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Original Research Paper

Existence of Three Weak Solutions for some Singular Elliptic Problems with Hardy Potential

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Abstract. In this paper, under growth conditions on the nonlinearity, we obtain the existence of at least three weak solutions for some singular elliptic problems involving the p-Laplacian, subject to Dirichlet boundary conditions in a smooth bounded domain in $\mathbb{R}^N$. The approach is based on variational methods and critical point theory.

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1 Introduction

In this paper, we want to investigate the following problem

$$\begin{cases} -\Delta_p u + \dfrac{|u|^{p-2}u}{|x|^p} = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where λ is positive parameter and $\Delta_p u$:=div$(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplace operator, Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ containing the origin and with smooth boundary $\partial\Omega$, $1 < p < N$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$(\mathrm{f}_1) \qquad |f(x,t)| \leq a_1 + a_2|t|^{q-1}, \qquad \forall (x,t) \in \Omega \times \mathbb{R},$$

for some non-negative constants a_1, a_2 and $q \in]1, p^*[$, where

$$p^* := \frac{pN}{N-p}.$$

Several results are known concerning the existence of solutions for singular elliptic problems, and we mention the works [5–10]. For example in [5], the authors obtained the existence of one solution for the problem

$$\begin{cases} -\Delta_p u = \dfrac{|u|^{p-2}u}{|x|^p} + \lambda f(x,u), & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

based on variational methods and critical point theory. Also, in [11] the authors have considered the problem (1) and they have obtained the existence of two distinct weak solutions requiring that the continuous and subcritical nonlinear term f satisfies the celebrated Ambrosetti-Rabinowitz condition.

Nonlinear singular elliptic equations are encountered in glacial advance, in transport of coal slurries down conveyor belts and in several other geophysical and industrial contents(see [3]).

In this work, our goal is to obtain the existence of at least three weak solutions for the problem (1) , by using variational methods.

Recall that a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be a Carathéodory function , if

(C_1) the function $x \to f(x,t)$ is measurable for every $t \in \mathbb{R}$;

(C_2) the function $t \to f(x,t)$ is continuous for a.e. $x \in \Omega$.

2 Preliminaries and Basic Definitions

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 2)$ containing the origin and with smooth boundary $\partial\Omega$. Further, denote by X the space $W_0^{1,p}$ (Ω)

endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p}.$$

Let $1 < p < N$, we recall classical Hardy's inequality, which says that

$$\int_\Omega \frac{|u(x)|^p}{|x|^p} dx \leq \frac{1}{H} \int_\Omega |\nabla u(x)|^p dx, \qquad (\forall u \in X) \tag{2}$$

where $H := (\frac{N-p}{p})^p$;(see, for instance, the paper [6]). By the compact embedding $X \hookrightarrow L^q(\Omega)$ for each $q \in [1, p^*[$, there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)} \leq c_q \|u\|, \qquad (\forall u \in X) \tag{3}$$

where c_q is the best constant.
Let us define $F(x,\xi) := \int_0^\xi f(x,t)dt$, for every (x,ξ) in $\Omega \times \mathbb{R}$. Moreover, we introduce the functional $I_\lambda : X \to \mathbb{R}$ associated with (1),

$$I_\lambda(u) := \Phi(u) - \lambda \Psi(u), \qquad (\forall u \in X)$$

where

$$\Phi(u) := \frac{1}{p} \left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega \frac{|u(x)|^p}{|x|^p} dx \right), \ \Psi(u) := \int_\Omega F(x, u(x))dx.$$

It is known that $\Phi, \Psi \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$, and

$$\Phi^{'}(u)(v) = \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla v dx + \int_\Omega \frac{|u|^{p-2}}{|x|^p} uv dx$$

and

$$\Psi^{'}(u)(v) = \int_\Omega f(x, u(x)) v(x) dx$$

for each $u, v \in W_0^{1,p}(\Omega)$.

Now we present one proposition that will be needed to prove the main theorem of this paper.

Proposition 2.1. *Let $T : X \to X^*$ be the operator defined by*

$$T(u)(v) := \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) dx + \int_\Omega \frac{|u(x)|^{p-2}}{|x|^p} u(x) v(x) dx,$$

for every $u, v \in X$. Then T is strictly monotone.

