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Original Research Paper

New Bellman-Type Inequalities

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Abstract. In this paper, we present a general version of operator Bellman inequality. Also, the refinement of inequality due to J. Aujla and F. Silva for the convex functions is given as well.

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1 Introduction and Preliminaries

Let $\mathcal{H}$ be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$, and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators acting on $\mathcal{H}$ equipped with the operator norm

$$\|A\| = \sup\left\{\|Ax\| : \ x \in \mathcal{H}, \ \ \langle x, x \rangle = 1\right\}.$$

Let the symbol I stand for the identity operator on $\mathcal{H}$. An operator A is said to be positive (denoted by $0 \leq A$) if $0 \leq \langle Ax, x \rangle$ for all $x \in \mathcal{H}$, and also an operator A is said to be strictly positive (denoted by $0 < A$) if

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A is positive and invertible. For two self-ajoint operators $A, B \in \mathcal{B}(\mathcal{H})$, we write $A \leq B$ if $0 \leq B - A$. A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is said to be positive if $0 \leq \Phi(A)$ when $0 \leq A$. If, in addition, $\Phi(I) = I$, it is said to be unital.

For any strictly positive operator $A, B \in \mathcal{B}(\mathcal{H})$ and $v \in [0,1]$, we write from [9]

$$\begin{aligned}
A \nabla_v B &= (1-v)A + vB, \\
A \sharp_v B &= A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^v A^{\frac{1}{2}}, \\
A !_v B &= \left(A^{-1} \nabla_v B^{-1}\right)^{-1}.
\end{aligned}$$

For the case $v = 1/2$, we write ∇, $\sharp$, and $!$, respectively. The weighted operator arithmetic-geometric-harmonic mean inequality asserts that

$$A !_v B \leq A \sharp_v B \leq A \nabla_v B,$$

for any positive operators $A, B \in \mathcal{B}(\mathcal{H})$ and any $v \in [0,1]$. Various inequalities improving this above inequality have been studied in [5, 11]. Let σ be an operator mean in the sense of Kubo and Ando [9]. A mean σ is called to be symmetric if $A \sigma B = B \sigma A$. According to the general theory of operator means [9], ∇ is the biggest and $!$ is the smallest among symmetric means.

A real valued function f defined on an interval $J \subseteq \mathbb{R}$ is said to be operator convex (resp. operator concave) if

$$f\left((1-v)A + vB\right) \leq (\text{resp. } \geq)\,(1-v) f(A) + v f(B), \tag{1}$$

for all self-adjoint operators A, B with spectra in J and all $v \in [0,1]$. A continuous real valued function f defined on an interval J is called operator monotone (more precisely, operator monotone increasing) if $A \leq B$ implies that $f(A) \leq f(B)$, and operator monotone decreasing if $A \leq B$ implies $f(B) \leq f(A)$ for all self-adjoint operators A, B with spectra in J.

The scalar Bellman inequality [4] says that if p is a positive integer and $a, b, a_i, b_i\,(1 \leq i \leq n)$ are positive real numbers such that $\sum_{i=1}^n a_i^p \leq a^p$ and $\sum_{i=1}^n b_i^p \leq b^p$, then

$$\left(a^p - \sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} + \left(b^p - \sum_{i=1}^n b_i^p\right)^{\frac{1}{p}} \leq \left((a+b)^p - \sum_{i=1}^n (a_i + b_i)^p\right)^{\frac{1}{p}}.$$

A multiplicative analogue of this inequality is due to Aczél [1]. In 1956, he proved that

$$\Big(a_1^2-\sum_{i=2}^{n}a_i^2\Big)\Big(b_1^2-\sum_{i=2}^{n}b_i^2\Big)\le\Big(a_1b_1-\sum_{i=2}^{n}a_ib_i\Big)^2,$$

where $a_i, b_i\,(1\le i\le n)$ are positive real numbers such that

$$a_1^2-\sum\nolimits_{i=2}^{n}a_i^2>0,\quad \text{or}\quad b_1^2-\sum\nolimits_{i=2}^{n}b_i^2>0.$$

