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On the Space of Real Valued Statistically Convergent Sequences

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Abstract. The aim of this paper is to introduce an equivalence relation on the space of real valued statistically convergence sequences, $\mathcal{C}_{st}$, and an inner product on the set of its equivalence classes. We equip $\mathcal{C}_{st}$ with the induced J- metric, d_J, by the given inner product. We prove that $\mathcal{C}_{st}$ is a complete J-metric space. We also show that the space of all real valued convergent sequences is a dense subspace of $(\mathcal{C}_{st}, d_J)$.

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1 Introduction

The concept of statistically convergence was introduced by Fast in [4] and Steinhaus in [10] independently in the same year 1951. It has

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many applications in various areas such as measure theory, trigonometric space, operator theory, etc. (e.g. [1, 2, 3, 5, 9, 11]). Let K be a subset of $\mathbb{N}$. The symbol $\delta(K)$ is called *natural density* of K and if the following limit exists, it is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}\left|\left\{k \leq n : \ k \in K\right\}\right| = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{\infty} \chi_{_K}(k),$$

where $\chi_{_K}$ is the characteristic function of K and $|\mathrm{A}|$ denotes the cardinality of the set A. Obviously, $0 \leq \delta(K) \leq 1$ holds for all $K \subseteq \mathbb{N}$. If $\delta(K) = 1$, we say K is a *natural dense subset* of $\mathbb{N}$ and $\delta(K^c) = 0$. If $\mathcal{K} = \{K \subseteq \mathbb{N} : \ \delta(K) = 1\}$, we have $K_1 \cap K_2$ and $K_1 \cup K_2$ are in $\mathcal{K}$ where K_1, K_2 are in $\mathcal{K}$ (see [10]). Let $x = \{x_n\}_n$ be a sequence. We say $\{x_n\}$ has *P-property* "for almost all n" (or x_n satisfies P-property, a.a.n) when the natural density set of all n that x_n has not P-property is zero.

Definition 1.1. ([6]) A real valued sequence $\{x_n\}_n$ is said to be *statistically converges* to l, if for every $\varepsilon > 0$, $\delta\left(\left\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\right\}\right) = 0$ holds. In this case, we write st-$\lim\limits_{n\to\infty} x_n = l$ or $x_n \xrightarrow{st.} l$ as $n \to \infty$.

Note that each finite set has zero density, therefore every convergent sequence is statistically convergent, but the converse of this statement is not true in general (see [6]).

Definition 1.2. ([6]) A sequence $x = \{x_n\}_n$ is called *statistically Cauchy* if for each $\varepsilon > 0$ there exists $N(\varepsilon)$ such that

$$\delta\left(\left\{k \in \mathbb{N} : |x_k - x_{N(\varepsilon)}| \geq \varepsilon\right\}\right) = 0.$$

Fridy in 1985 ([6]) showed that the structure of statistically convergent sequence is analogous to the structure of convergent subsequence that the set of its index is a natural dense subset of $\mathbb{N}$.

Theorem 1.3. *([6]) A sequence $x = \{x_n\}_n$ is statistically convergent if and only if it is statistically Cauchy sequence. Also, st-$\lim\limits_{n\to\infty} x_n = l$ if and only if there exists a convergent sequence $y = \{y_n\}_n$ such that $x_n = y_n$ a.a.n.*

Kaya et. al in ([7]) represented a different proof from that of [6] to show that a sequence x is statistically convergent to l if and only if there exists $K \subseteq \mathbb{N}$ with $\delta(K) = 1$ such that x is converges to l in K, i.e

$$\text{st-}\lim_{n\to\infty} x_n = \lim_{\substack{n \to \infty \\ n \in K}} x_n = l.$$

Let $\mathcal{C}_{st}$ be the space of all real valued statistically convergent sequences. Following lemma guarantees that the statistically convergent sequencses space $\mathcal{C}_{st}$ is a vector space under addition and scalar multiplication over $\mathbb{R}$.

