Construction of iterative adaptive methods with memory with 100% improvement of convergence order

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Abstract. An efficient family of the recursive methods of adaptive is proposed for solving nonlinear equations and is developed such that all previous information is applied. These methods have reached the maximum degree of convergence improvement of 100% and also have an efficiency index of 2. Three families have been examined from Steffensen-Like single, two, and three-step methods that have used 2, 3 and 4 parameters, respectively. Numerical comparisons have made with other existing methods one-, two-, three-, and four-point to show the performance of the convergence speed of the proposed method and confirm theoretical results.

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1 Introduction

It is often necessary for scientific and engineering practices to find a root of a polynomial or a nonlinear equation. Undoubtedly, Traub is the pioneer in classifying iterative methods for solving such equations as one or multi-point [46]. It is well-known that Newton’s method is one of the most common iterative methods to approximate the solution \( \alpha \) of \( f(x) = 0 \) is of great importance [30]. However, the condition of derivative existence for function \( f \) in a neighborhood of the required root is mandatory indeed for convergence of Newton’s method, which restricts its applications in practice. To overcome this problem, Steffensen replaced the first derivative of the function in Newton’s iterate by forward finite difference approximation. Steffensen-type methods without using derivatives only compute divided differences and can be used for nondifferentiable problems. Traub his book proved the best one-point iterative method should achieve the order of convergence \( n \) using \( n \) function evaluations. In trying to furnish multi-point methods of various orders, the Kung-Traub conjecture [23] is a crucial part of the development. Based on this hypothesis, a multi-point iteration without memory using \( n \) evaluations per full cycle possesses the maximal order of convergence \( 2^{n-1} \), which is called the optimal order. Following the Kung and Traub conjecture, many authors tried to construct optimal multipoint methods without memory [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 17, 19, 20, 21, 22, 23, 29, 31, 33, 34, 36, 37, 38, 40, 41, 47, 50]. Traub developed the first method with memory by applying Steffensen’s method [40], and increased the order of convergence of this method from 2 to 2.41 (improvement 20.5\%) without using any new information, and only by reusing the information of the previous step. Traub presented his memory method by entering a free parameter to Steffensen’s method as follows:

\[
\begin{align*}
\gamma_k &= \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}, \quad k = 1, 2, 3, \ldots, \\
x_{k+1} &= x_k - \frac{\gamma_k f(x_k)^2}{f(x_k + \gamma_k f(x_k)) - f(x_k)} \quad k = 0, 1, 2, \ldots.
\end{align*}
\]

(1)

To see more related papers in the with memory methods of study, the readers might refer to [10, 12, 14, 15, 16, 24, 25, 26, 27, 28, 32, 39, 43, 44, 45, 46, 48, 49]. We develop an adaptive method with memory i.e., that uses the information not only from the last two-steps, but also from
all previous iterations. This technique enables us to achieve the highest efficiency both theoretically and practically. Adaptive methods with memory have efficiency index 2, hence competes for all the existing methods without and with memory in the literature. It should be noted that improvement of the degree of convergence up to 100% has been mentioned in references [42], but these families are different from the mentioned methods. The computational efficiency in the sense of Ostrowski-Traub [31, 46], of an iterative method of the order $p$, requiring $n$ function evaluations per iteration, is frequently calculated using Ostrowski-Traub’s efficiency index $E(p, n) = p^{1/n}$. We later compare both numerical performances and efficiency index of the proposed method with some significant methods to show our claims. To achieve and remodify the optimal one-, two-, three- steps methods, we approximate and update the introduced accelerator parameters in each iteration by suitable kind and optimal of Newton’s interpolation.

The main objective of this paper is to achieve the highest efficiency index, 2, without imposing an evaluation of the function. Contents of the paper are summarized in what follows. The next section deals with modifying the optimal one-, two-, three-points methods without memory introduced by Zheng et al.[50], Soleymani et al.[39], and Lotfi-Assari.[27]. In Section 3, with-memory methods with maximum self-referential parameters (one, two, and three) are presented for one, two, and three-step methods, respectively. The new class of recursive with methods of adaptive is supported with detailed proof in this section to verify the construction theoretically. Numerical examples are given in Section 4 to illustrate the convergence behavior of our methods for simple roots. Finally, a short conclusion has been given in the last section.

2 Modified Steffensen-Like Methods

2.1 One step method by Zheng et al.

In this section, we deal with modifying the one-point method without memory by Zheng et al. [50], such that the error equation has two ac-
celsators. Zheng et al.’s method has the iterative expression as follows:

\[
\begin{align*}
  w_k &= x_k + \gamma f(x_k), \quad k = 0, 1, 2, \ldots, \\
  x_{k+1} &= x_k - \frac{f(x_k)}{f'[x_k, w_k]},
\end{align*}
\]

where $\gamma \in \mathbb{R}$ is nonzero arbitrary parameter. To transform Eq. (2) into a method with memory, with two accelerators:

\[
\begin{align*}
  w_k &= x_k + \gamma_k f(x_k), \quad k = 0, 1, 2, \cdots, \\
  x_{k+1} &= x_k - \frac{f(x_k)}{f'[x_k, w_k] + q_k f[w_k]},
\end{align*}
\]

where $\gamma$ and $q$ are arbitrary nonzero parameters. In what follows, we present the error equation of Eq. (3).

**Theorem 2.1.** Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ be a scalar function which has a simple root $\alpha$ in the open interval $I$, and also the initial approximation $x_0$ is sufficiently close the simple zero, then, the one-step iteration method (3) has two-order satisfies the following error equation:

\[
e_{k+1} = (1 + f'(\alpha)\gamma)(q + c_2)e_k^2 + O(e_k^3).
\]

**Proof.** Using symbolic computation following code written in the computational software package Mathematica has given.

In [1]: $f[e_-] = fla(e + \sum_{i=2}^{3} c_i e^i);
In [2]: ew = e + \gamma Series[f[e], \{e, 0, 3\}]//FullSimplify
Out [2]: $(1 + \gamma fla)e + O(e) ^2$
In [3]: $f\{x, y\} = \frac{f[x] - f[y]}{x - y};$
In [4]: $e_{k+1} = e - Series[f[e], \{e, 0, 3\}]//FullSimplify
Out [4]: $(1 + f'(\alpha)\gamma)(q + c_2)e^2 + O(e^3).$

\[\square\]

### 2.2 Acceleration of modified Soleymani et al.’s method

This Section concerns modifying Soleymani et al.’s method (SLTKM) [39] so that it could be considered for the proposed scheme in the next
Section. Let recall the mentioned method:

\[
\begin{align*}
    w_k &= x_k + \gamma f(x_k), \quad y_k = x_k - \frac{f(x_k)}{f(y_k)}, \quad k = 0, 1, 2, \ldots, \\
    x_{k+1} &= y_k - \frac{f(y_k)}{f[w_k, y_k] + qf(w_k)} (1 + \frac{f(y_k)}{f(x_k)}),
\end{align*}
\]

where \( w_k = x_k + \gamma f(x_k), 0 \neq \gamma, q \) and \( \gamma, q \in \mathbb{R} \), and \( f[x, y] = \frac{f(x) - f(y)}{x - y} \) stands for the divided difference of the first order. This method is an optimal-method without-memory. In order words, it uses three function evaluations per iteration and has optimal convergence order 4. It is possible to adapt-method (5) in some ways that it remains optimal in the sense of Kung and Traub conjecture [23] as follows:

\[
\begin{align*}
    w_k &= x_k + \gamma_k f(x_k), \quad y_k = x_k - \frac{f(x_k)}{f(y_k)}, \quad k = 0, 1, 2, \ldots, \\
    x_{k+1} &= y_k - \frac{f(y_k)}{f[w_k, y_k] + q_k f(w_k)} (1 + \frac{f(y_k)}{f(x_k)}),
\end{align*}
\]

where \( \gamma, \lambda \) and \( q \) are arbitrary nonzero real parameters. The next theorem states the error equation of the method (6).

**Theorem 2.2.** Let \( I \subseteq \mathbb{R} \) be an open interval, \( f : I \rightarrow \mathbb{R} \) be a differentiable function, and has a simple zero, say \( \alpha \). If \( x_0 \) is an initial guess to \( \alpha \), then the error equation of the method (6) is given by

\[
e_{k+1} = \frac{1}{f'(\alpha)} ((1 + \gamma f'(\alpha))^2 (q + c_2) (\lambda + f'(\alpha) q (1 + \gamma f'(\alpha)) + f'(\alpha) c_2) (2q(2 + \gamma f'(\alpha)) + (3 + \gamma f'(\alpha)) c_2) - f'(\alpha) c_3) e_k + O(e_k^5),
\]

where \( c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)} \) for \( k = 2, 3, \ldots \).