The operator theory related to inequalities in Hilbert space is studied in many papers. In [10, Corollary 2.2], Morassaei et al. showed the following non-commutative version of classical Bellman inequality:

$$\Phi\big((I-A)^{\frac{1}{p}}\nabla_v(I-B)^{\frac{1}{p}}\big)\le(\Phi\left(I-A\nabla_vB\right))^{\frac{1}{p}},\tag{2}$$

where $0\le v\le1$, $p>1$ and $A,B\in\mathcal{B}(\mathcal{H})$ are two contractions (i.e., $0<A,B\le I$) and $\Phi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})$ is a unital positive linear map. We refer the reader to [15] for some fresh discussion of Bellman inequality.

In this paper, we extend inequality (2) to two arbitrary operator means σ and τ. Naturally, this generalization imposes additional constant. Some related inequalities for operator concave functions are also presented.

2 Main Results

In order to prove our desired inequalities, we need the following lemmas. The first lemma is the celebrated Choi-Davis-Jensen inequality (see, e.g. [14, Theorem 1.20]).

Lemma 2.1. *Let $A\in\mathcal{B}(\mathcal{H})$ be a self-adjoint operator with the spectra on the interval J, $f:J\to\mathbb{R}$ be an operator concave, then for any unital positive linear mapping Φ,*

$$\Phi\left(f\left(A\right)\right)\le f\left(\Phi\left(A\right)\right),\tag{3}$$

or equivalently,

$$\Phi\left(A\sigma B\right)\le\Phi\left(A\right)\sigma\Phi\left(B\right),\tag{4}$$

where σ is an arbitrary operator mean in the Kubo-Ando sense.

The second lemma contains a multiplicative reverse for the weighted arithmetic-geometric-harmonic mean inequality. See [6], [7], and references therein.

Lemma 2.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$, then*

$$\frac{m\sharp_\lambda M}{m\nabla_\lambda M} A\nabla_v B \leq A\sharp_v B \leq \frac{m\nabla_\lambda M}{m\sharp_\lambda M} A!_v B,$$

where $\lambda = \min\{v, 1-v\}$ and $0 \leq v \leq 1$.

The following lemma contains a well known property for operator monotone increasing (resp. decreasing). However, we give the proof to keep the present paper self-contained.

Lemma 2.3. *Let $f : (0,\infty) \to (0,\infty)$ be a given function and let $\alpha \geq 1$.*

(a) If f is operator monotone, then

$$f(\alpha t) \leq \alpha f(t). \tag{5}$$

(b) If f is operator monotone decreasing, then

$$f(\alpha t) \geq \frac{1}{\alpha} f(t).$$

Proof. (a) Since $f(t)$ is operator monotone, then $\frac{t}{f(t)}$ is operator monotone too [14, Corollary 1.14]. Hence for $\alpha \geq 1$,

$$\frac{t}{f(t)} \leq \frac{\alpha t}{f(\alpha t)} \ \Rightarrow \ \alpha f(t) \geq f(\alpha t).$$

(b) If f is operator monotone decreasing, then $1/f$ is operator monotone. Applying inequality (5) for $1/f$, we infer that

$$f(\alpha t)^{-1} \leq \alpha f(t)^{-1} \Rightarrow f(\alpha t) \geq \frac{1}{\alpha} f(t).$$

□

It should be mentioned here that part (b) of Lemma 2.3 will not be used in this paper. It has been given for the sake of completeness. On making use of the above lemmas, we reach the next result.

Theorem 2.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$, and τ, σ be two arbitrary operator means such that $!_v \leq \tau, \sigma \leq \nabla_v$, and let $f : (0,\infty) \to (0,\infty)$ be an operator monotone, then for any unital positive linear mapping Φ,*

$$\Phi\big(f(A)\tau f(B)\big) \leq \Big(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\Big)^2 f\big(\Phi(A\sigma B)\big),$$

where $\lambda = \min\{v, 1-v\}$ and $0 \leq v \leq 1$.

