{n-i}J_iK_i$. Thus we have

$$(0 :_E J_nK_n) = (0 :_E \sum_{i=0}^{d} I_{n-i}H_{n-i}J_iK_i) \quad \textit{for every } n \geq 1$$

and so $\mathcal{FH}$ is a reduction of $\mathcal{GK}$ relative to E. $\square$

Now we mention a useful notation from [2]. Let I be an ideal of R. For a subset $\mathcal{P}$ of $Spec(R)$, the notation $I(\mathcal{P})$ denotes (I if $I = R$ and), if I is proper, the intersection of those primary terms in a minimal primary decomposition of I which are contained in at least one member of $\mathcal{P}$. We know $I(\mathcal{P}) = \bigcap\limits_{P\in\mathcal{P}} I(\{P\})$ we shall abbreviate $I(\{P\})$ (for $P \in Spec(R)$) by $I(P)$. Note that $I(P)$ is just the contraction back to R of the extension of I to R_P under the natural ring homomorphism.

Remark 2.6. (See [2, 1.6].) Let $P \in Spec(R)$, I and J be ideals of R. Let $E = E(R/P)$. Then the following statements are equivalent:

(a) $(0 :_E I) \subseteq (0 :_E J)$;

(b) $IR_P \subseteq JR_P$;

(c) $I(P) \subseteq J(P)$.

Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on R. For every prime ideal P of R, $\{I_nR_P\}_{n\geq 0}$ is a filtration on R_P. We will denote this filtration on R_P by $\mathcal{F}_P$.

Lemma 2.7. *Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a Noetherian filtration on R. Then for every prime ideal P of R,*

$$(\mathcal{F}_P)^- = (\mathcal{F}^-)_P.$$

Proof. Let $\mathcal{F}^- = \{U_n\}_{n\geq 0}$ and $(\mathcal{F}_P)^- = \{H_n\}_{n\geq 0}$ where $\mathcal{F}^-$ and $(\mathcal{F}^-)_P$ are filtrations on R and R_P respectively. Let $\frac{x}{1} \in U_n R_P$. Then there exist $u \in U_n$ and $t \in R \setminus P$ such that $\frac{x}{1} = \frac{u}{t}$. Thus there exists $s \in R \setminus P$ such that $su = stx$. Since $u \in U_n$ we have $u^k \in \sum_{i=1}^{k} u^{k-i} I_{ni}$ and so $(su)^k \in \sum_{i=1}^{k} (su)^{k-i} I_{ni}$ for a positive integer k. But $su = stx$ implies that $(\frac{stx}{1})^k \in \sum_{i=1}^{k} (\frac{stx}{1})^{k-i} I_{ni} R_P$. Now by Remark 2.6 and $\sum_{i=1}^{k} (stx)^{k-i} I_{ni} \subseteq \sum_{i=1}^{k} x^{k-i} I_{ni}$, we have

$$(0 :_{E(R/P)} \sum_{i=1}^{k} x^{k-i} I_{ni}) \subseteq (0 :_{E(R/P)} \sum_{i=1}^{k} (stx)^{k-i} I_{ni}) \subseteq (0 :_{E(R/P)} (stx)^k).$$

Now since $s, t \in R \setminus P$ we can see that

$$(0 :_{E(R/P)} \sum_{i=1}^{k} x^{k-i} I_{ni}) \subseteq (0 :_{E(R/P)} x^k).$$

So by Remark 2.6, $(\frac{x}{1})^k \in \sum_{i=1}^{k} (\frac{x}{1})^{k-i} I_{ni} R_P$. In other words $\frac{x}{1} \in H_n$ and so $(\mathcal{F}^-)_P \subseteq (\mathcal{F}_P)^-$.