**Proof.** We use the self-explained mathematical approach to avoid the tedious and humdrum algebraic manipulation. First, we define the Taylor’s series of \( f(x) \) as follows:

\[
In[1] : f[e_\cdot] = f[a(e + c_2 e^2 + c_3 e^3 + c_4 e^4)];
\]

where \( e = x - \alpha, f[a] = f'(\alpha) \). Note that since \( \alpha \) is a simple zero of \( f(x) \), then \( f'(\alpha) \neq 0, f(\alpha) = 0 \). We define

\[
In[2] : f[x_\cdot, y_\cdot] = \frac{f[x] - f[y]}{x - y};
\]
In[3]: \( ew = e + \gamma f[e] \);
In[4]: \( ey = e - \text{Series}[\frac{f[e]}{f[ey, ew] + qf[ew]}, \{e, 0, 4\}] \);

\[
\begin{align*}
In[5]: e_{k+1} &= ey - \text{Series}\left[\frac{f[ey]}{f[ey, ew] + qf[ew] + \lambda(ey - e)(ey - ew)} \right. \nonumber \\
&\quad \left. (1 + \frac{f[ey]}{f[e]}), \{e, 0, 4\}\right]/\text{FullSimplify} \\
Out[5]: e_{k+1} &= \frac{1}{f'(\alpha)}(1 + \gamma f'(\alpha))^2(q + c_2)(\lambda + f'(\alpha)q^2(1 + \gamma f'(\alpha)) \\
&+ f'(\alpha)c_2(2q(2 + \gamma f'(\alpha)) + (3 + \gamma f'(\alpha))c_2) - f'(\alpha)c_3e_k^4 + O(e_k^5) \nonumber.
\end{align*}
\]
This completes the proof. □

2.3 Acceleration of the modified Lotfi and Assari’s method

This Section concerns with modifying Lotfi and Assari’s method (LAM) [27], so that it could be considered for the proposed scheme in the next Section. Let me recall the mentioned method:

\[
\begin{align*}
w_k &= x_k + \gamma f(x_k), \ y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \ z_k = y_k - \frac{f(y_k)}{f'(y_k)}, \ x_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}, \ k = 0, 1, 2, \ldots \nonumber.
\end{align*}
\]
(7)

This optimal method without memory uses four function evaluations per iteration and has convergence order 8. To transform Eq. (7) in a with-memory method with four accelerators, we consider the following modification of (7) [27]:

\[
\begin{align*}
w_k &= x_k + \gamma f(x_k), \ k = 1, 2, 3, \ldots \\
y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \ k = 0, 1, 2, \ldots \\
z_k &= y_k - \frac{f(y_k)}{f'(y_k)}, \ k = 0, 1, 2, \ldots \\
x_{k+1} &= z_k - \frac{f(z_k)}{f'(z_k)}.
\end{align*}
\]
(8)
where $\gamma_k, \beta_k, \lambda_k$ and $q_k$ are nonzero arbitrary parameters. We give the following convergence theorem for the proposed method (8) as follows:

**Theorem 2.3.** Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ be a differentiable function, and has a simple zero, say $\alpha$. If $x_0$ is an initial guess to $\alpha$, then the error equation of the method (8) is given by

$$e_{k+1} = (1 + \gamma f'(\alpha))^4(q + c_2)^2(\lambda + f'(\alpha)c_2(q + c_2) - f'(\alpha)c_3)(-\beta + c_2)
\begin{align*}
lambda + f'(\alpha)c_2(q + c_2) - f'(\alpha)c_3 + f'(\alpha)c_4) & f'(\alpha)^{-2}c_k^8 + O(c_k^9).
\end{align*}
(9)

**Proof.** First, we define Taylor’s series of $f(x)$ as follows:

$$In[1]: f[e_] = fla(e + c_2e^2 + \cdots + c_8e^8),$$

where $e = x - \alpha$, $f[1]a = f'(\alpha)$. Note that since $\alpha$ is a simple zero of $f(x)$, the $f'(\alpha) \neq 0, f(\alpha) = 0$. We define

$$In[2]: f[x_-, y_] = \frac{f[x] - f[y]}{x - y};$$
$$In[3]: f[x_-, y_, z_] = \frac{f[x, y] - f[y, z]}{x - z};$$
$$In[4]: f[x_-, y_, z_, t_] = \frac{f[x, y, z] - f[y, z, t]}{x - t};$$
$$In[5]: ew = e + \gamma f[e];$$
$$In[6]: ey = e - \text{Series}[f[e] \frac{f[e]}{f[e, ew] + qf[ew]}, \{e, 0, 8}];$$
$$In[7]: ez = ey - \text{Series}[f[ew][e] \frac{f[e]}{f[e, ew] + qf[ew]}, \{e, 0, 8}];$$
$$In[8]: e_{k+1} = ez - \text{Series}[f[ew][e] \frac{f[e]}{f[e, ew] + qf[ew]}, \{e, 0, 8}]]/\text{FullSimplify}$$

$Out[8]: e_{k+1} = ((1 + \gamma f'(\alpha))^4(q + c_2)^2(\lambda + f'(\alpha)c_2(q + c_2) - f'(\alpha)c_3)(-\beta + c_2)
\begin{align*}
&\lambda + f'(\alpha)c_2(q + c_2) - f'(\alpha)c_3 + f'(\alpha)c_4) & f'(\alpha)^{-2}c_k^8 + O(c_k^9)
\end{align*}

And thus proof is completed. □
3 Development the recursive adaptive method with memory

This Section deals with the main contribution of this work. In other words, it has attempted to introduce a recursive adaptive method with a memory so that it has the highest possible efficiency index as proposed to methods with memory in the literature. It is worth mentioning that some special-cases of this new method cover the existing-methods.

3.1 One step adaptive method

This Section concerns with extracting the novel method with-memory from \( (3) \) by using two self-accelerating parameters. Theorem \( (2.1) \) states that the modified method \( (3) \) has the order of convergence 2 if \( \gamma \neq \frac{1}{f'(\alpha)} \) and \( q \neq -c_2 \). Now, we pose some questions: Is it possible to increase the order of convergence of this method? If so, how can it be done, and what is the new convergence order? For answering these questions, we look at the error equation \( (4) \). As can be seen that if we set \( \gamma = \frac{1}{f'(\alpha)} \) and \( q = -c_2 = \frac{f''(\alpha)}{2f'(\alpha)} \), then at least the coefficient of \( e_k^2 \) disappears. However, since \( \alpha \) is not determined and consequently, \( f'(\alpha) \) and \( f''(\alpha) \) cannot be computed. On the other hand, we can approximate \( \alpha \) using available data and therefore improve the order of convergence. Following the same idea in the methods with memory, this issue can be resolved. However, we are going to do it more efficiently, say recursive adaptively. Let us describe it a little more. If we use information from the current and only the last iteration, we come up with the method with memory introduced in \([27, 28]\). Also, note that we have considered the best approximations. Hence, to this end, the following approximates are applied

\[
\gamma_k = \frac{-1}{N_2'(x_k)} \approx \frac{-1}{f'(\alpha)}, \quad q_k = \frac{N_3''(w_k)}{-2N_3'(w_k)} \approx -\frac{f''(\alpha)}{2f'(\alpha)},
\]

where \( k = 1, 2, \cdots \), the \( N_2'(x_k), N_3'(w_k) \) are Newton’s interpolating polynomials of two and third degree, set through three and four best available approximations (nodes) \((x_k, x_{k-1}, w_{k-1})\) and \((w_k, x_k, x_{k-1}, w_{k-1})\), respectively. It should be noted that if one uses lower Newton’s interpo-
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In the iterative formula (4), lower accelerators are obtained.

Replacing the fixed parameters \( q \) and \( \gamma \) in the iterative formula (4) by the varying \( \gamma_k \) and \( q_k \) calculated by (4), we propose the following new methods with memory,

\[
\begin{align*}
x_0, q_0, \gamma_0 \text{ are given, and } \quad w_0 &= x_0 + \gamma_0 f(x_0) \\
g_k &= \frac{-1}{N_{2k}(x_k)}, \quad q_k = \frac{N''_{2k+1}(w_k)}{-\left(2N'_{2k+1}(w_k)\right)}, \quad k = 1, 2, \ldots, \\
w_k &= x_k + \gamma_k f(x_k), \quad x_{k+1} = x_k - \frac{f(x_k)}{f(x_k, w_k) + q_k f(w_k)}, \quad k = 0, 1, 2, \ldots.
\end{align*}
\]

Here, we answer the second question regarding the order of convergence of the method with memory (11). In what follows, we discuss the general convergence analysis of the recursive adaptive method with memory (11). It should be noted that the convergence order varies as the iteration go ahead. First, we need the following lemma.