Proof. Since f is operator concave, by [14, Corollary 1.12], Lemma 2.2 and Lemma 2.3 (a) we have

$$\begin{aligned} f(A)\nabla_v f(B) &\leq f(A\nabla_v B) \leq f\Big(\Big(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\Big)^2 A!_v B\Big) \\ &\leq \Big(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\Big)^2 f\left(A!_v B\right) \leq \Big(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\Big)^2 f\left(A\sigma B\right). \end{aligned}$$

Applying positive linear mapping Φ, we get

$$\begin{aligned} \Phi\big(f(A)\big)\tau\Phi\big(f(B)\big) &\leq \Phi\big(f(A)\big)\nabla_v\Phi\big(f(B)\big) \\ &= \Phi\big(f(A)\nabla_v f(B)\big) \\ &\leq \Phi\Big(\Big(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\Big)^2 f(A\sigma B)\Big) \qquad (6) \\ &= \Big(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\Big)^2 \Phi\big(f(A\sigma B)\big). \end{aligned}$$

By (3), we know that

$$\Phi\big(f(A\sigma B)\big) \leq f\big(\Phi(A\sigma B)\big)$$

so,

$$\Phi\big(f(A)\big)\tau\Phi\big(f(B)\big) \leq \Big(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\Big)^2 f\big(\Phi(A\sigma B)\big).$$

Now, by the inequality (4) we obtain

$$\begin{aligned} \Phi\big(f(A)\tau f(B)\big) &\leq \Phi\big(f(A)\big)\tau\Phi\big(f(B)\big) \\ &\leq \Big(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\Big)^2 f\big(\Phi(A\sigma B)\big). \end{aligned}$$

This completes the proof of the theorem. $\square$

Now we are in a position to present our promised generalization of (2).

Corollary 2.5. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M < 1$, and τ, σ be two arbitrary operator means such that $!_v \leq \tau, \sigma \leq \nabla_v$, then for any unital positive linear mapping Φ and $p > 1$,*

$$\Phi\big((I-A)^{\frac{1}{p}}\tau(I-B)^{\frac{1}{p}}\big) \leq \Big(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\Big)^2 \big(\Phi(I - A\sigma B)\big)^{\frac{1}{p}},$$

where $\lambda = \min\{v, 1-v\}$ and $0 \leq v \leq 1$.

Proof. By [14, Corollary 1.16] the function $f(t) = t^r$ is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$. Thus, the function $f(t) = (1-t)^r$ on $(0,1)$ is operator concave, for $0 \leq r \leq 1$. Now, since Φ is unital, it follows from Theorem 2.4 that

$$\begin{aligned}
\Phi\big((I-A)^{\frac{1}{p}}\tau(I-B)^{\frac{1}{p}}\big) &\leq \Big(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\Big)^2 \big(I - \Phi(A\sigma B)\big)^{\frac{1}{p}} \\
&= \Big(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\Big)^2 \big(\Phi(I) - \Phi(A\sigma B)\big)^{\frac{1}{p}} \\
&= \Big(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\Big)^2 \big(\Phi(I - A\sigma B)\big)^{\frac{1}{p}}.
\end{aligned}$$

Hence the proof is completed. $\square$

Corollary 2.6. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$, and τ, σ be two arbitrary operator means such that $!_v \leq \tau, \sigma \leq \nabla_v$, then for any unital positive linear mapping Φ,*

$$\Phi\big(\log A \;\tau\; \log B\big) \leq \Big(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\Big)^2 \log \Phi(A\sigma B),$$

where $\lambda = \min\{v, 1-v\}$ and $0 \leq v \leq 1$.