Lemma 1.4. *([4]) Assume that* $\{x_n\}_n$ *and* $\{y_n\}_n$ *are in* $\mathcal{C}_{st}$ *such that st-*$\lim_{n\to\infty} x_n = L_1$ *and st-*$\lim_{n\to\infty} y_n = L_2$*, for some* L_1 *and* L_2 *in* $\mathbb{R}$*. Then,*

(i) *st-*$\lim_{n\to\infty}(x_n + y_n) = L_1 + L_2$.

(ii) *st-*$\lim_{n\to\infty}(x_n\, y_n) = L_1\, L_2$.

(iii) *st-*$\lim_{n\to\infty}(c\, x_n) = c\, L_1$*, for any* $c \in \mathbb{R}$.

In this paper we introduce an equivalence relation "$\sim$" on the vector space $\mathcal{C}_{st}$ to give an inner product on the quotient space $\mathcal{C}_{st}/\sim$. We define the induced J-metric d_J on $\mathcal{C}_{st}$ and prove that $(\mathcal{C}_{st}, d_J)$ is a complete metric space. We also show that the subspace $\mathcal{C}$ of all real valued convergent sequences is a dense subspace of metric space $(\mathcal{C}_{st}, d_J)$.

2 Main Results

Remember that a mapping $f : \mathbb{R} \to \mathbb{R}$ is called *sequentially continuous* if $\lim_{n\to\infty} f(x_n) = f(x)$ whenever $\lim_{n\to\infty} x_n = x$. Analogous to the concept of sequentially contiuous we give the following definition which is used in the sequel.

Definition 2.1. A mapping $f : \mathbb{R} \to \mathbb{R}$ is said to be *statistically sequentially continuous*, when

$$\text{st-}\lim_{n\to\infty} x_n = x \quad \text{implies that} \quad \text{st-}\lim_{n\to\infty} f(x_n) = f(x).$$

Lemma 2.2. *Every sequentially continuous function on $\mathbb{R}$ is statistically sequentially continuous.*

Proof. Let $\{x_n\}_n$ be a sequence which is statistically converges to l and f be a sequentially continuous map. Then there exists a subset K of $\mathbb{N}$ with $\delta(K) = 1$ such that

$$\text{st-}\lim_{n\to\infty} x_n = \lim_{\substack{n \to \infty \\ n \in K}} x_n = l.$$

Since f is a sequential continuous map then

$$\text{st-}\lim_{n\to\infty} f(x_n) = \lim_{\substack{n \to \infty \\ n \in K}} f(x_n) = f(l).$$

□

For example by Lemma 2.2 for the absolute value function we have

$$\text{st-}\lim_{n\to\infty} |x_n| = |\text{st-}\lim_{n\to\infty} x_n|, \text{ where } \{x_n\}_n \in \mathcal{C}_{st}. \tag{1}$$

Recently in ([8]) Kucukaslan et al. have defined an equivalence relation on the set of all sequences of points from a metric space by introducing $x \sim y$ if and only if there is a natural dense subset M of $\mathbb{N}$ such that $x_n = y_n$ for all $n \in M$. So $x \sim y$ if and only if $x_n = y_n$ a.a.n. Note that if $x_n = y_n$ for almost all n and the two sequences are statistically convergent, then $\text{st-}\lim_{n\to\infty} x_n = \text{st-}\lim_{n\to\infty} y_n$. The converse is not necessarily true. For example;

$$\text{st-}\lim_{n\to\infty} \frac{1}{n} = \text{st-}\lim_{n\to\infty} 0 = 0,$$

but $\frac{1}{n} \neq 0$ for all $n \in \mathbb{N}$.

In the following we want to give an equivalence relation that some how generalizes the equivalence relation introduced by Kucukaslan et al. on $\mathcal{C}_{st}$. We will use this to introduce an inner product known as *J-inner product* on $\mathcal{C}_{st}$ and finally we will show that the J-metric space $\mathcal{C}_{st}$ is complete, hence it is a Hilbert space.