Lemma 3.1. If \( \gamma_k = \frac{-1}{N_{2k}(x_k)} \) and \( q_k = \frac{N''_{2k+1}(w_k)}{-2N'_{2k+1}(w_k)} \), then

\[
\begin{align*}
(1 + \gamma_k f'(\alpha)) &\sim \prod_{s=0}^{k-1} e_s e_{s,w}, \\
(q_k + c_2) &\sim \prod_{s=0}^{k-1} e_s e_{s,w},
\end{align*}
\]

where \( e_s = x_s - \alpha, e_{s,w} = w_s - \alpha. \)

Proof. The proof is similar to the Lemma 4 mentioned in [44]. □

The following result determines the order of convergence of the one-point iterative method with memory (11).

Theorem 3.2. If an initial estimation \( x_0 \) is close enough to a simple root \( \alpha \) of \( f(x) = 0 \), and \( \gamma_0 \) and \( q_0 \) are uniformly bounded above, being \( f \) a real sufficiently differentiable function, then the R-order of convergence of the one-point method adaptive with memory (11) is obtained from the following system of nonlinear equations.

\[
\begin{align*}
\begin{cases}
r^k p - (1 + p)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - r^k = 0, \\
r^{k+1} - 2(1 + p)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 2r^k = 0,
\end{cases}
\end{align*}
\]

(14)
where \( r \) and \( p \) are the convergence order of the sequences \( \{x_k\} \) and \( \{w_k\} \), respectively. Also, \( k \) indicates the number of iterations.

**Proof.** Let \( \{x_k\} \) and \( \{w_k\} \) be convergent with orders \( r \) and \( p \), respectively. Then:

\[
\begin{align*}
    e_{k+1} &\sim e_k^r \sim e_{k-1}^{r^2} \sim \ldots \sim e_0^{r^{k+1}}, \\
    e_{k,w} &\sim e_k^p \sim e_{k-1}^{rp} \sim \ldots \sim e_0^{pr^k},
\end{align*}
\]

(15)

where \( e_k = x_k - \alpha \) and \( e_{k,w} = w_k - \alpha \). Now, by Lemma (3.1) and Eq (15), we obtain

\[
(1 + \gamma_k f'(\alpha)) \sim \prod_{s=0}^{k-1} e_{s}\,e_{s,w} = (e_0 e_{0,w}) \ldots (e_{k-1} e_{k-1,w})
\]

\[
= (e_0 e_0^p)(e_0^{r_0} e_0^{rp}) \ldots (e_0^{r_{k-1}} e_0^{r_{k-1}p})
\]

\[
= e_0^{(1+p)+(1+p)r+\ldots+(1+p)r^{k-1}}
\]

\[
= e_0^{(1+p)(1+r+\ldots+r^{k-1})}.
\]

(16)

Similarly, we get

\[
(q_k + c_2) \sim e_0^{(1+p)(1+r+\ldots+r^{k-1})}.
\]

(17)

By considering the errors of \( w_k \) and \( x_{k+1} \) in Eq. (15), and Eqs. (16)-(17). We conclude:

\[
e_{k,w} \sim (1 + \gamma_k f'(\alpha))e_k \sim e_0^{(1+p)(1+r+\ldots+r^{k-1})} e_0^{r_k},
\]

(18)

\[
e_{k+1} \sim (1 + \gamma_k f'(\alpha))(q_k + c_2)e_k^2 \sim e_0^{(1+p)(1+r+\ldots+r^{k-1})} e_0^{2r^k}.
\]

(19)

To obtain the desire result, it is enough to match the right-hand-side of the Eqs(15),(18), and (19):

\[
\begin{align*}
    r^k p - (1 + p)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - r^k & = 0, k = 1, 2, \ldots, \\
    r^{k+1} - 2(1 + p)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 2r^k & = 0.
\end{align*}
\]

This completes the proof of the Theorem. \( \square \)
Remark 3.3. For $k = 1$, the order of convergence of the method with memory (11) can be computed from the following system of equations:

\[
\begin{align*}
 rp - (1 + p) - r &= 0, \\
r^2 - 2(1 + p) - 2r &= 0.
\end{align*}
\]  

This system of equations has the solution $p = \frac{1}{4}(3 + \sqrt{17}) \approx 1.78078$, and $r = \frac{1}{2}(3 + \sqrt{17}) \approx 3.56155$. This special case gives the given result by Dzunic [14] and denoted by DM. If $k = 2$, the system of equations (14) becomes:

\[
\begin{align*}
 r^2p - (1 + p + rp + r + r^2) &= 0, \\
r^3 - 2(1 + p + rp + r + r^2) &= 0.
\end{align*}
\]  

This system of equations has the solution: $p \approx 1.95029$ and $r \approx 3.90057$. Also, Positive solution of the system (14) for $k = 3$, is given by $p \approx 1.98804$ and $r \approx 3.97609$. And, positive solution of the system (14) for $k = 4$, is given by (has been shown by TAM4) $p \approx 1.99705$ and $r \approx 3.99941$.

As can be seen, the order of convergence is very close to 4, so its efficiency index is very close to 2. This efficiency is astonishingly remarkable.

3.2 Two-steps adaptive method

Let us look at the error equation of the modified method (6). It is clear that there are some possibilities to vanish the coefficient of $e_k^4$. For example, if $1 + \gamma f'(\alpha) = 0$, $q + c_2 = 0$, or $(\lambda + f'(\alpha))q^2(1 + \gamma f'(\alpha)) + f'(\alpha)c_2(2q(2 + \gamma f'(\alpha)) + (3 + \gamma f'(\alpha))c_2 - f'(\alpha)c_3 = 0$, then the coefficient of $e_k^4$ vanishes at once. To get the best result, we suggest that all these relations hold simultaneously. We note that this can happen theoretically. To be more precise, it can be seen that these relations lead to $\gamma = \frac{-1}{f'(\alpha)}$, $q = -c_2 = \frac{f''(\alpha)}{2f'(\alpha)}$, and $\lambda = f'(\alpha)c_3 = \frac{f'''(\alpha)}{6}$. Since $\alpha$ is not known at hand, it is impossible to compute $f'(\alpha)$, $f''(\alpha)$, and $f'''(\alpha)$. Even worse, if we assume that $\alpha$ is known, computing $f'(\alpha)$, $f''(\alpha)$, and $f'''(\alpha)$ is not suggested since it increases these function evaluations, and therefore, it spoils that optimality of the method (6). Following the same idea in the methods with memory, this issue can be resaved. However, we are going
to do it more efficiently, say recursive adaptively. Note that we have considered the best approximations. Hence

\[
\begin{align*}
\gamma_k &= -\frac{1}{N_3'(x_k)} \simeq \frac{f'(\alpha)}{f''(\alpha)}, \\
q_k &= -\frac{N_4''(w_k)}{2N_4(w_k)} \simeq -\frac{f''(\alpha)}{2f'(\alpha)}, \\
\lambda_k &= \frac{N_5''(y_k)}{6} \simeq \frac{f''(\alpha)}{6},
\end{align*}
\]  

(22)

where \(N_3'(x_k), N_4''(w_k)\) and \(N_5''(y_k)\) are Newton’s interpolation polynomials go through the nodes \({x_k, x_{k-1}, w_{k-1}, y_{k-1}}\), \({w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}}\), and \({y_k, w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}}\), respectively. This situation has been studied in [10, 11, 27, 28, 39]. Such methods are not adaptive. To construct a recursive adaptive method with memory, we use the information not only in the current and its previous iterations but also in all the previous iterations, i.e., from the beginning to the current iteration. Thus, as iterations proceed, the degree of interpolation polynomials increases, and the best-updated approximations for computing the self-accelerator \(\gamma_k, q_k, \text{and } \lambda_k\) are obtained. Let \(x_0, \gamma_0, q_0, \text{and } \lambda_0\) be given suitably. Then,

\[
\begin{align*}
\gamma_k &= -\frac{1}{N_3'(x_k)}, \\
q_k &= -\frac{N_4''(w_k)}{2N_4(w_k)} \simeq -\frac{f''(\alpha)}{2f'(\alpha)}, \\
\lambda_k &= \frac{N_5''(y_k)}{6} \simeq \frac{f''(\alpha)}{6}, \\
y_k &= x_k - \frac{f(x_k)}{f(x_k) + q_k f(w_k)}, \\
x_{k+1} &= y_k - \frac{f(y_k)}{f(y_k) + q_k f(w_k) + \lambda_k (y_k - x_k) (y_k - w_k)} (1 + \frac{f(y_k)}{f(x_k)}).
\end{align*}
\]  

(23)

In what follows, we discuss the general convergence analysis of the recursive adaptive method with memory (23). It should be noted that the convergence order varies as the iteration go ahead. First, we need the following lemma.