Proof. Since the function $f(t) = \log t$ is operator concave on $(0,\infty)$ [14, Example 1.7], we infer the desired result from Theorem 2.4. $\square$

In [13, Theorem 2.2], Moslehian noted the following inequalities for non-negative operator decreasing and operator concave f,

$$f(A)\,\tau f(B) \le f(A\tau B)\,. \tag{7}$$

In particular, he obtained if $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$f(A^p)\,\sharp_{\frac{1}{q}} f(B^q) \le f\big(A^p \sharp_{\frac{1}{q}} B^q\big), \tag{8}$$

and

$$\big\langle f(A^p)x, x\big\rangle^{\frac{1}{p}} \big\langle f(B^q)x, x\big\rangle^{\frac{1}{q}} \le \big\langle f(A^p \sharp_{\frac{1}{q}} B^q)x, x\big\rangle, \tag{9}$$

for all $x \in \mathcal{H}$. As it is mentioned in [2, Theorem 2.1], the condition operator decreasing is equivalent to operator convexity. Thus, the inequalities (7), (8), and (9) are valid just for the trivial case $f(t) = t$. We conclude this paper with the following considerable generalization of (8).

Theorem 2.7. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators such that $mI \le A, B \le MI$ for some scalars $0 < m < M$, and τ, σ be two arbitrary operator means such that $!_v \le \tau, \sigma \le \nabla_v$, and let $f : (0, \infty) \to (0, \infty)$ be an operator concave, then*

$$f(A)\tau f(B) \le \left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^2 f(A\sigma B). \tag{10}$$

In particular,

$$f(A)\sharp_v f(B) \le \frac{m\nabla_\lambda M}{m\sharp_\lambda M} f(A\sharp_v B), \tag{11}$$

where $\lambda = \min\{v, 1-v\}$ and $0 \le v \le 1$.

Proof. Letting $\Phi(T) = T$ for any $T \in \mathcal{B}(\mathcal{H})$ in Theorem 2.4, we get (10). To prove (11), since f is concave, by the arithmetic-geometric mean inequality and Lemma 2.2, Lemma 2.3(a) we have

$$\begin{aligned} f(A)\sharp_v f(B) &\le f(A)\nabla_v f(B) \le f(A\nabla_v B) \\ &\le f\left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M} A\sharp_v B\right) \le \frac{m\nabla_\lambda M}{m\sharp_\lambda M} f(A\sharp_v B)\,, \end{aligned}$$

as desired. $\square$

Remark 2.8. A result similar to (11) can be found in [7, Theorem 2].

Corollary 2.9. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators such that $m \leq A^p, B^q \leq M$ for some scalars $0 < m < M$, $1/p + 1/q = 1$, $p, q > 1$. Let $f : (0, \infty) \to (0, \infty)$ be an operator concave, then*

$$f(A^p)\sharp_{\frac{1}{q}} f(B^q) \leq \frac{m\nabla_\gamma M}{m\sharp_\gamma M} f(A^p \sharp_{\frac{1}{q}} B^q). \tag{12}$$

In particular, for any vector $x \in \mathcal{H}$

$$\langle f(A^p)x, x\rangle^{\frac{1}{p}} \langle f(B^q)x, x\rangle^{\frac{1}{q}} \leq \left(\frac{m\nabla_\gamma M}{m\sharp_\gamma M}\right)^2 \left\langle f\left(A\sharp_{1/q} B\right) x, x\right\rangle, \tag{13}$$

where $\gamma = \min\{1/p, 1/q\}$.

Proof. By replacing $v = 1/q$, $A = A^p$, and $B = B^q$ in Theorem 2.7, we get the inequality (12).