Conversely, let $\frac{x}{1} \in H_n$. Then $(\frac{x}{1})^k \in \sum_{i=1}^{k} (\frac{x}{1})^{k-i} I_{ni} R_P$ for a positive integer k. Then there are $a_1 \in I_{n1}, \ldots, a_k \in I_{nk}$ and $s_1, \ldots, s_k \in R \setminus P$ such that

$$(\frac{x}{1})^k + \frac{a_1}{s_1}(\frac{x}{1})^{k-1} + \cdots + \frac{a_{k-1}}{s_{k-1}}(\frac{x}{1})^1 + \frac{a_k}{s_k} = 0.$$

Let $s = s_1 \ldots s_k$. Then there exists $t \in R \setminus P$ such that

$$(tsx)^k \in \sum_{i=1}^{k} (tsx)^{k-i} I_{ni}.$$

This shows $tsx \in U_n$. But $\frac{x}{1} = \frac{tsx}{ts} \in U_n R_P$ and so $(\mathcal{F}_P)^- \subseteq (\mathcal{F}^-)_P$ and this completes the proof. $\square$

By well-known work of Matlis and Gabriel, We know for every injective $R-$module E, there is a family $\{P_\lambda : \lambda \in \Lambda\}$ of prime ideals of R such that $E = \bigoplus_{\lambda\in\Lambda} E(R/P_\lambda)$ (we use E(L) to denote the injective envelope of an $R-$module L). Further we know the set $\{P_\lambda : \lambda \in \Lambda\}$ is the set of all associated prime ideals of R which is denoted by $Ass_R(E)$.

Remark 2.8. Let $E = \bigoplus_{\lambda\in\Lambda} E(R/P_\lambda)$ be an injective $R-$module. Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on R. Let U_n be the set of all $x \in R$ such that

$$(0 :_E \sum_{i=1}^{k} x^{k-i} I_{ni}) \subseteq (0 :_E x^k)$$

for a positive integer k. Since R is a Noetherian ring, $Ass_R(E)$ is a finite set. We know

$$(0 :_E \sum_{i=1}^{k} x^{k-i} I_{ni}) \subseteq (0 :_E x^k)$$

for the positive integer k if and only if for every $P \in Ass_R(E)$,

$$(0 :_{E(R/P)} \sum_{i=1}^{k} x^{k-i} I_{ni}) \subseteq (0 :_{E(R/P)} x^k).$$

But by Remark 2.6, for every $P \in Ass_R(E)$,

$$(0 :_{E(R/P)} \sum_{i=1}^{k} x^{k-i} I_{ni}) \subseteq (0 :_{E(R/P)} x^k)$$

for the positive integer k if and only if

$$(\frac{x}{1})^k \in \sum_{i=1}^{k} (\frac{x}{1})^{k-i} I_{ni} R_P.$$

By Lemma 2.7, we have $(\mathcal{F}_P)^- = (\mathcal{F}^-)_P$. Let $\mathcal{F}^- = \{J_n\}_{n\geq 0}$. Then we see $x \in U_n$ if and only if $\frac{x}{1} \in J_n R_P$ for every $P \in Ass_R(E)$. Since $(\mathcal{F}_P)^- = \{J_n R_p\}_{n\geq 0}$ is a filtration of ideals on R_P, it is easy to see that $\{U_n\}_{n\geq 0}$ is a filtration of ideals on R.

Definition 2.9. Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on R and let E be an injective $R-module$. For every $n \geq 0$, we assume that U_n contains all $x \in R$ such that

$$(0 :_E \sum_{i=1}^{k} x^{k-i} I_{ni}) \subseteq (0 :_E x^k)$$

for a positive integer k. By Remark 2.8, we know $\{U_n\}_{n\geq 0}$ is a filtration on R. This filtration is denoted by $\mathcal{F}^{*(E)}$ and is called the integral closure of a filtration $\mathcal{F} = \{I_n\}_{n\geq 0}$ relative to an injective $R-$module E. By Remark 2.8, we can see $(\mathcal{F}^{*(E)})_P = (\mathcal{F}^{-})_P$ for every $P \in Ass_R(E)$.

For example, let I be an ideal of R and E be an injective $R-$module. For $\mathcal{F} = \{I^n\}_{n\geq 0}$ we have $\mathcal{F}^{*(E)} = \{(I^n)^{*(E)}\}_{n\geq 0}$ (please note, if I and J are two ideals of R and E is an injective $R-$module then we have $I^{*(E)} J^{*(E)} \subseteq (IJ)^{*(E)}$).