**Lemma 3.4.** If \(\gamma_k = -\frac{1}{N_3'(x_k)}, q_k = -\frac{N_4''(w_k)}{2N_4(w_k)}, \text{and } \lambda_k = \frac{N_5''(y_k)}{6},\) then:
CONSTRUCTION OF ITERATIVE ADAPTIVE METHODS WITH MEMORY

\[(1 + \gamma_k f'(\alpha)) \sim \prod_{s=0}^{k-1} e_s e_{s,w} e_{s,y}, \quad (24)\]

\[(c_2 + q_k) \sim \prod_{s=0}^{k-1} e_s e_{s,w} e_{s,y}, \quad (25)\]

\[(\lambda_k + f'(\alpha)q^2(1 + \gamma_k f'(\alpha)) + f'(\alpha)c_2(2q(2 + \gamma_k f'(\alpha))) + (3 + \gamma_k f'(\alpha)c_2) e_{s,w} e_{s,y}, \quad (26)\]

where \(e_s = x_s - \alpha, e_{s,w} = w_s - \alpha, e_{s,y} = y_s - \alpha.\)

**Proof.** The proof is similar to Lemmas 2.1 and 2.2 in [44]. □

**Theorem 3.5.** Let \(x_0\) be a suitable initial guess to the simple root \(\alpha\) of \(f(x) = 0.\) Also, suppose the initial values \(\gamma_0, q_0,\) and \(\lambda_0\) are chosen appropriately. Then the \(R\)-order of the recursive adaptive method with memory \((23)\) can be obtained from the following system of nonlinear equations:

\[
\begin{align*}
& r^k p_1 - (1 + p_1 + p_2)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - r^k = 0, \\
& r^k p_2 - 2(1 + p_1 + p_2)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 2r^k = 0, \\
& r^{k+1} - 4(1 + p_1 + p_2)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 4r^k = 0,
\end{align*}
\]

where \(r, p_1\) and \(p_2\) are the order of convergence of the sequences \(\{x_k\}, \{w_k\},\) and \(\{y_k\},\) respectively. Also, \(k\) indicates the number of iterations.

**Proof.** Let \(\{x_k\}, \{w_k\},\) and \(\{y_k\},\) be convergent with orders \(r, p_1,\) and \(p_2,\) respectively. Then:

\[
\begin{align*}
&e_{k+1} \sim e_k^r \sim e_{k-1}^{r^2} \sim \ldots \sim e_0^{r^{k+1}}, \\
&e_{k,w} \sim e_k^{p_1} \sim e_{k-1}^{p_{1r}} \sim \ldots \sim e_0^{p_{1r}k}, \\
&e_{k,y} \sim e_k^{p_2} \sim e_{k-1}^{p_{2r}} \sim \ldots \sim e_0^{p_{2r}k},
\end{align*}
\]
where \( e_k = x_k - \alpha, e_{k,w} = w_k - \alpha \) and \( e_{k,y} = y_k - \alpha \). Now, by Lemma (3.4) and Eq (27), we obtain

\[
(1 + \gamma_k f'(\alpha)) \sim \prod_{s=0}^{k-1} e_s e_s, w e_s, y = (e_0 e_0, w e_0, y) \cdots (e_{k-1} e_{k-1}, w e_{k-1}, y)
\]

\[
= (e_0 e_0 p_1 e_0 p_2) (e_0 e_0 r p_1 r p_2) \cdots (e_0 e_0 r^{k-1} p_1 e_0 r^{k-1} p_2)
\]

\[
= e_0 (1+p_1+p_2+r^2+\cdots+r^{k-1}).
\]

Similarly, we get:

\[
(\lambda_k + f'(\alpha)q^2 (1 + \gamma_k f'(\alpha)) + f'(\alpha) e_0 (1+p_1+p_2+r^2+\cdots+r^{k-1})
\]

\[
= (\lambda_k + f'(\alpha)q^2 (1 + \gamma_k f'(\alpha)) + f'(\alpha) e_0 (1+p_1+p_2+r^2+\cdots+r^{k-1})
\]

By considering the errors of \( w_k, y_k, \) and \( x_{k+1} \) in Eq. (27), and Eqs. (29)-(31), we conclude:

\[
\begin{align*}
e_{k,w} & \sim (1 + \gamma_k f'(\alpha)) e_k \sim e_0^{(1+p_1+p_2+r^2+\cdots+r^{k-1})} e_0^{r_k} \\
e_{k,y} & \sim (1 + \gamma_k f'(\alpha)) (q_k + c_2) e_k^2 \sim e_0^{(1+p_1+p_2+r^2+\cdots+r^{k-1})^2} e_0^{2r_k} \\
e_{k+1} & \sim (1 + \gamma_k f'(\alpha))^2 (q_k + c_2) (\lambda_k + f'(\alpha)q^2 (1 + \gamma_k f'(\alpha)) + f'(\alpha) e_0 (1+p_1+p_2+r^2+\cdots+r^{k-1}) e_0^{4r_k}
\end{align*}
\]

To obtain the desired result, it is enough to match the right-hand-side of the Eqs. (27),
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(32), (33), and (34). Then

\[
\begin{aligned}
& r^k p_1 - (1 + p_1 + p_2)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - r^k = 0, \quad k = 1, 2, \ldots, \\
& r^k p_2 - 2(1 + p_1 + p_2)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 2r^k = 0, \\
& r^{k+1} - 4(1 + p_1 + p_2)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 4r^k = 0.
\end{aligned}
\]

This completes the proof of the Theorem. □

Remark 3.6. Positive solution of system (23), (k = 1), is specified through (It been shown by TAM7): \( p_1 \approx 1.88, p_2 \approx 2.76 \), and \( r \approx 7.53 \).

And, if \( k = 2 \), we obtain the order of convergence: \( p_1 \approx 3.96, p_2 \approx 3.99, r \approx 7.94 \). Also, \( k = 3 \), the solution of equations (27) has the solution: \( p_1 \approx 4.98, p_2 \approx 3.99, r \approx 7.99 \). Likewise, for \( k = 4 \), we obtain the order of convergence:

\[
p_1 \approx 1.99, p_2 \approx 3.99, r \approx 7.99
\]

(been shown by TAM8). In this case the efficiency index is \( 7.99 + 1 = 2 \) which shows that our developed method compels all the existing methods with memory.

3.3 Three-steps adaptive method

This Section introduced a new efficient adaptive method with memory. We continued as before and developed a three steps method with memory with the best efficiency index. Indeed, we achieve efficiency index 2. Also, note that we have considered the best approximations (8).

Hence

\[
\begin{aligned}
\gamma_k &= -\frac{N_1'(x_k)}{N_1(x_k)} \approx -\frac{f'(\alpha)}{f(\alpha)}, \\
q_k &= \frac{N_2''(w_k)}{2N_2'(w_k)} \approx -\frac{f''(\alpha)}{2f'(\alpha)}, \\
\lambda_k &= \frac{N_6''(y_k)}{N_6'(y_k)} \approx f'(\alpha)c_4 = \frac{f'''(\alpha)}{f'(\alpha)c_4}, \\
\beta_k &= \frac{N_7''(y_k)}{24} \approx f'(\alpha)c_4 = \frac{f'''(\alpha)}{24}.
\end{aligned}
\]

where \( N_1'(x_k), N_2''(w_k), N_6''(y_k) \) and \( N_7''(z_k) \) are Newton’s interpolation polynomials go through the nodes \( \{x_k, x_{k-1}, w_{k-1}, y_{k-1}, z_{k-1}\} \).
\[ \{w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}, z_{k-1}\}, \{y_k, w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}, z_{k-1}\}, \]

and \[ \{z_k, y_k, w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}, z_{k-1}\}, \]

respectively. The degree of interpolation polynomials increases, and the best-updated approximations for computing the self-accelerator \( \gamma_k, q_k, \lambda_k \) and \( \beta_k \) are obtained. Now, we can present the first three-step adaptive memory method as follows:

\[
\begin{align*}
\gamma_k &= -\frac{1}{N_{4k}(x_k)}, 
q_k &= N'_{4k+1}(w_k) + N''_{4k+2}(y_k),
\lambda_k &= \frac{N''_{4k+2}(y_k)}{6}, 
\beta_k &= \frac{N''_{4k+2}(z_k)}{24}, \quad k = 1, 2, 3, \ldots,
\end{align*}
\]

\[w_k = x_k + \gamma_k f(x_k), \quad y_k = x_k - \frac{f(x_k)}{f(w_k)}, \quad k = 0, 1, 2, \ldots, \]

\[z_k = y_k - \frac{f(w_k - y_k)}{f(w_k - y_k)}, \quad \lambda_k = (y_k - x_k)(y_k - w_k), \]

\[x_{k+1} = x_k + (f(w_k, x_k, y_k) - f(w_k, x_k, z_k))(y_k - z_k) + \beta_k (y_k - w_k)(x_k - x_k)(y_k - w_k). \]

(37)

In what follows, we discuss the general convergence analysis of the recursive adaptive method with memory (37). It should be noted that the convergence order varies as the iteration go ahead. We need the following lemma.