To prove (13), take $\tau = \nabla_{1/q}$, $\sigma = \sharp_{1/q}$, $A = A^p$, and $B = B^q$ in Theorem 2.7, we have

$$f(A^p)\nabla_{1/q} f(B^q) \leq \left(\frac{m\nabla_\gamma M}{m\sharp_\gamma M}\right)^2 f\left(A\sharp_{1/q} B\right).$$

Hence for any vector $x \in \mathcal{H}$,

$$\begin{aligned}
\langle f(A^p)x, x\rangle^{\frac{1}{p}} \langle f(B^q)x, x\rangle^{\frac{1}{q}} &\leq \frac{1}{p}\langle f(A^p)x, x\rangle + \frac{1}{q}\langle f(B^q)x, x\rangle \\
&= \left\langle \left(\frac{1}{p} f(A^p) + \frac{1}{q} f(B^q)\right) x, x\right\rangle \\
&= \left\langle \left(f(A^p)\nabla_{1/q} f(B^q)\right) x, x\right\rangle \\
&\leq \left(\frac{m\nabla_\gamma M}{m\sharp_\gamma M}\right)^2 \left\langle f\left(A\sharp_{1/q} B\right) x, x\right\rangle
\end{aligned}$$

where we have used weighted arithmetic-geometric mean inequality. □

Remark 2.10. It follows from McCarthy inequality [14, Theorem 1.4],

$$\begin{aligned}
\left\langle f(A^p)^{\frac{1}{p}} x, x\right\rangle \left\langle f(B^q)^{\frac{1}{q}} x, x\right\rangle &\leq \langle f(A^p)x, x\rangle^{\frac{1}{p}} \langle f(B^q)x, x\rangle^{\frac{1}{q}} \\
&\leq \left(\frac{m\nabla_\gamma M}{m\sharp_\gamma M}\right)^2 \left\langle f\left(A\sharp_{1/q} B\right) x, x\right\rangle
\end{aligned}$$

or equivalently,

$$\left\| f(A^p)^{\frac{1}{2p}} x \right\| \left\| f(B^q)^{\frac{1}{2q}} x \right\| \leq \frac{m \nabla_\gamma M}{m \sharp_\gamma M} \left\| f\left(A \sharp_{1/q} B\right)^{\frac{1}{2}} x \right\|$$

To prove (2), Morassaei et al. [10] have proved that if $A, B \in \mathcal{B}(\mathcal{H})$ are two positive operators and $f : (0, \infty) \to (0, \infty)$ is an operator convex (resp. concave), then for any unital positive linear mapping Φ and $0 \leq v \leq 1$

$$f\left(\Phi\left((1-v)A + vB\right)\right) \leq (\text{resp. } \geq)\, \Phi\left((1-v) f(A) + v f(B)\right).$$

Trivially, the above inequality can be considered as an extension of inequality (1) (choose $\Phi(T) = T$ for any $T \in \mathcal{B}(\mathcal{H})$). Another generalization of the inequality (1) for operators in $\mathcal{B}(\mathcal{H})$ (see, [3]) asserts that if $A, B \in \mathcal{B}(\mathcal{H})$ are positive, and f is a non-negative convex (not necessary operator convex) function on $(0, \infty)$, then

$$\left\| f\left((1-v)A + vB\right) \right\| \leq \left\| (1-v) f(A) + v f(B) \right\|. \tag{14}$$

We refer the reader to [12, 16] as a sample of the extensive use of this inequality.

In the next theorem, we improve (14). In order to reach our purpose, we need the following lemma [8, Theorem 2.2].

Lemma 2.11. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators, and let f be a non-negative convex function on $[0, \infty)$ with $f(0) = 0$. Then,*

$$\left\| f(A) + f(B) \right\| \leq \left\| f(A+B) \right\|.$$

Before stating our theorem we recall that if $T \in \mathcal{B}(\mathcal{H})$ is a positive operator, and if f is a non-negative increasing function on $[0, \infty)$, then

$$\left\| f(T) \right\| = f\left(\|T\|\right).$$

Theorem 2.12. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators, and let $f(t)$ be a non-negative function on $[0, \infty)$ such that $g(t) = f\left(\sqrt{t}\right)$ is convex and $g(0) = 0$. Then for any $0 \leq v \leq 1$,*

$$\left\| f((1-v)A + vB) + f\left(\sqrt{2\lambda\Big(\frac{A^2+B^2}{2} - \big(\frac{A+B}{2}\big)^2\Big)} \right) \right\| \leq \left\| (1-v) f(A) + v f(B) \right\|,$$

where $\lambda = \min\{v, 1-v\}$.