Theorem 2.10. *Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on R and let E be an injective $R-module$. Let $\mathcal{F}^{*(E)} = \{U_n\}_{n\geq 0}$. Further for a non negative integer n and $x \in R$, let $L_k = Rx^k + x^{k-1} I_{n1} + x^{k-2} I_{n2} + \cdots + x I_{n(k-1)} + I_{nk}$ and $H_k = I_{nk}$. Then $x \in U_n$ if and only if the filtration $\{H_k\}_{k\geq 0}$ is a reduction of filtration $\{L_k\}_{k\geq 0}$ relative to E.*

Proof. $(\Rightarrow)$ Let $x \in U_n$. Then there exists a positive integer k such that

$$(0 :_E \sum_{i=1}^{k} x^{k-i} I_{ni}) \subseteq (0 :_E x^k).$$

Since $x^{k-i} I_{ni} \subseteq H_{k-(k-i)} L_{k-i}$ for every $1 \leq i \leq k$,

$$(0 :_E \sum_{i=0}^{k} H_{k-i} L_i) = (0 :_E \sum_{i=0}^{k} H_{k-(k-i)} L_{k-i}) \subseteq (0 :_E \sum_{i=1}^{k} x^{k-i} I_{ni}).$$

But $(0 :_E \sum\limits_{i=1}^{k} x^{k-i} I_{ni}) \subseteq (0 :_E x^k)$ and so

$$(0 :_E \sum_{i=0}^{k} H_{k-i} L_i) \subseteq (0 :_E \sum_{i=0}^{k} x^{k-i} I_{ni})) = (0 :_E L_k).$$

Also we know $\sum_{i=0}^{k} H_{k-i}L_i \subseteq L_k$. Then

$$(0 :_E L_k) = (0 :_E \sum_{i=0}^{k} H_{k-i}L_i).$$

Now, we will show that

$$(0 :_E L_t) = (0 :_E \sum_{i=0}^{k} H_{t-i}L_i) \quad \textit{for every } t \geq 1.$$

First let $t < k$. Since $t < k$,

$$(0 :_E \sum_{i=0}^{k} H_{t-i}L_i) \subseteq (0 :_E H_0L_t) = (0 :_E L_t).$$

Also we know $\sum_{i=0}^{k} H_{t-i}L_i \subseteq L_t$. Thus we have

$$(0 :_E L_t) = (0 :_E \sum_{i=0}^{k} H_{t-i}L_i) \quad \textit{for every } t < k.$$

Now let $t > k$. This is clear that $\sum_{i=0}^{k} H_{t-i}L_i = \sum_{i=0}^{k} x^i I_{n(t-i)})$ and so

$$(0 :_E \sum_{i=0}^{k} H_{t-i}L_i) = (0 :_E \sum_{i=0}^{k} x^i I_{n(t-i)}).$$

Since $(0 :_E \sum_{i=1}^{k} x^{k-i}I_{ni}) \subseteq (0 :_E x^k)$, we can see that

$$(0 :_E \sum_{i=1}^{k} x^{k-i}I_{n(r+i)}) \subseteq (0 :_E x^{k+r}).$$

But by

$$(0 :_E x^{k+r}I_{n(t-(k+r))}) \supseteq (0 :_E \sum_{i=1}^{k} x^{k-i}I_{n(r+i)}I_{n(t-(k+r))})$$

$$\supseteq (0 :_E \sum_{i=1}^{k} x^{k-i} I_{n(t-k+i)}) \supseteq (0 :_E \sum_{i=0}^{k} x^i I_{n(t-i)})$$

we have

$$(0 :_E L_t) = (0 :_E \sum_{i=k+1}^{t} x^i I_{n(t-i)})) \cap (0 :_E \sum_{i=0}^{k} x^i I_{n(t-i)})$$

$$= (0 :_E \sum_{i=0}^{k} x^i I_{n(t-i)}) = (0 :_E \sum_{i=0}^{k} H_{t-i} L_i)).$$

Then

$$(0 :_E L_t) = (0 :_E \sum_{i=0}^{k} H_{t-i} L_i) \quad \textit{for every } t \geq 1.$$