**Lemma 3.7.** If \( \gamma_k = -\frac{1}{N_{4k}(x_k)}, \) \( q_k = -\frac{N'_{4k+1}(w_k) + N''_{4k+2}(y_k)}{2N_{4k+1}(w_k)}, \) \( \lambda_k = \frac{N''_{4k+2}(y_k)}{6}, \)

and \( \beta_k = \frac{N''_{4k+2}(z_k)}{24}, \) then:

\[
(1 + \gamma_k f'(\alpha)) \sim \prod_{s=0}^{k-1} e_s e_{s,w} e_{s,y} e_{s,z}, \quad (38)
\]

\[
(c_2 + q_k) \sim \prod_{s=0}^{k-1} e_s e_{s,w} e_{s,y} e_{s,z}, \quad (39)
\]

\[
(\lambda_k + f'(\alpha)c_2(q_k + c_2) - f'(\alpha)c_3) \sim \prod_{s=0}^{k-1} e_s e_{s,w} e_{s,y} e_{s,z}, \quad (40)
\]

\[
(\beta_k + c_2(\lambda_k + f'(\alpha)c_2(q_k + c_2) - f'(\alpha)c_3) - f'(\alpha)c_4) \sim \prod_{s=0}^{k-1} e_s e_{s,w} e_{s,y} e_{s,z}, \quad (41)
\]

where \( e_s = x_s - \alpha, e_{s,w} = w_s - \alpha, e_{s,y} = y_s - \alpha, e_{s,z} = z_s - \alpha. \)

**Proof.** The proof is similar to Lemma 1 in[44]. \( \square \)
Theorem 3.8. Let $x_0$ be a suitable initial guess to the simple root $\alpha$ of $f(x) = 0$. Also, suppose the initial values $\gamma_0, q_0, \lambda_0,$ and $\beta_0$ are chosen appropriately.

Then the $R$-order of the recursive adaptive method with memory (37) can be obtained from the following system of nonlinear equations:

$$
\begin{align*}
0 &= r^k p_1 - (1 + p_1 + p_2 + p_3) (1 + r + r^2 + r^3 + \ldots + r^{k-1}) - r^k, \\
0 &= r^k p_2 - 2(1 + p_1 + p_2 + p_3) (1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 2r^k, \\
0 &= r^k p_3 - 4(1 + p_1 + p_2 + p_3) (1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 4r^k, \\
0 &= r^{k+1} - 8(1 + p_1 + p_2 + p_3) (1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 8r^k,
\end{align*}
$$

(42)

where $r$, $p_1$, $p_2$, and $p_3$ are the order of convergence of the sequences $\{x_k\}$, $\{w_k\}$, $\{y_k\}$, and $\{z_k\}$, respectively. Also, $k$, indicates the number of iterations.

Proof. Let $\{x_k\}$, $\{w_k\}$, $\{y_k\}$ and $\{z_k\}$ be convergent with orders $r$, $p_1$, $p_2$ and $p_3$ respectively. Then:

$$
\begin{align*}
(1 + \gamma_0 f(\alpha)) &\sim \prod_{s=0}^{k-1} e_s e_{s,w} e_{s,y} e_{s,z} = (e_0 e_{0,w} e_{0,y} e_{0,z}) \cdots (e_{k-1} e_{k-1,w} e_{k-1,y} e_{k-1,z}) \\
&= (e_0 e_0^{p_1} e_0^{p_2} e_0^{p_3}) (e_0 e_0^{p_1} e_0^{p_2} e_0^{p_3}) \cdots (e_0 e_0^{r^{k-1} p_1} e_0^{r^{k-1} p_2} e_0^{r^{k-1} p_3}) \\
&= e_0 (1 + p_1 + p_2 + p_3 + (1 + p_1 + p_2 + p_3) r + \ldots + (1 + p_1 + p_2 + p_3) r^{k-1}) \\
&= e_0 (1 + p_1 + p_2 + p_3) (1 + r + \ldots + r^{k-1}).
\end{align*}
$$

(44)

Similarly, we get

$$(q_k + c_2) \sim e_0 (1 + p_1 + p_2 + p_3) (1 + r + \ldots + r^{k-1}),$$

(45)
and

\[(\lambda_k + f'(\alpha)c_2(q_k + c_2) - f'(\alpha)c_3) \sim e_0^{(1+p_1+p_2+p_3)(1+r+\ldots+r^{k-1})}, \quad (46)\]

\[(\beta_k + c_2(\lambda_k + f'(\alpha)c_2(q_k + c_2) - f'(\alpha)c_3) - f'(\alpha)c_4) \sim e_0^{(1+p_1+p_2+p_3)(1+r+\ldots+r^{k-1})}. \quad (47)\]

By considering the errors of \(w_k, y_k, z_k\) and \(x_{k+1}\) in Eq. (37) and Eqs. (44)-(47), we conclude:

\[e_{k,w} \sim (1 + \gamma_k f'(\alpha))e_k \sim e_0^{(1+p_1+p_2+p_3)(1+r+\ldots+r^{k-1})}e_0^{2r_k}, \quad (48)\]

\[e_{k,y} \sim (1 + \gamma_k f'(\alpha))(q_k + c_2)e_k^2 \sim e_0^{(1+p_1+p_2+p_3)(1+r+\ldots+r^{k-1})}e_0^{2r_k}, \quad (49)\]

\[e_{k,z} \sim (1 + \gamma_k f'(\alpha))^2(q_k + c_2)(\lambda_k + f'(\alpha)c_2(q_k + c_2) - f'(\alpha)c_3)e_k^3 \sim e_0^{(1+r_1+r_2+r_3)(1+r+\ldots+r^{k-1})}e_0^{2r_k}, \quad (50)\]

\[e_{k+1} \sim (1 + \gamma_k f'(\alpha))^4(q_k + c_2)^2(\lambda_k + f'(\alpha)c_2(q_k + c_2) - f'(\alpha)c_3)(\beta_k + c_2(\lambda_k + f'(\alpha)c_2(q_k + c_2) - f'(\alpha)c_3) - f'(\alpha)c_4)e_k^5 \sim e_0^{(1+p_1+p_2+p_3)(1+r+\ldots+r^{k-1})}e_0^{2r_k}. \quad (51)\]

To obtain the desired result, it is enough to match the right-hand-side of the Eqs. (43), (48), (49), (50) and (51). Then:

\[
\begin{align*}
    r^kp_1 - (1 + p_1 + p_2 + p_3) & (1 + r + r^2 + r^3 + \ldots + r^{k-1}) - r^k = 0, \quad k \in \mathbb{N} \\
    r^kp_2 - 2(1 + p_1 + p_2 + p_3) & (1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 2r^k = 0, \\
    r^kp_3 - 4(1 + p_1 + p_2 + p_3) & (1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 4r^k = 0, \\
    r^{k+1} - 8(1 + p_1 + p_2 + p_3) & (1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 8r^k = 0.
\end{align*}
\]

This completes the proof of the Theorem. \(\square\)
Remark 3.9. \( k = 1 \), we use the information from the current and the one previous steps. In this case, the order of convergence of the with memory method (37) can be computed from the following system

\[
\begin{align*}
rp_1 - (1 + p_1 + p_2 + p_3) - r &= 0, \\
rp_2 - 2(1 + p_1 + p_2 + p_3) - 2r &= 0, \\
rp_3 - 4(1 + p_1 + p_2 + p_3) - 4r &= 0, \\
r^2 - 8(1 + p_1 + p_2 + p_3) - 8r &= 0.
\end{align*}
\]

After solving these equations, we have: \( p_1 = \frac{1}{16}(15 + \sqrt{257}) \approx 1.93945 \), \( p_2 = \frac{2}{3}(15 + \sqrt{257}) \approx 3.8789 \), \( p_3 = \frac{4}{3}(15 + \sqrt{257}) \approx 7.7578 \) and \( r = \frac{1}{2}(15 + \sqrt{257}) \approx 15.5156 \).

This special case determines the given result by Lotfi-Assari [27].

If \( k = 2 \), we obtain the order of convergence: \( r_1 \approx 1.99632 \), \( r_2 \approx 3.99265 \), \( r_3 \approx 7.9853 \) and \( r \approx 15.9706 \).

And, if \( k = 3 \), The positive real solution (42) is: \( p_1 \approx 1.99977 \), \( p_2 \approx 3.99954 \), \( p_3 \approx 7.99908 \) and \( r \approx 15.9982 \).