Definition 2.3. Suppose $x = \{x_n\}_n$ and $y = \{y_n\}_n$ are in $\mathcal{C}_{st}$. Define the relation "$\sim$" on the set $\mathcal{C}_{st}$ as follows

$$''x \sim y'' \text{ if and only if } \text{st-}\lim_{n\to\infty} x_n = \text{st-}\lim_{n\to\infty} y_n.$$

Obviously the relation "$\sim$" is an equivalence relation on $\mathcal{C}_{st}$. The quotient space $\mathcal{C}_{st}/\sim$ (briefly $\tilde{\mathcal{C}_{st}}$) is the collection of all equivalence classes

$$\tilde{x} = [x] := \left\{ \{y_n\}_n \in \mathcal{C}_{st} : \text{ st-}\lim_{n\to\infty} x_n = \text{ st-}\lim_{n\to\infty} y_n \right\}.$$

Note that $\tilde{x} = \tilde{y}$ if and only if st-$\lim_{n\to\infty} x_n =$ st-$\lim_{n\to\infty} y_n$. Also $\tilde{x} = \tilde{0}$ if and only if st-$\lim_{n\to\infty} x_n = 0$ (i.e $x \in \tilde{0}$).

Define the real valued function φ on $\mathcal{C}_{st} \times \mathcal{C}_{st}$ by

$$\varphi(x, y) = \text{st-}\lim_{n\to\infty} (x_n\, y_n), \tag{2}$$

where $x = \{x_n\}_n \in \mathcal{C}_{st}$ and $y = \{y_n\}_n \in \mathcal{C}_{st}$. Clearly, the function φ is a semi-inner product on the vector space $\mathcal{C}_{st}$ over $\mathbb{R}$. Note that if $x = \{0\}_n$, then $\varphi(x, x) = 0$. But the converse is not true in general for example $\varphi(x, x) = 0$, where $0 \neq x = \{x_n\}_n = \{\frac{1}{n}\}_n$.

The following lemma shows that the semi inner product φ induces an inner product on $\tilde{\mathcal{C}_{st}}$.

Lemma 2.4. *Let $\tilde{F}$ be a function on $\tilde{\mathcal{C}_{st}} \times \tilde{\mathcal{C}_{st}}$ defined by $\tilde{F}(\tilde{x}, \tilde{y}) = \varphi(x, y)$ where $x = \{x_n\}, y = \{y_n\}$ are in $\mathcal{C}_{st}$ and φ is defined by (2). Then $\tilde{F}$ is an inner product on $\tilde{\mathcal{C}_{st}}$.*

Proof. Let $x = \{x_n\}$ and $y = \{y_n\}$ be in $\mathcal{C}_{st}$ such that $s = \{s_n\} \in \tilde{x}$ and $t = \{t_n\} \in \tilde{y}$. So

$$\text{st-}\lim_{n\to\infty} x_n = \text{st-}\lim_{n\to\infty} s_n \quad \text{and} \quad \text{st-}\lim_{n\to\infty} y_n = \text{st-}\lim_{n\to\infty} t_n.$$

Then

$$\varphi(x, y) = \text{st-}\lim_{n\to\infty} x_n\, y_n = \text{st-}\lim_{n\to\infty} s_n\, t_n = \varphi(s, t).$$

This shows that $\tilde{F}$ is well defined. Clearly, $\tilde{F}$ is a bi-linear functional on $\tilde{\mathcal{C}_{st}}$ over $\mathbb{R}$ with $\tilde{F}(\tilde{x}, \tilde{x}) \geq 0$ and $F(\tilde{x}, \tilde{x}) = 0$ if and only if $\tilde{x} = \tilde{0}$. Therefore $\tilde{F}$ is an inner product on $\tilde{\mathcal{C}_{st}}$. $\square$

From now on, we refer to the bi-linear function φ as the J-inner product on $\mathcal{C}_{st}$ and use $(\mathcal{C}_{st}, \varphi)$ to denote the J-inner product space. Note that $\varphi(x, x) = 0$ if and only if $\tilde{x} = \tilde{0}$.

Define $\|.\|_J : \mathcal{C}_{st} \to \mathbb{R}$ by

$$\|x\|_J = \varphi(x,x)^{1/2} = \text{st-}\lim_{n\to\infty} |x_n|,$$

where $x = \{x_n\}_n \in \mathcal{C}_{st}$ and φ is the J-inner product on $\mathcal{C}_{st}$.