Proof. Clearly T is coercive .Taking into account (2.2) of [12] for $p > 1$ there exists a positive constant C_p such that if $p \geq 2$, then

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq C_p|x - y|^p,$$

if $1 < p < 2$, then

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq C_p \frac{|x - y|^2}{(|x| + |y|)^{2-p}},$$

where $\langle ., .\rangle$ denotes the usual inner product in $\mathbb{R}^N$. Thus, it is easy to see that, if $p \geq 2$, then, for any $u, v \in X, with \quad u \neq v$,

$$\langle Tu - Tv, u - v\rangle \geq C_p \int_\Omega |\nabla u(x) - \nabla v(x)|^p dx = C_p \|(u - v)\|^p > 0,$$

and if $1 < p < 2$ then,

$$\langle Tu - Tv, u - v\rangle \geq C_p \int_\Omega \frac{|\nabla u(x) - \nabla v(x)|^2}{(|\nabla u| + |\nabla v|)^{2-p}} dx > 0,$$

for every $u, v \in X$, which means that T is strictly monotone. □

Moreover, by Theorem 3.1 of [4] and proposition 2.1 ,Φ is weakly lower semicontinuous and $\Phi' : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ is a homeomorphism. Condition (f_1) and compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ imply that the functional Ψ has compact derivative. From the Hardy's inequality (see (2)), it follows that

$$\frac{\|u\|^p}{p} \leq \Phi(u) \leq \left(\frac{H + 1}{pH}\right) \|u\|^p,$$

for every $u \in X$.

Fixing the real parameter λ, a function $u : \Omega \to \mathbb{R}$ is said to be a weak solution of (1) if $u \in X$ and

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx + \int_\Omega \frac{|u(x)|^{p-2}}{|x|^p}u(x)v(x)dx$$

$$-\lambda\int_\Omega f(x,u(x))v(x)dx = 0,$$

for every $v \in X$. Hence, the critical points of I_λ are exactly the weak solutions of (1).

Our main tools are the following critical point theorems.

Theorem 2.2 ([2], Theorem 3.6)**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_{x\in X}\Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that:
(a_1) $\frac{\sup_{\Phi(x)\le r}\Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$;
(a_2) for each $\lambda \in \Lambda_r := \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r}\Psi(x)}\right[$ the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2.3 ([1], Corollary 3.1)**.** *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

1. $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

2. for each $\lambda > 0$ and for every $u_1, u_2 \in X$ which are local minima for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are two positive constants r_1, r_2 and $\overline{v} \in X$, with $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$, such that

(b_1) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \dfrac{2}{3}\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$;

(b_2) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} < \dfrac{1}{3}\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$.

Then, for each

$$\lambda \in \left]\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})},\ \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)},\ \frac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}\right\}\right[$$

, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.

3 Main results

In this section we establish the main results of this paper. Now, fix $x_0 \in \Omega$ and pick $D > 0$ such that $B(x_0, D) \subset \Omega$ not containing origin, where $B(x_0, D)$ denotes the ball with center x_0 and radious D.

Theorem 3.1. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that condition* (f_1) *holds. Moreover, assume that*

(f_2) *there exist $\alpha \in [0, +\infty[$ and $1 < \gamma < p$ such that*

$$F(x,t) \leq \alpha(1 + |t|^\gamma),$$

for each $(x,t) \in \Omega \times \mathbb{R}$;

(f_3) $F(x,t) \geq 0$ *for each $(x,t) \in \Omega \times \mathbb{R}^+$;*

(f_4) *there exist $r > 0$ and $\delta > 0$ with $r < \frac{1}{p}(\frac{2\delta}{D})^p m(D^N - (\frac{D}{2})^N)$ such that*

$$\bar{\omega_r} := \frac{1}{r}\left(a_1 c_1 (pr)^{\frac{1}{p}} + \frac{a_2}{q}(c_q)^q (pr)^{\frac{q}{p}}\right) < \frac{p\inf_{x\in\Omega} F(x,\delta)}{\left(\frac{H+1}{H}\right)(\frac{2\delta}{D})^p(2^N-1)}$$

where c_1 and c_q are the best constants in (3).

Then, for each $\lambda \in \Lambda_{r,\delta} = \left]\frac{\left(\frac{H+1}{H}\right)(\frac{2\delta}{D})^p(2^N-1)}{p\inf_{x\in\Omega} F(x,\delta)}, \frac{1}{\bar{\omega_r}}\right[$, the problem (1) admits at least three weak solutions.