Also, for \( k = 4 \), the system (42) has the solution (shown by TAM16)

\[ p_1 = 2, \ p_2 = 4, \ p_3 = 8, \ \text{and} \ r = 16. \]

This shows that the R-order of convergence for (37) is 16.

Remark 3.10. As can be easily seen that the improvement in the order of convergence from 2, 4 and 8 to 4, 8 and 16 (100\% of an improvement) is attained without any additional functional evaluations, which points to the very high computational efficiency of the proposed methods. Therefore, the efficiency index of the proposed method (11), (23) and (37) is \( 4^{1/2} = 8^{1/3} = 16^{1/4} = 2 \).

4 Numerical results and comparisons

The errors \( |x_k - \alpha| \) of approximations to the sought zeros, produced by the different methods at the first three iterations are given in Table 2.
where \(m \times n\) stands for \(m \times 10^{-n}\). Tables 2 – 4 also include, for each test function, the initial estimation values and the last value of the computational order of convergence \(COC\) [18] computed by the expression

\[
COC = \frac{\log |f(x_n)/f(x_{n-1})|}{\log |f(x_{n-1})/f(x_{n-2})|},
\]

The package Mathematica 10, with 5000 arbitrary precision arithmetic, has been used in our computations. Iterative methods with and without memory, for comparing with our proposed scheme has been chosen as comes next.

Four-step without memory Geum et al. order 16 (GKM)[17]:

\[
\begin{align*}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)}; u_k = \frac{f(y_k)}{f'(x_k)}; m_k = \frac{1+2u_k-4u_k^2}{1-3u_k^2}, k = 0, 1, 2, \ldots, \\
z_k &= y_k - m_k \frac{f(y_k)}{f'(x_k)}, \quad \nu_k = \frac{f(z_k)}{f(y_k)}, \quad \lambda_k = \frac{f(z_k)}{f(x_k)}, \quad h_k = \frac{1}{1-3u_k^2}; \quad t_k = \frac{f(s_k)}{f(z_k)}; \\
W_k &= \frac{1}{1-3u_k^2}; \quad \frac{1}{2}(u_k w_k (6 + 12 u_k + 2 u_k^2 + 48 u_k^3 - 8)) + (-2 u_k + 2) w_k^3; \\
s_k &= z_k - h_k \frac{f(z_k)}{f(x_k)}; \quad x_k+1 = s_k - W_k \frac{f(s_k)}{f(x_k)}.
\end{align*}
\] (55)

One-step with memory Dzunic order 3.56 (DM)[14]:

\[
\begin{align*}
\gamma_k &= \frac{-1}{N_2(x_k)}, \quad q_k = \frac{N''_1(w_k)}{2N'_2(w_k)}, \quad k = 1, 2, \ldots, \\
w_k &= x_k - \gamma_k f(x_k), \quad x_{k+1} = x_k - \frac{f(x_k)}{f(x_k)} f'(x_k); \quad \gamma_k = \frac{N''_1(w_k)}{f'(x_k)}; \quad k = 0, 1, 2, \ldots.
\end{align*}
\] (56)

One-step Abbasbandy’s method order 3 (AM) [1]:

\[
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{f^2(x_k) f''(x_k)}{2 f'^3(x_k)} + \frac{f^3(x_k) f'''(x_k)}{2 f'^5(x_k)}.
\] (57)

Two-step with memory Soleymani et al. order 7.22 (SLTKM) [39]:

\[
\begin{align*}
\gamma_k &= \frac{-1}{N_3(x_k)}, \quad q_k = \frac{N''_1(w_k)}{2N'_2(w_k)}; \quad \gamma_k = \frac{N''_1(w_k)}{6}, \quad k = 1, 2, 3, \ldots, \\
y_k &= x_k - \frac{f(x_k)}{f(x_k)}; \quad \lambda_k = \frac{N''_1(w_k)}{6}; \quad k = 0, 1, 2, \ldots, \\
x_{k+1} &= y_k - \frac{f(y_k)}{f(y_k)}; \quad \nu_k = \frac{f(y_k)}{f(y_k)} + \lambda_k (y_k-x_k)(y_k-y_k) (1 + \frac{f(y_k)}{f(x_k)}).
\end{align*}
\] (58)
Three-step without memory Thukral-Petković order 8 (TPM)[41]:

\[
\begin{align*}
    y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \quad u_k = \frac{f(y_k)}{f'(x_k)} + b f(y_k), \quad k = 0, 1, 2, \ldots, \\
    z_k &= y_k - \frac{f(y_k)}{f'(x_k)} \frac{f'(x_k) + b f(y_k)}{f(x_k) + (k - 2) f(y_k)}, \\
    \phi_k &= 1 + 2 u_k + (5 - 2 b) u_k^2 + (2 b^2 - 12 b + 12) u_k^3, \\
    x_{k+1} &= z_k - \frac{f(z_k)}{f'(x_k)} (\phi_k + \frac{f(z_k)}{f(y_k) - a f(z_k)} + \frac{4 f(z_k)}{f(x_k)}).
\end{align*}
\]

Table 1 lists the exact roots \(\alpha\) and initial approximations \(x_0\). Tables 2–4 show that the proposed methods compete with the previous methods. Besides, its efficiency index is much better than the previous works. In other words, TAM4, TAM7, TAM8, and TAM16 have efficiency indices \(4^{\frac{1}{2}} = 2\), \(7.53^{\frac{1}{2}} \simeq 1.96\), \(8^{\frac{1}{3}} = 2\), and \(16^{\frac{1}{4}} = 2\), respectively. To check the effectiveness of the proposed iterative methods, we have considered tenth-test nonlinear functions. All the numerical computations are carried out on the computer algebra system MATHEMATICA 10 using 3000 digits floating-point arithmetic. The results of comparisons are given in Tables 2–4. The errors \(|x_k - \alpha|\) of approximations to the sought zeros, produced by the different methods at the first, two, and three iterations. These tables also include, for each test function, the initial estimation values and the last value of the computational order of convergence in companion with convergence rate and EI each method. A comparison between without-memory, with-memory and adaptive methods in terms of the maximum convergence order alongside the number of steps per cycle has been given in Figure 1.

**Table 1:** Test functions

<table>
<thead>
<tr>
<th>Nonlinear function</th>
<th>Zero</th>
<th>Initial guess</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f_1(x) = \log(1 + x \sin(x)) )</td>
<td>(\alpha = 0)</td>
<td>(x_0 = 0.6)</td>
</tr>
<tr>
<td>(f_2(x) = 1 + \frac{1}{x^2} - \frac{1}{x} - x^2)</td>
<td>(\alpha = 1)</td>
<td>(x_0 = 1.4)</td>
</tr>
<tr>
<td>(f_3(x) = e^{-x^2} - x^2 + x^3 + 1)</td>
<td>(\alpha = -1)</td>
<td>(x_0 = -1.65)</td>
</tr>
<tr>
<td>(f_4(x) = \frac{-5x^2}{2} + x^4 + x^5 + \frac{1}{1+x^2})</td>
<td>(\alpha = 1)</td>
<td>(x_0 = 1.5)</td>
</tr>
<tr>
<td>(f_5(x) = \log(1 + x^2) + e^{-3x^2 + x^3} \sin(x))</td>
<td>(\alpha = 0)</td>
<td>(x_0 = 0.5)</td>
</tr>
<tr>
<td>(f_6(x) = x \log(1 - \pi + x^2) - \frac{1+x^2}{1+x} \sin(x^2) + \tan(x^2))</td>
<td>(\alpha = \sqrt{\pi})</td>
<td>(x_0 = 1.7)</td>
</tr>
<tr>
<td>(f_7(x) = x^3 + 4x^2 - 10)</td>
<td>(\alpha = 1)</td>
<td>(x_0 = 1.3652)</td>
</tr>
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</table>
Figure 1: Comparison of methods without memory, with memory and adaptive (%25, %50, %75, and %100 of improvements) in terms of highest possible convergence order.