The pair $(\mathcal{C}_{st}, \|.\|_J)$ is called *J-normed space.*

The $\|.\|_J$ enables us to introduce a metric, $\tilde{d_J}$, on the set $\tilde{\mathcal{C}_{st}}$ as $\tilde{d_J}(\tilde{x},\tilde{y}) = d_J(x,y) = \|x-y\|_J$. In this case $(\tilde{\mathcal{C}_{st}}, \tilde{d_J})$ and $(\mathcal{C}_{st}, d_J)$ have identical metric. So from now on we interchangeably use the metric space $(\mathcal{C}_{st}, d_J)$. Note that $d_J(x,x) = 0$ if and only if $\tilde{x} = \tilde{0}$.

Remark 2.5. For any $x = \{x_n\}_n$ and $y = \{y_n\}_n$ in the J-metric space $(\mathcal{C}_{st}, d_J)$, $d_J(x,y) = 0$ if and only if "$x \sim y$".

In fact, if st-$\lim\limits_{n\to\infty} x_n = \ell_1$ and st-$\lim\limits_{n\to\infty} y_n = \ell_2$, for some ℓ_1 and ℓ_2 in $\mathbb{R}$, then by (1)

$$\begin{aligned} |\ell_1 - \ell_2| &= |\text{st-}\lim_{n\to\infty}(x_n - y_n)| \\ &= st-\lim_{n\to\infty}|x_n - y_n| \\ &= d_J(x,y). \end{aligned}$$

Definition 2.6. Let $\{y^m\}_m$ be a sequence such that for all $m \in \mathbb{N}$, $y^m = \{y^m_k\}_k \in \mathcal{C}_{st}$. A $\mathcal{C}_{st}$-valued sequence $\{y^m\}_m$ is said to be

(*i*) J$-$*convergent* to $y = \{y_k\}_k \in \mathcal{C}_{st}$ if $\lim_{m\to\infty} d_J(y^m, y) = 0$, i.e.,

$$\lim_{m\to\infty}\left(\text{st-}\lim_{k\to\infty}|y^m_k - y_k|\right) = 0$$

holds. In this case, we write $y^m \overset{d_J}{\to} y$ as $m \to \infty$.

(*ii*) *J-Cauchy* if $\lim\limits_{m,n\to\infty} d_J(y^n, y^m) = 0$.

(*iii*) The J- metric space $\mathcal{C}_{st}$ is *complete* if every J-Cauchy sequence in $\mathcal{C}_{st}$ is a J-convergent sequence to an element of $\mathcal{C}_{st}$.

(*iv*) A subset K of the J-metric space $\mathcal{C}_{st}$ is called *d_J-dense*, if for each $x \in \mathcal{C}_{st}$ there exists a K- valued sequence $\{y^n\}_n$ such that $\lim_{n\to\infty} d_J(y^n, x) = 0$.

Similar to a metric space, we refer to τ_{d_J} as the topology on $\mathcal{C}_{st}$ that is induced by the generalized metric d_J. Then

$$\begin{aligned} B(x,r) &= \{y \in \mathcal{C}_{st} : d_J(x,y) < r\} \\ &= \{y \in \mathcal{C}_{st} : st-\lim_{n\to\infty} |x_n - y_n| < r\} \end{aligned}$$

is an open ball with center $x = \{x_n\}$ and radius $r > 0$.

If $x, y \in \mathcal{C}_{st}$ and "$x \sim y$", then $B(x,r) = B(y,r)$ holds for all $r > 0$.

Proposition 2.7. *Let $\{x^k\}_k$ be a $\mathcal{C}_{st}$-valued sequence with $x^k = \{x^k_m\}_m \in \mathcal{C}_{st}$. Then, for any $y^1 = \{y^1_m\}$ and $y^2 = \{y^2_m\}_m$ in $\mathcal{C}_{st}$ the following statements hold.*

(a) If $x^k \overset{d_J}{\to} y^1$ as $k \to \infty$ and $y^1 \sim y^2$, then $x^k \overset{d_J}{\to} y^2$, as $k \to \infty$.

(b) If $x^k \overset{d_J}{\to} y^1$ and $x^k \overset{d_J}{\to} y^2$ as $k \to \infty$, then $y^1 \sim y^2$.