Proof. Our aim is to apply Theorem 2.2 to problem (1). To this end let $X := W_0^{1,p}(\Omega)$ with the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p dx\right)^{1/p},$$

and the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{p}\left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega \frac{|u(x)|^p}{|x|^p}dx\right),$$

and

$$\Psi(u) := \int_\Omega F(x, u(x))dx,$$

for all $u \in X$.
As seen before, the functionals Φ and Ψ satisfy the regularity assumptions requested in Theorem 2.2. Now, let $\bar{v} \in X$ be defined by

$$\bar{v}(x) = \begin{cases} 0 & \text{x} \in \Omega \setminus B(x_0, D) \\ \frac{2\delta}{D}(D - |x - x_0|) & \text{x} \in B(x_0, D) \setminus B(x_0, \frac{D}{2}), \\ \delta & \text{x} \in B(x_0, \frac{D}{2}) \end{cases}$$

where $|.|$ denotes the Euclidean norm on $\mathbb{R}^{\mathbb{N}}$. We have

$$\frac{1}{p}\left(\frac{2\delta}{D}\right)^p m\left(D^N - \left(\frac{D}{2}\right)^N\right) \le \Phi(\bar{v})$$

$$\leq \left(\frac{H+1}{pH}\right)\left(\frac{2\delta}{D}\right)^p m\left(D^N - \left(\frac{D}{2}\right)^N\right)$$

where $m := \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})}$ is the measure of unit ball of $\mathbb{R}^{\mathbb{N}}$ and Γ is the Gamma function. Thanks to (f_3),

$$\Psi(\bar{v}) \geq \int_{B(x_0,\frac{D}{2})} F(x,\bar{v}(x))dx \geq \inf_{x\in\Omega} F(x,\delta) m\left(\frac{D}{2}\right)^N$$

and so

$$\frac{\Psi(\bar{v})}{\Phi(\bar{v})} \geq \frac{p\inf_{x\in\Omega} F(x,\delta)}{\left(\frac{H+1}{H}\right)\left(\frac{2\delta}{D}\right)^p (2^N-1)}.$$

From $r < \frac{1}{p}\left(\frac{2\delta}{D}\right)^p m\left(D^N - \left(\frac{D}{2}\right)^N\right)$, one has $r < \Phi(\bar{v})$. Bearing in mind define the functional Φ, we see that

$$\begin{aligned} \Phi^{-1}(]-\infty,r]) &= \{u\in X;\ \Phi(u)\leq r\} \\ &\subseteq \left\{u\in X;\ \frac{\|u\|^p}{p}\leq r\right\}. \end{aligned}$$

So, the compact embedding $X \hookrightarrow L^q(\Omega)$ and (f_1) imply that, for each $u \in \Phi^{-1}(]-\infty,r])$, we have

$$\Psi(u) \leq a_1\int_\Omega |u(x)|dx + \frac{a_2}{q}\int_\Omega |u(x)|^q dx \leq a_1c_1\|u\| + \frac{a_2}{q}(c_q\|u\|)^q$$

$$\leq a_1c_1(pr)^{\frac{1}{p}} + \frac{a_2}{q}(c_q)^q(pr)^{\frac{q}{p}}$$

and so

$$\frac{1}{r}\sup_{\Phi(u)\leq r}\Psi(u) \leq \frac{1}{r}\left(a_1c_1(pr)^{\frac{1}{p}} + \frac{a_2}{q}(c_q)^q(pr)^{\frac{q}{p}}\right)$$

and so condition (a_1) of Theorem 2.2 is verified. Now, let us introduce the integral functional related to problem (1)

$$I_\lambda(.) := \Phi(.) - \lambda\Psi(.)$$

and we prove that, for each $\lambda > 0$, I_λ is coercive. By arguments similar to those used before, we obtain

$$\int_\Omega |u(x)|^\gamma dx \leq (c_\gamma \|u\|)^\gamma$$

and so, for each $u \in X$ with $\|u\| \geq \max\{1, \frac{1}{c_\gamma}\}$, from (f_2) one has

$$\Psi(u) = \int_\Omega F(x, u(x))dx \leq \int_\Omega \alpha(1 + |u(x)|^\gamma)dx \leq \alpha\left(|\Omega| + (c_\gamma\|u\|)^\gamma\right).$$

This leads to

$$I_\lambda(u) \geq \frac{1}{p}\|u\|^p - \lambda\alpha\left(|\Omega| + (c_\gamma\|u\|)^\gamma\right)$$

and, since $\gamma < p$, coercivity of I_λ is obtained. Taking into account that

$$\Lambda_{r,\delta} \subseteq \left]\frac{\Phi(\bar{v})}{\Psi(\bar{v})}, \frac{r}{\sup_{\Phi(u)\leq r}\Psi(u)}\right[,$$

Theorem 2.2 ensures that, for each $\lambda \in \Lambda_{r,\delta}$, the functional I_λ admits at least three critical points in X that are weak solutions of the problem (1). $\square$

Remark 3.2. In Theorem 3.1, if we consider $f(x, 0) \neq 0$, then we obtain the existence of at least three non-zero weak solutions.