5 Conclusion

In this study, we have increased convergence-order methods 2, 4 and 8 without imposing new evaluation on different recursive methods, with a convergence order of 4, 8 and 16, 100% improvement, respectively. To this end, based on Newton’s interpolation, the parameters of self-evaluation are interpolated. The numerical results show that the proposed method is very useful to find an acceptable approximation of the exact solution of nonlinear equations, especially when the function is non-differentiable. Table 2 compares one-step iterative with and without memory and the proposed method on functions \( f_i(t), i = 1, 2, \ldots, 7 \). Similarly, Table 3 compares two-step iterative methods. Also, Table 4 compares three- and four-step iterative methods with the proposed schemes. The last column of Tables shows the efficiency index defined by \( EI = \frac{COC}{n} \), which is asymptotically 2. In other words, the proposed adaptive method with memory (11), (23), and (37) show behavior as optimal \( n \)-point methods without memory. Therefore, we have developed a family iterative methods adaptive with memory which have efficiency index 2. The efficiency index of the proposed adaptive family with memory is \( 4^\frac{1}{2} = 8^\frac{1}{3} = 16^\frac{1}{4} = 2 \), which is much better than optimal one-,..., five-point optimal methods without-memory having efficiency indexes \( 2^{1/2} \approx 1.414, 4^{1/3} \approx 1.587, 8^{1/4} \approx 1.681, 16^{1/5} \approx 1.741, 32^{1/6} \approx 1.781, 64^{1/7} \approx 1.811 \), respectively. Adaptive methods with memory have minimum evaluation...
function, not evaluation derivative, and most efficiency index, hence, competes
with existing methods with- and without memory.
Table 2: Comparison of the absolute error of proposed method with one-step methods at first, second and third iterations for the test functions

| Methods  | $|x_1-\alpha|$ | $|x_2-\alpha|$ | $|x_3-\alpha|$ | COC   | E1     |
|----------|----------------|----------------|----------------|-------|--------|
| AM [1]   | 0.60000(0)     | 0.44377(0)     | 0.10028(0)     | 3.0000| 1.44225|
| DM [14]  | 0.60000(0)     | 0.36450(0)     | 0.54166(−1)    | 3.4590| 1.85984|
| TM [46]  | 0.60000(0)     | 0.47811(0)     | 0.56230(−1)    | 2.3950| 1.54758|
| TAM4 (11) k=4 | 0.36450(0) | 0.54166(−1) | 0.23973(−5) | 4.0148| 2.00370|

$f_2(x) = 1 + \frac{1}{x^4} - \frac{1}{x^3} - x^2, \alpha = 1, x_0 = 1.4, q_0 = \gamma_0 = 0.1$

| Methods  | $|x_1-\alpha|$ | $|x_2-\alpha|$ | $|x_3-\alpha|$ | COC   | E1     |
|----------|----------------|----------------|----------------|-------|--------|
| AM [1]   | 0.40000(0)     | 0.69117(−1)    | 0.84282(−3)    | 3.0000| 1.44225|
| DM [14]  | 0.40000(0)     | 0.46538(−1)    | 0.12681(−3)    | 3.5552| 1.88552|
| TM [46]  | 0.40000(0)     | 0.60801(−1)    | 0.28094(−2)    | 2.4157| 1.55425|
| TAM4 (11) k=4 | 0.46538(−1) | 0.12681(−3) | 0.35998(−15) | 3.9986| 1.99965|

$f_3(x) = e^{x^2-x} - \cos(x^2-1) + x^3 + 1, \alpha = -1, x_0 = -1.5, q_0 = \gamma_0 = 0.1$

| Methods  | $|x_1-\alpha|$ | $|x_2-\alpha|$ | $|x_3-\alpha|$ | COC   | E1     |
|----------|----------------|----------------|----------------|-------|--------|
| AM [1]   | 0.50000(0)     | 0.48941(−1)    | 0.16236(−3)    | 3.0000| 1.44225|
| DM [14]  | 0.50000(0)     | 0.15659(−1)    | 0.10877(−5)    | 3.4075| 1.84594|
| TM [46]  | 0.50000(0)     | 0.22068(−1)    | 0.12109(−5)    | 2.3993| 1.54897|
| TAM4 (11) k=4 | 0.15659(−1) | 0.10877(−5) | 0.83019(−24) | 4.0020| 2.00050|

$f_4(x) = \frac{-5x^4}{x^2} + x^4 + x^3 + \frac{1}{1+x^2}, \alpha = 1, x_0 = 1.5, q_0 = \gamma_0 = 0.1$

| Methods  | $|x_1-\alpha|$ | $|x_2-\alpha|$ | $|x_3-\alpha|$ | COC   | E1     |
|----------|----------------|----------------|----------------|-------|--------|
| AM [1]   | 0.50000(0)     | 0.18311(0)     | 0.33638(−1)    | 3.0000| 1.44225|
| DM [14]  | 0.50000(0)     | 0.41826(0)     | 0.71739(−1)    | 3.5453| 1.88290|
| TM [46]  | 0.50000(0)     | 0.41554(0)     | 0.13304(0)     | 2.2867| 1.51218|
| TAM4 (11) k=4 | 0.41826(0) | 0.71739(−1) | 0.30353(−3) | 3.9997| 1.99992|

$f_5(x) = \log(1 + x^2) + e^{-3x^2-x} \sin(x), \alpha = 0, x_0 = 0.5, q_0 = \gamma_0 = 0.1$

| Methods  | $|x_1-\alpha|$ | $|x_2-\alpha|$ | $|x_3-\alpha|$ | COC   | E1     |
|----------|----------------|----------------|----------------|-------|--------|
| AM [1]   | 0.50000(0)     | 0.88441(−4)    | 0.45840(−12)   | 3.0000| 1.44225|
| DM [14]  | 0.50000(0)     | 0.64108(−1)    | 0.12721(−2)    | 3.5500| 1.88414|
| TM [46]  | 0.50000(0)     | 0.42599(−1)    | 0.11207(−2)    | 2.4134| 1.55351|
| TAM4 (11) k=4 | 0.64108(−1) | 0.12721(−2) | 0.10199(−10) | 3.9993| 3.99982|

$f_6(x) = x \log(1 - \pi + x^2) - \frac{1+x^2}{1+x^2} \sin(x^2) + \tan(x^2), \alpha = \sqrt{\pi}, x_0 = 1.7, q_0 = \gamma_0 = 0.1$

| Methods  | $|x_1-\alpha|$ | $|x_2-\alpha|$ | $|x_3-\alpha|$ | COC   | E1     |
|----------|----------------|----------------|----------------|-------|--------|
| AM [1]   | 0.72454(−1)    | 0.58079(−3)    | 0.33518(−9)    | 3.0000| 1.44225|
| DM [14]  | 0.72454(−1)    | 0.10543(−1)    | 0.44438(−6)    | 3.5005| 1.87096|
| TM [46]  | 0.72454(−1)    | 0.11486(−1)    | 0.28170(−5)    | 2.4090| 1.55210|
| TAM4 (11) k=4 | 0.10543(−1) | 0.44438(−6) | 0.48786(−24) | 3.9992| 1.99980|

$f_7(x) = x^3 + 4x^2 - 10, \alpha = 1.3652, x_0 = 1, q_0 = \gamma_0 = 0.1$

| Methods  | $|x_1-\alpha|$ | $|x_2-\alpha|$ | $|x_3-\alpha|$ | COC   | E1     |
|----------|----------------|----------------|----------------|-------|--------|
| AM [1]   | 0.36520(0)     | 0.47568(−1)    | 0.22845(−4)    | 3.0000| 1.44225|
| DM [14]  | 0.36520(0)     | 0.36340(0)     | 0.34044(−3)    | 3.7222| 1.92930|
| TM [46]  | 0.36520(0)     | 0.27996(0)     | 0.64692(−2)    | 2.4053| 1.55090|
| TAM4 (11) k=4 | 0.36340(0) | 0.34044(−3) | 0.30013(−4) | 4.0000| 2.00000|
### Table 3: Comparison evaluation function and efficiency index of the proposed method with two-step methods with and without memory

| Methods    | $|x_1 - \alpha|$ | $|x_2 - \alpha|$ | $|x_3 - \alpha|$ | COC | El |
|------------|----------------|----------------|----------------|-----|----|
| SLTKM [39] | 0.60000(0)     | 0.22353(0)     | 0.30292(−5)    | 7.1871 | 1.92982 |
| TAM7 [23]  | 0.60000(0)     | 0.22353(0)     | 0.24015(−5)    | 7.5523 | 1.96197 |
| TAM8 [23]  | 0.22353(0)     | 0.24015(−5)    | 0.61245(−39)   | 8.1640 | 2.01357 |

$f_2(x) = 1 + \frac{1}{x} - \frac{1}{x^2}, \alpha = 1, x_0 = 1.4, q_0 = \gamma_0 = \lambda_0 = 0.1$

| Methods    | $|x_1 - \alpha|$ | $|x_2 - \alpha|$ | $|x_3 - \alpha|$ | COC | El |
|------------|----------------|----------------|----------------|-----|----|
| SLTKM [39] | 0.40000(0)     | 0.37115(−2)    | 0.28982(−15)   | 7.2315 | 1.93379 |
| TAM7 [23]  | 0.40000(0)     | 0.37115(−2)    | 0.37500(−16)   | 7.5218 | 1.95933 |
| TAM8 [23]  | 0.37115(−2)    | 0.37500(−16)   | 0.79732(−112)  | 8.1885 | 2.01559 |