($\Leftarrow$) Let $\{H_k\}_{k\geq 0}$ be a reduction of filtration $\{L_k\}_{k\geq 0}$ relative to M. Then there exists a positive integer d such that

$$(0 :_E L_k) = (0 :_E \sum_{i=0}^{d} H_{k-i} L_i) \quad \textit{for every } k \geq 1.$$

Particularly, we have $(0 :_E L_{d+1}) = (0 :_E \sum_{i=0}^{d} H_{d+1-i} L_i)$. But by $\sum_{i=0}^{d} H_{d+1-i} L_i = \sum_{i=0}^{d} x^i I_{n(d+1-i)} \subseteq \sum_{i=0}^{d} x^i I_{n(d-i)}$ we have

$$(0 :_E x^{d+1}) \supseteq (0 :_E L_{d+1}) = (0 :_E \sum_{i=0}^{d} H_{d+1-i} L_i) \supseteq (0 :_E \sum_{i=0}^{d} x^i I_{n(d-i)}).$$

Hence $x \in U_n$. $\square$

The following theorem shows that $\mathcal{F} \to \mathcal{F}^{*(E)}$, is a semi-prime operation.

Theorem 2.11. (See [4, 2.4].) *Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ and $\mathcal{G} = \{J_n\}_{n\geq 0}$ be filtrations on R. Then for every injective $R-$module E, we have* *(a)* $\mathcal{F} \subseteq \mathcal{F}^{*(E)}$*;*

(b) if $\mathcal{F} \subseteq \mathcal{G}$ *then* $\mathcal{F}^{*(E)} \subseteq \mathcal{G}^{*(E)}$*;*

(c) $(\mathcal{F}^{*(E)})^{*(E)} = \mathcal{F}^{*(E)}$*;*

(d) $\mathcal{F}^{*(E)}\mathcal{G}^{*(E)} \subseteq (\mathcal{F}\mathcal{G})^{*(E)}$.

Proof. (a) and (b) are clear.

(c) By (a) and (b), we have $\mathcal{F}^{*(E)} \subseteq (\mathcal{F}^{*(E)})^{*(E)}$. Let $\mathcal{F} = \{I_n\}_{n\geq 0}$, $\mathcal{F}^{*(E)} = \{U_n\}_{n\geq 0}$, $\mathcal{F}^{-} = \{J_n\}_{n\geq 0}$ and $(\mathcal{F}^{*(E)})^{*(E)} = \{K_n\}_{n\geq 0}$. Let $x \in K_n$. By Lemma 2.7 and Remark 2.8, we have

$$\begin{aligned}((\mathcal{F}^{*(E)})^{*(E)})_P &= ((\mathcal{F}^{*(E)})^{-})_P = ((\mathcal{F}^{*(E)})_P)^{-}\\ &= ((\mathcal{F}^{-})_P)^{-} = ((\mathcal{F}^{-})^{-})_P\end{aligned}$$

for every $P \in Ass_R(E)$. Also we know from [4, 2.4.3], $(\mathcal{F}^{-})^{-} = \mathcal{F}^{-}$. Thus there are $a \in J_n$ and $s \in R \setminus P$ such that $\frac{x}{1} = \frac{a}{s}$. Hence $tsx = ta$ for $t \in R \setminus P$. Since $tsx = ta \in J_n$, there exists a positive integer k such that $(tsx)^k \in \sum_{i=1}^{k}(tsx)^{k-i}I_{ni}$. Now since $ts \in R \setminus P$, it is easy to see that

$$(0 :_{E(R/P)} \sum_{i=1}^{k} x^{k-i}I_{ni}) \subseteq (0 :_{E(R/P)} x^k)$$

for every $P \in Ass_R(E)$. Then $x \in U_n$ and so $(\mathcal{F}^{*(E)})^{*(E)} \subseteq \mathcal{F}^{*(E)}$. This follows (c).