Remark 3.3. According to the Sobolev embedding theorem there is a positive constant c such that

$$\|u\|_{L^{p^*}(\Omega)} \leq c\|u\|, \qquad (\forall u \in X). \tag{4}$$

The best approximation for constant c in (4) is

$$c := \frac{1}{N\sqrt{\pi}}\left(\frac{N!\Gamma(\frac{N}{2})}{2\Gamma(\frac{N}{p})\Gamma(N+1-\frac{N}{p})}\right)^{1/N}\eta^{1-\frac{1}{p}},$$

where

$$\eta := \frac{N(p-1)}{N-p},$$

(see, for instance, [13]). The consequences of using the Hölder's inequality, in (3), is as follows

$$c_q \leq \frac{\mathrm{meas}(\Omega)^{\frac{p^*-q}{p^*q}}}{N\sqrt{\pi}} \left(\frac{N!\Gamma(\frac{N}{2})}{2\Gamma(\frac{N}{p})\Gamma(N+1-N/p)} \right)^{1/N} \eta^{1-1/p},$$

where meas (Ω) denotes the Lebesgue measure of the set Ω.

Another the main result of this section is as follows.

Theorem 3.4. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory function with assumption (f_1) that satisfies the condition $f(x,t) \geq 0$ for every $(x,t) \in \Omega \times \mathbb{R}$. Moreover, assume that there exist three positive constants r_1, r_2 and δ with $r_1 < \frac{1}{2p}\left(\frac{2\delta}{D}\right)^p m\left(D^N - \left(\frac{D}{2}\right)^N\right)$ and $2\left(\frac{H+1}{pH}\right)\left(\frac{2\delta}{D}\right)^p m\left(D^N - \left(\frac{D}{2}\right)^N\right) < r_2$. Furthermore, suppose that*

(B_1) $\bar{\omega}_{r_1} := \frac{1}{r_1}\left(a_1 c_1 (pr_1)^{\frac{1}{p}} + \frac{a_2}{q}(c_q)^q (pr_1)^{\frac{q}{p}}\right) < \frac{2}{3}\frac{p\inf_{x\in\Omega} F(x,\delta)}{\left(\frac{H+1}{H}\right)\left(\frac{2\delta}{D}\right)^p (2^N-1)}$;

(B_2) $\bar{\omega}_{r_2} := \frac{1}{r_2}\left(a_1 c_1 (pr_2)^{\frac{1}{p}} + \frac{a_2}{q}(c_q)^q (pr_2)^{\frac{q}{p}}\right) < \frac{1}{3}\frac{p\inf_{x\in\Omega} F(x,\delta)}{\left(\frac{H+1}{H}\right)\left(\frac{2\delta}{D}\right)^p (2^N-1)}$.

Then, for each $\lambda \in \left]\frac{3}{2}\frac{(\frac{H+1}{H})(\frac{2\delta}{D})^p(2^N-1)}{p\inf_{x\in\Omega} F(x,\delta)}, \min\{\frac{1}{\bar{\omega}_{r_1}}, \frac{1}{2\bar{\omega}_{r_2}}\}\right[$, the problem (1) admits at least three weak solutions u_i for $i = 1, 2, 3$, such that $\|u_i\| < (p\, r_2)^{\frac{1}{p}}$.

Proof. Take Φ and Ψ as in the proof of Theorem 3.1. Our aim is to verify (b_1) and (b_2) in Theorem 2.3. To this end, choose $\bar{v}$ as given in Theorem 3.1. Using

$$\frac{1}{p}\left(\frac{2\delta}{D}\right)^p m\left(D^N - \left(\frac{D}{2}\right)^N\right) \leq \Phi(\bar{v}) \leq$$

$$\left(\frac{H+1}{pH}\right)\left(\frac{2\delta}{D}\right)^p m\left(D^N - \left(\frac{D}{2}\right)^N\right)$$

and theorem data it is clear that we have $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$. Now we have

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} = \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \int_\Omega F(x,u(x))dx}{r_1} \leq \bar{\omega}_{r_1} < \frac{1}{\lambda} < \frac{2}{3}\frac{\Psi(\bar{v})}{\Phi(\bar{v})}$$

and

$$\frac{2\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} = \frac{2\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \int_\Omega F(x,u(x))dx}{r_2} \leq 2\,\bar{\omega}_{r_2} < \frac{1}{\lambda} < \frac{2}{3}\frac{\Psi(\bar{v})}{\Phi(\bar{v})} .$$

Therefore, (b_1) and (b_2) of Theorem 2.3 are established. Finally, we verify that $\Phi - \lambda\Psi$ satisfies the assumption *2.* of Theorem 2.3. Let u_1 and u_2 be two local minima for $\Phi-\lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi-\lambda\Psi$, and so, they are weak solutions for the problem (1). We will show that they are nonnegative.