Proof. First we assume $0 \leq v \leq 1/2$. Since $h(t) = t^2$ is a convex function on $(0, \infty)$ [14, Corollary 1.16], we have

$$\begin{aligned}
(1-v)A^2 + vB^2 - 2\lambda\Big(\frac{A^2+B^2}{2} - \big(\frac{A+B}{2}\big)^2\Big) \\
&= (1-2v)A^2 + 2v\big(\frac{A+B}{2}\big)^2 \\
&\geq \Big((1-2v)A + 2v\big(\frac{A+B}{2}\big)\Big)^2 \\
&= \big((1-v)A + vB\big)^2.
\end{aligned}$$

A similar argument for $1/2 \leq v \leq 1$ implies the above inequality. Thus we have

$$\big((1-v)A + vB\big)^2 + 2\lambda\Big(\frac{A^2+B^2}{2} - \big(\frac{A+B}{2}\big)^2\Big) \leq (1-v)A^2 + vB^2, \qquad (15)$$

where $\lambda = \min\{v, 1-v\}$. Obviously, the operator inequality (15) implies the following norm inequality

$$\Big\|\big((1-v)A + vB\big)^2 + 2\lambda\Big(\frac{A^2+B^2}{2} - \big(\frac{A+B}{2}\big)^2\Big)\Big\| \leq \big\|(1-v)A^2 + vB^2\big\|.$$

Since g is non-negative and convex on $[0, \infty)$, it follows that g is increasing. Now, we obtain

$$\begin{aligned}
&\Big\|f\big((1-v)A + vB\big) + f\Big(\sqrt{2\lambda\Big(\frac{A^2+B^2}{2} - \big(\frac{A+B}{2}\big)^2\Big)}\ \Big)\Big\| \\
=&\Big\|g\Big(\big((1-v)A + vB\big)^2\Big) + g\Big(2\lambda\Big(\frac{A^2+B^2}{2} - \big(\frac{A+B}{2}\big)^2\Big)\Big)\Big\| \\
\leq&\Big\|g\Big(\big((1-v)A + vB\big)^2 + 2\lambda\Big(\frac{A^2+B^2}{2} - \big(\frac{A+B}{2}\big)^2\Big)\Big)\Big\| \\
=&g\Big(\Big\|\big((1-v)A + vB\big)^2 + 2\lambda\Big(\frac{A^2+B^2}{2} - \big(\frac{A+B}{2}\big)^2\Big)\Big\|\Big) \\
\leq&g\big(\big\|(1-v)A^2 + vB^2\big\|\big) = \big\|g\big((1-v)A^2 + vB^2\big)\big\| \\
\leq&\big\|(1-v)g(A^2) + vg(B^2)\big\| = \big\|(1-v)f(A) + vf(B)\big\|,
\end{aligned}$$

where the first and the last equality follows from the assumption $g(t) = f\left(\sqrt{t}\right)$, the first inequality obtained from Lemma 2.11, and the inequality (14) implies the last inequality. $\square$

Applying the above theorem to the power function, we get:

Corollary 2.13. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators. Then for any $0 \le v \le 1$, and $p \ge 2$*

$$\left\| ((1-v)\,A + vB)^p + \left(2\lambda\left(\frac{A^2+B^2}{2} - \left(\frac{A+B}{2}\right)^2\right)\right)^{\frac{p}{2}} \right\| \le \left\| (1-v)\,A^p + vB^p \right\|,$$

where $\lambda = \min\{v, 1-v\}$.

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