$f_3(x) = e^{x^2} - \cos(x^2 - 1) + x^3 + 1, \alpha = -1, x_0 = -1.5, q_0 = \gamma_0 = \lambda_0 = 0.1$

| Methods    | $|x_1 - \alpha|$ | $|x_2 - \alpha|$ | $|x_3 - \alpha|$ | COC | El |
|------------|----------------|----------------|----------------|-----|----|
| SLTKM [39] | 0.50000(0)     | 0.48587(−4)    | 0.32385(−4)    | 7.2038 | 1.93132 |
| TAM7 [23]  | 0.50000(0)     | 0.48587(−4)    | 0.67957(−26)   | 7.4904 | 1.95712 |
| TAM8 [23]  | 0.48587(−4)    | 0.67957(−26)   | 0.54183(−180)  | 8.2048 | 2.01692 |

$f_4(x) = \frac{\sqrt{x}}{2} + x^2 + x^3 + \frac{1}{\sqrt{x^2}}, \alpha = 1, x_0 = 1.5, q_0 = \gamma_0 = \lambda_0 = 0.1$

| Methods    | $|x_1 - \alpha|$ | $|x_2 - \alpha|$ | $|x_3 - \alpha|$ | COC | El |
|------------|----------------|----------------|----------------|-----|----|
| SLTKM [39] | 0.50000(0)     | 0.31600(0)     | 0.42113(−2)    | 7.2133 | 1.93217 |
| TAM7 [23]  | 0.50000(0)     | 0.31600(0)     | 0.38872(−2)    | 7.9861 | 1.99884 |
| TAM8 [23]  | 0.31600(0)     | 0.38872(−2)    | 0.22990(−16)   | 8.0000 | 2.00000 |

$f_5(x) = \log(1 + x^2) + e^{-3x^2 + \sin(x)}, \alpha = 0, x_0 = 0.5, q_0 = \gamma_0 = \lambda_0 = 0.1$

| Methods    | $|x_1 - \alpha|$ | $|x_2 - \alpha|$ | $|x_3 - \alpha|$ | COC | El |
|------------|----------------|----------------|----------------|-----|----|
| SLTKM [39] | 0.50000(0)     | 0.22780(−1)    | 0.13805(−10)   | 7.2390 | 1.93446 |
| TAM7 [23]  | 0.50000(0)     | 0.22780(−1)    | 0.91585(−11)   | 7.4955 | 1.95704 |
| TAM8 [23]  | 0.22780(−1)    | 0.91585(−11)   | 0.54892(−74)   | 8.0000 | 2.00000 |

$f_6(x) = \log(1 + \pi + x^2) - \frac{1+x^2}{\sqrt{\pi^2}} \sin(x^2) + \tan(x^2), \alpha = \sqrt{x}, x_0 = 1.7, q_0 = \gamma_0 = \lambda_0 = 0.1$

| Methods    | $|x_1 - \alpha|$ | $|x_2 - \alpha|$ | $|x_3 - \alpha|$ | COC | El |
|------------|----------------|----------------|----------------|-----|----|
| SLTKM [39] | 0.73454(−1)    | 0.70222(−5)    | 0.25138(−29)   | 7.2120 | 1.93205 |
| TAM7 [23]  | 0.73454(−1)    | 0.70222(−5)    | 0.25789(−31)   | 7.4793 | 1.95563 |
| TAM8 [23]  | 0.70222(−5)    | 0.25789(−31)   | 0.33555(−214)  | 7.9996 | 1.99997 |

$f_7(x) = x^2 + 4x^2 - 10, \alpha = 1.3652, x_0 = 1, q_0 = \gamma_0 = \lambda_0 = 0.1$

| Methods    | $|x_1 - \alpha|$ | $|x_2 - \alpha|$ | $|x_3 - \alpha|$ | COC | El |
|------------|----------------|----------------|----------------|-----|----|
| SLTKM [39] | 0.36520(0)     | 0.61026(0)     | 0.23768(−3)    | 8.0000 | 2.00000 |
| TAM7 [23]  | 0.36520(0)     | 0.61026(0)     | 0.23768(−3)    | 8.0000 | 2.00000 |
| TAM8 [23]  | 0.61026(0)     | 0.23768(−3)    | 0.23001(−3)    | 8.0000 | 2.00000 |
Table 4: Comparison evaluation function and efficiency index of the proposed method by three- and four-step methods with and without memory

| Methods       | $|x_1 - \alpha|$ | $|x_2 - \alpha|$ | $|x_3 - \alpha|$ | COC  | El      |
|---------------|-----------------|-----------------|-----------------|------|---------|
| TPM $[41]$    | 0.600000(0)     | 0.15946(1)      | 0.31506(13)     | 8.000000 | 1.68179 |
| TAM16 $[37]$  | 0.19386(1)      | 0.12850(-28)    | 0.373(-64)      | 16.00010 | 2.00003 |
| TPM $[41]$    | 0.400000(0)     | 0.20684(-3)     | 0.94711(-27)    | 8.000000 | 1.68179 |
| TAM16 $[37]$  | 0.19386(1)      | 0.12850(-28)    | 0.373(-64)      | 16.00010 | 2.00003 |
| TPM $[41]$    | 0.500000(0)     | 0.22661(-5)     | 0.12409(-44)    | 8.000000 | 1.68179 |
| TAM16 $[37]$  | 0.32415(-6)     | 0.34397(-105)   | 0.70235(-1693)  | 16.000000 | 2.000000 |
| TPM $[41]$    | 0.500000(0)     | 0.52236(-1)     | 0.29347(-5)     | 8.000000 | 1.68179 |
| TAM16 $[37]$  | 0.32415(-6)     | 0.34397(-105)   | 0.70235(-1693)  | 16.000000 | 2.000000 |
| TPM $[41]$    | 0.500000(0)     | 0.54581(-2)     | 0.75733(-15)    | 8.000000 | 1.68179 |
| TAM16 $[37]$  | 0.32415(-6)     | 0.34397(-105)   | 0.70235(-1693)  | 16.000000 | 2.000000 |
| TPM $[41]$    | 0.500000(0)     | 0.55075(-3)     | 0.30763(-48)    | 15.508000 | 1.98444 |
| TAM16 $[37]$  | 0.32415(-6)     | 0.34397(-105)   | 0.70235(-1693)  | 16.000000 | 2.000000 |
| TPM $[41]$    | 0.500000(0)     | 0.56311(6)      | 0.28713(-94)    | 16.000000 | 1.74110 |
| TAM16 $[37]$  | 0.32415(-6)     | 0.34397(-105)   | 0.70235(-1693)  | 16.000000 | 2.000000 |
| TPM $[41]$    | 0.500000(0)     | 0.55428(-3)     | 0.10344(-42)    | 16.000000 | 1.74110 |
| TAM16 $[37]$  | 0.32415(-6)     | 0.34397(-105)   | 0.70235(-1693)  | 16.000000 | 2.000000 |

\( f_0(x) = x \log(1 + x^2) + e^{-12x^2} \sin(\pi x), \alpha = 0, x_0 = 0.5, q_0 = \gamma_0 = \lambda_0 = \beta_0 = 0.1 \)

| Methods       | $|x_1 - \alpha|$ | $|x_2 - \alpha|$ | $|x_3 - \alpha|$ | COC  | El      |
|---------------|-----------------|-----------------|-----------------|------|---------|
| TPM $[41]$    | 0.72454(-1)     | 0.21456(-8)     | 0.94711(-69)    | 8.000000 | 1.68179 |
| TAM16 $[37]$  | 0.27167(-6)     | 0.15849(-97)    | 0.19365(-1557)  | 16.000000 | 2.000000 |
| TPM $[41]$    | 0.36520(0)      | 0.60970(-3)     | 0.30013(-4)     | 8.000000 | 1.68179 |
| TAM16 $[37]$  | 0.27167(-6)     | 0.15849(-97)    | 0.19365(-1557)  | 16.000000 | 2.000000 |
| TPM $[41]$    | 0.36520(0)      | 0.75262(-3)     | 0.30013(-4)     | 16.000000 | 2.000000 |
| TAM16 $[37]$  | 0.27167(-6)     | 0.15849(-97)    | 0.19365(-1557)  | 16.000000 | 2.000000 |
| TPM $[41]$    | 0.36520(0)      | 0.84783(-3)     | 0.30013(-4)     | 16.000000 | 2.000000 |
| TAM16 $[37]$  | 0.27167(-6)     | 0.15849(-97)    | 0.19365(-1557)  | 16.000000 | 2.000000 |
| TPM $[41]$    | 0.36520(0)      | 0.94171(-1)     | 0.30013(-4)     | 16.000000 | 2.000000 |
| TAM16 $[37]$  | 0.27167(-6)     | 0.15849(-97)    | 0.19365(-1557)  | 16.000000 | 2.000000 |
| TPM $[41]$    | 0.36520(0)      | 0.95262(-3)     | 0.23001(-3)     | 16.000000 | 2.000000 |
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