Let $\bar{u}$ be a weak solution of problem (1). Using the argument of contradiction, assume that the set $T = \{x \in \Omega : \bar{u}(x) < 0\}$ is non-empty and of positive measure. Put $u^*(x) = \min\{0, \bar{u}(x)\}$ for all $x \in \Omega$. It is clear that, $u^* \in X$ and hence we have

$$\int_\Omega |\nabla\bar{u}(x)|^{p-2}\nabla\bar{u}(x)\nabla u^*(x)dx + \int_\Omega \frac{|\bar{u}(x)|^{p-2}}{|x|^p}\bar{u}(x)u^*(x)dx$$

$$-\lambda\int_\Omega f(x,\bar{u}(x))u^*(x)dx = 0.$$

Thus, from our sign assumptions on the data, we have

$$0 \leq \int_T |\nabla\bar{u}(x)|^p dx + \int_T \frac{|\bar{u}(x)|^p}{|x|^p}dx \leq 0.$$

Hence, $\bar{u} = 0$ in T and this is antithesis. Hence, $u_1(x) \geq 0$ and $u_2(x) \geq 0$ for every $x \in \Omega$. Thus, it follows that $su_1 + (1-s)u_2 \geq 0$ for all $s \in [0,1]$, and that

$$f(x, su_1 + (1-s)u_2) \geq 0,$$

and consequently, $\Psi(su_1 + (1-s)u_2) \geq 0$, for every $s \in [0,1]$.

By using Theorem 2.3, for every

$$\lambda \in \left]\frac{3}{2}\frac{\left(\frac{H+1}{H}\right)(\frac{2\delta}{D})^p(2^N-1)}{p\inf_{x\in\Omega} F(x,\delta)}, \min\{\frac{1}{\bar{\omega}_{r_1}}, \frac{1}{2\,\bar{\omega}_{r_2}}\}\right[\subseteq$$

$$\left]\frac{3}{2}\frac{\Phi(w)}{\Psi(w)},\ \min\left\{\frac{r_1}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}, \frac{r_2/2}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which are the weak solutions of the problem (1) and the proof is complete. $\square$

To illustrate the Theorem 3.2, we provide an example.

Example 3.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function with $F(t) = \int_0^t f(\xi)d\xi$ and $f(t) \leq t^2$ for every $t \in \mathbb{R}$. Also suppose that there exist positive constants r_1 , r_2 and δ such that the following inequalities hold.

(i_1) $r_1 < \min\{\delta^{\frac{3}{2}}\pi, \frac{4}{3(c_3)^3}\}$, $4\left(\frac{H+1}{H}\right)\delta^{\frac{3}{2}}\pi < r_2 < \frac{2}{3(c_3)^3}$

(i_2) $3\left(\frac{H+1}{H}\right)\delta^{\frac{3}{2}} < \int_0^\delta f(\xi)d\xi$

where c_3 is best constant in (3).In this case problem

$$\begin{cases} -\Delta_{\frac{3}{2}} u + \dfrac{|u|^{\frac{-1}{2}}u}{|x|^{\frac{3}{2}}} = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω be a bounded domain in $\mathbb{R}^2$ containing the origin and containing the ball with radious 2 not containing origin and with smooth boundary $\partial\Omega$, has at least three weak solutions. To this end according to conditions $f(t) \leq |t|^2$ and (f_1) we can consider $a_1 = 0$, $a_2 = 1$ and $q = 3$. also consider $p = \frac{3}{2}$ and $D = 2$. On the other hand, from $N = 2$ we have $m = \pi$. Hence according to Theorem 3.2, it is enough to show that

$$\lambda = 1 \in \left]\frac{3}{2}\frac{\left(\frac{H+1}{H}\right)(\frac{2\delta}{D})^p(2^N-1)}{p\inf_{x\in\Omega} F(x,\delta)}, \min\{\frac{1}{\bar{\omega}_{r_1}}, \frac{1}{2\,\bar{\omega}_{r_2}}\}\right[$$

and this, according to the inequalities (i_1) and (i_2) can be easily researched.

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