Proof. (a) Let $\ell \in \mathbb{R}$ be the statistical limit of y^1 and y^2 . Let $\varepsilon > 0$.

By Theorem 1.3 there are two natural dense subset K_1 and K_2 of $\mathbb{N}$ such that

$$|y^i_m - \ell| < \frac{\varepsilon}{3},\ m \in K_i,\ i = 1, 2. \tag{3}$$

According to the assumption there exists a positive number N such that st-$\lim_{m\to\infty} |x^k_m - y^1_m| < \frac{\varepsilon}{3}$ for all $k \geq N$. By considering the natural dense subset $K = K_1 \cap K_2$ of $\mathbb{N}$ we have

$$|x^k_m - y^2_m| \leq |x^k_m - y^1_m| + |y^1_m - \ell| + |y^2_m - \ell|, \tag{4}$$

for all $k \geq N$ and $m \in K, i = 1, 2,$.

By (3) and (4) st-$\lim_{m\to\infty} |x^k_m - y^2_m| < \varepsilon$ for all $k \geq \mathbb{N}$. So $\lim_{k\to\infty} d_J(x^k, y^2) = 0$.

(b) Let $\varepsilon > 0$, $x^k \overset{d_J}{\to} y^1$ and $x^k \overset{d_J}{\to} y^2$ as $k \to \infty$. There are two natural numbers N_1 and N_2 such that for all $k \geq N_i$, ,$i = 1, 2$

$$d_J(x^k, y^i) < \frac{\varepsilon}{2}. \tag{5}$$

Let $k \geq \max\{N_1, N_2\}$. By the triangle inequality of d_J and (5) we have

$$0 \leq d_J(y^1, y^2) < d_J(y^1, x^k) + d_J(x^k, y^2) < \varepsilon.$$

This completes the proof. □

Theorem 2.8. *The space $\mathcal{C}$ of real valued convergent sequences is a d_J-dense subspace of $\mathcal{C}_{st}$.*

Proof. Let $\varepsilon > 0$ and $x = \{x_k\}_k \in \mathcal{C}_{st}$ be a sequence statistically convergent to $t \in \mathbb{R}$. Then, by Theorem 1.3 there is a natural dense subset K of $\mathbb{N}$ such that

$$\text{st-}\lim_{k\to\infty} x_k = \lim_{\substack{k \to \infty \\ k \in K}} x_k = t$$

holds. From the last equality, there exists a positive number $N_0(\varepsilon) \in \mathbb{N}$ such that

$$|x_k - t| < \varepsilon \tag{6}$$

holds for all k in the natural dense subset $\{k \geq \mathbb{N}_0 : \; k \in K\}$. Let N_1 be a positive number such that $\frac{1}{k} < \varepsilon$ when $k \geq N_1$ and let N be the $\max\{N_0, N_1\}$. Then the subset $A = \{k \in K : \; k \geq N\}$ is a natural dense subset of $\mathbb{N}$.

Now we define the $\mathcal{C}$-valued sequence $\{y^m\}_m$, where $y^m = \{y^m_k\}_k$ as follows

$$y^m_k = \begin{cases} x_k + \frac{1}{mk} & k \in A,\, m \geq \mathbb{N} \\ t & k \notin A \, or \, m < \mathbb{N}. \end{cases} \tag{7}$$

So by (6) and (7) we have

$$|y^m_k - t| = |x_k - t + \frac{1}{mk}| < (1 + \frac{1}{m})\varepsilon$$

for all k, thus

$$\lim_{k \to \infty} |y^m_k - t| = 0,$$

hence $y^m \in \mathcal{C}$ for all m.
On the other hand by definition of y^m, $|y^m_k - x_k| < \frac{1}{m}\varepsilon$ for all k in the natural dense subset A and $m \geq N$. Also

$$d_J(y^m, x) < \frac{1}{m}\varepsilon.$$

So the proof is completed. □

We use the following lemma to prove the completeness of $(\mathcal{C}_{st}, d_J)$.