(d) Let $\mathcal{F}^{*(E)} = \{U_n\}_{n\geq 0}$ and $\mathcal{G}^{*(E)} = \{V_n\}_{n\geq 0}$. Let $x \in U_n$ and $y \in V_n$. Further let $P \in Ass_R(E)$. We have

$$(\mathcal{F}^{*(E)})_P(\mathcal{G}^{*(E)})_P = (\mathcal{F}^{-})_P(\mathcal{G}^{-})_P = (\mathcal{F}^{-}\mathcal{G}^{-})_P.$$

But by [4, 2.4.4], we know $\mathcal{F}^{-}\mathcal{G}^{-} \subseteq (\mathcal{F}\mathcal{G})^{-}$. This shows if $(\mathcal{F}\mathcal{G})^{-} = \{H_n\}_{n\geq 0}$ then $\frac{x}{1}\frac{y}{1} \in H_nR_P$. Thus there are $a \in H_n$ and $s, t \in R \setminus P$ such that $tsxy = ta$. Since $ta \in H_n$, there is a positive integer k such that $(stxy)^k \in \sum_{i=1}^{k}(tsxy)^{k-i}I_{ni}J_{ni}$. This implies that

$$(0 :_{E(R/P)} \sum_{i=1}^{k}(xy)^{k-i}I_{ni}J_{ni}) \subseteq (0 :_{E(R/P)} \sum_{i=1}^{k}(tsxy)^{k-i}I_{ni}J_{ni})$$

$$\subseteq (0 :_{E(R/P)} (tsxy)^k).$$

Since $s, t \in R \setminus P$, for every $P \in Ass_R(E)$ we have

$$(0 :_{E(R/P)} \sum_{i=1}^{k} (xy)^{k-i} I_{ni} J_{ni}) \subseteq (0 :_{E(R/P)} (xy)^k).$$

Then $(0 :_E \sum_{i=1}^{k} (xy)^{k-i} I_{ni} J_{ni}) \subseteq (0 :_E (xy)^k)$ and so if $(\mathcal{F}\mathcal{G})^{*(E)} = \{W_n\}_{n\geq 0}$ then $xy \in W_n$. $\square$

3 Main results

Let I be an ideal of R and E be an injective $R-$module. In [2], it is shown that $I^{*(E)} = I^{-}(Ass_R(E))$. In this section we will prove a similar theorem for the integral closure of a filtration $\mathcal{F}$ relative to an injective $R-$module E. First we introduce the following notation.

Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on R and let $\mathcal{P}$ be a subset of $Spec(R)$. For every $P \in \mathcal{P}$, we have

$$I_n(\mathcal{P})I_m(\mathcal{P}) \subseteq I_n(P)I_m(P) = (I_nR_P)^c(I_mR_P)^c \subseteq (I_nI_mR_P)^c$$

$$\subseteq (I_{n+m}R_P)^c = I_{n+m}(P).$$

Then

$$I_n(\mathcal{P})I_m(\mathcal{P}) \subseteq \bigcap_{P\in\mathcal{P}} I_{n+m}(P) = I_{n+m}(\mathcal{P}).$$

This shows that $\{I_n(\mathcal{P})\}_{n\geq 0}$ is a filtration on R. We denote this filtration by $\mathcal{F}(\mathcal{P})$.

Now we are ready to prove the main proposition of this section.

Theorem 3.1. (See [2, 2.6].) *Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on R and let E be an injective $R-$module. Then $\mathcal{F}^{*(E)} = \mathcal{F}^{-}(Ass_R(E))$.*

Proof. Let $\mathcal{F}^{*(E)} = \{U_n\}_{n\geq 0}$ and $(\mathcal{F})^{-} = \{J_n\}_{n\geq 0}$. We will show

$$U_n = J_n(Ass_R(E)) \quad for \quad every \quad n.$$

Let $x \in U_n$. Then for every $P \in Ass_R(E)$, $\frac{x}{1} \in J_nR_P$. Let $J_n = Q_1 \cap \cdots \cap Q_k$ be a minimal primary decomposition of J_n. Let