Lemma 2.9. *Let $\{y^m\}_m$ be a $\mathcal{C}_{st}$-valued J-Cauchy sequence and let the real number y_0^m be the statistical limit of y^m, for all $m \in \mathbb{N}$. Then, the real valued sequence $\{y_0^m\}_m$ is a convergence sequence.*

Proof. Let $\varepsilon > 0$ and N be a positive number such that

$$d_J(y^m, y^n) < \varepsilon \text{ for all } m > n \geq N. \tag{8}$$

According to the assumption for any $m \in \mathbb{N}$ there is a dense subset K_m such that

$$\text{st-}\lim_{k\to\infty} y_k^m = \lim_{\substack{k \to \infty \\ k \in K_m}} y_k^m = y_0^m \tag{9}$$

where $y^m = \{y_k^m\}_k \in \mathcal{C}_{st}$. Put $K = K_m \cap K_n$ when $m > n \geq N$. Then by (8) and (9) we have

$$\begin{aligned} |y_0^m - y_0^n| &= |\text{st-}\lim_{k\to\infty} (y_k^m - y_k^n)| \\ &= |\lim_{\substack{k \to \infty \\ k \in K}} (y_k^m - y_k^n)| \\ &= \lim_{\substack{k \to \infty \\ k \in K}} |y_k^m - y_k^n| \\ &= d_J(y^m, y^n) \\ &< \varepsilon. \end{aligned} \tag{10}$$

So (10) implise $\{y_0^m\}_m$ is a real valued Cauchy sequence and then it converges to some $y_0 \in \mathbb{R}$. □

Theorem 2.10. *The pair $(\mathcal{C}_{st}, d_J)$ is a J-complete metric space.*

Proof. Suppose $\{y^m\}_m$ is a $\mathcal{C}_{st}$-valued J-Cauchy sequence, when $y^m = \{y_k^m\}_k$. Let ε be an arbitrary positive number. By Lemma 2.9 the following statements hold.

(i) There are a positve number N_0 and a natural dense subset K_0 such that $|y_k^m - y_k^n| < \varepsilon$ hold for all $m > n \geq N_0$ and $k \in K_0$.

(ii) There exisits a real number $y_0^m \in \mathbb{R}$ such that st-$\lim_{k\to\infty} y_k^m = y_0^m$ $m = 1, 2, 3, ...$ Also for each m there is a natural dense subset $K_m^{'}$ of $\mathbb{N}$ and a positive number N_m so that $|y_k^m - y_0^n| < \varepsilon$ hold for $k \in \{k \in K_m^{'} : k \geq N_m\}$.

(iii) The real valued sequence $\{y_0^m\}_m$ converges to a real number y_0. So there is a positve number $N_0^{'}$ such that $|y_0^m - y_0| < \varepsilon$ holds for all $m \geq N_0^{'}$.

Denote the natural dense subset $K_m := K_0 \cap \{k \in K_m^{'} : k \geq N_m\}$ m=1,2,3,... and $N = max\{N_0, N_0^{'}\}$.

Define a sequence $x = \{x_k\}_k$ by

$$x_k = \begin{cases} y_k^N & k \in \cup_{m=N}^{\infty} K_m \\ \\ y_0 & otherwise. \end{cases}$$

We must show that $x \in \mathcal{C}_{st}$ and $\lim\limits_{m\to\infty} d_J(y^m, x) = 0$. If $k \in \cup_{m=N}^{\infty} K_m$, then $k \in K_m$ for some $m \geq N$ and

$$|x_k - y_0| \leq |y_k^N - y_k^m| + |y_k^m - y_0^m| + |y_0^m - y_0| < 3\varepsilon.$$

Therefore st-$\lim_{k\to\infty} x_k = y_0$ $m = 1, 2, 3, ...$ that is $x \in \mathcal{C}_{st}$. Suppose $m \geq N$ and $k \in K_m$ then

$$|y_k^m - x_k| = |y_k^m - y_k^N| < \varepsilon. \tag{11}$$

So (11) holds for $k \in \cup_{m=N}^{\infty} K_m$. Hence $d_J(y^m, x) < \varepsilon$ for all $m \geq N$ and it completes the proof. □

Remark 2.11. According to the introduced inner product on $\mathcal{C}_{st}$ and the induced metric d_J on $\mathcal{C}_{st}$, by Theorem 2.10 the space $\mathcal{C}_{st}$ is a Hilbert space.

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