$$\sqrt{Q_i} \cap P = \emptyset \ \ for \ \ every \ \ 1 \leq i \leq l$$

and

$$\sqrt{Q_i} \cap P \neq \emptyset \ \ for \ \ every \ \ l+1 \leq i \leq k.$$

Then $\frac{x}{1} \in J_nR_P = (Q_1 \cap \cdots \cap Q_l)R_P$ and so there are $y \in Q_1 \cap \cdots \cap Q_l$ and $s, t \in R \setminus P$ such that $stx = sy \in Q_1 \cap \cdots \cap Q_l$. Since $st \notin \sqrt{Q_i}$ for every $1 \leq i \leq l$, $x \in Q_1 \cap \cdots \cap Q_l$ and so $x \in J_n(P)$ for every $P \in Ass_R(E)$. Hence $x \in J_n(Ass_R(E))$ and so $U_n \subseteq J_n(Ass_R(E))$. For converse inclusion, let $x \in J_n(P)$ for every $P \in Ass_R(E)$. Then $\frac{x}{1} \in J_nR_P$ for every $P \in Ass_R(E)$. Hence there are $y \in J_n$ and $s, t \in R \setminus P$ such that $stx = sy \in J_n$. Then there is a positive integer k such that $(stx)^k \in \sum\limits_{i=1}^{k}(tsx)^{k-i}I_{ni}$. By $s, t \in R \setminus P$ we can see

$$\begin{aligned}(0 :_{E(R/P)} \sum_{i=1}^{k} x^{k-i}I_{ni}) &\subseteq (0 :_{E(R/P)} \sum_{i=1}^{k}(tsx)^{k-i}I_{ni}) \\ &\subseteq (0 :_{E(R/P)} (stx)^n) \\ &\subseteq (0 :_{E(R/P)} x^n).\end{aligned}$$

Thus we have

$$(0 :_{E(R/P)} \sum_{i=1}^{k} x^{k-i}I_{ni}) \subseteq (0 :_{E(R/P)} x^n)$$

for every $P \in Ass_R(E)$. Thus $x \in U_n$ and so $J_n(Ass_R(E)) \subseteq U_n$. This follows $U_n = J_n(Ass_R(E))$. □

Definition 3.2. (See [4, 3.1(2)].) Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on R and $\mathcal{F}^- = \{J_n\}_{n\geq 0}$. Members of

$$A^-(\mathcal{F}) = \{P : P \in Ass(R/J_n) \ for \ some \ n \geq 1\}$$

are called the asymptotic prime divisors of $\mathcal{F}$.

Let E be an injective $R-$module. We know from [1, 2.2], for each ideal I of R, the module $(0 :_E I)$ has a secondary representation, and so we can form the finite set of prime ideals $Att_R(0 :_E I)$. In fact,

$$Att_R(0 :_E I) = \{P' \in ass(I)) : P' \subseteq P \ for\ some\ P \in Ass_R(E)\}.$$

Definition 3.3. Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on R and E be an injective $R-$module. Let $\mathcal{F}^{*(E)} = \{U_n\}_{n\geq 0}$. We will show the set

$$\{P : P \in Att(0 :_E U_n) \ for\ some\ n \geq 1\}$$

by $At^*(\mathcal{F}, E)$.

Theorem 3.4. (See [2, 3.2].) *Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a Noetherian filtration on R. Let E be an injective $R-$module. Then $At^*(\mathcal{F}, E)$ is a finite set.*

Proof. Let $\mathcal{F}^{*(E)} = \{U_n\}_{n\geq 0}$ and $\mathcal{F}^- = \{J_n\}_{n\geq 0}$. By Note 3, we know

$$At^*(\mathcal{F}, E) = \{P' \in ass(U_n) : P' \subseteq P \ for\ some\ P \in Ass_R(E)\}.$$

But we know from Theorem 3.1, $\mathcal{F}^{*(E)} = \mathcal{F}^-(Ass_R(E))$. Then

$$At^*(\mathcal{F}, E) = \{P' \in ass(J_n(Ass_R(E)) : P' \subseteq P \ for\ some\ P \in Ass_R(E)\}$$

$$= \{P' \in A^-(\mathcal{F}) : P' \subseteq P \ for\ some\ P \in Ass_R(E)\}.$$

Now the proof is completed by [4, 3.3]. □

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Farhad Dorostkar
Assistant Professor of Mathematics
Department of Pure Mathematics
Faculty of Mathematical Sciences
University of Guilan
Rasht, Iran
E-mail: dorostkar@guilan.ac.ir