Journal of Mathematical Extension
Vol. 15, No. 3, (2021) (13)1-18
URL: https://doi.org/10.30495/JME.2021.1566
ISSN: 1735-8299
Original Research Paper

Some New Results for Functional Fractional Differential Inclusions with Impulses Effect

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Abstract. This study aims to investigate the existence of solutions for nonlocal functional differential inclusions with impulses effect in Banach spaces. We examine the case when the multivalued function is non-convex, and the linear term generates a semigroup not necessarliy compact. The significant results are obtained by applying NCHM (non-compactness Hausdorff measure) and theorems of fixed point. Eventually, we provide an example to elaborate the outcomes.

AMS Subject Classification: 34A60, 34B37, 34G10
Keywords and Phrases: Semilinear impulsive differential inclusions; Nonlocal conditions; Fixed point theorems; Mild solutions

1 Introduction

Impulsive differential models, in both forms as inclusions and equations, have significantly participated as a storng tool in advancing variety of disciplines, physics, chemistry, biology, economics, control theory, technology and so on. One can find some applications in [1, 5, 25].

Received: February 2020; Accepted: September 2020
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The essential of general theory and the applied developments of such problems have been discussed in details, see [7, 11, 13].

In this work, we consider the functional impulsive differential inclusion which is given by the form:

$$(P_\Psi)\begin{cases} {}^cD^\alpha x(t) \in Ax(t) + F(t,\tau(t)x), & t \in J = [0,b],\ t \neq t_i, \\ x(t) = \Psi(t) - g(x), & t \in [-r,0], \\ x(t_i^+) - x(t_i) = I_i(x(t_i^-)), & i = 1,...,m, \end{cases}$$

where ${}^cD^\alpha$ is the Caputo derivative $(0 < \alpha < 1)$, A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on E where E is a Banach space which is real and separable, $F : J \times \Theta \to 2^E$ is a lower Carathéodory multifunction, $\Psi : [-r,0] \to E$, for every $1 \leq i \leq m$, $I_i : E \to E$, $g : \Lambda \to E$, and $x(t_i^+) = \lim_{s\to t_i^+} x(s)$, $x(t_i^-) = \lim_{s\to t_i^-} x(s)$. Finally, for any $t \in J, x \in \Lambda$, the element $\tau(t)x$ of Θ defined by $\tau(t)x(\theta) = x(t+\theta)$, $\theta \in [-r,0]$, where $\tau(t)x$ represents the history of the state from $-r$ to the present time t.

Nonlocal conditions problems were essentially emerged from physics, see [4, 10, 14]. The topic of abstract differential problems with nonlocal conditions was initially taken into investigation by Byszewski [10]. On the other hand, when it comes to dealing with such nonlocal problems, the compactness of the operator of solution at zero still the main obstacle. Various techniques and methods have been developed by many authors in this direction, for further specifics, one can see [2, 11, 12, 13, 17, 19, 20, 22, 24, 27, 28, 29]. For instance, Wang et.al. [28] gave a new definition of solutions to (P_Ψ) without delay. Furthermore, the authors concluded their results when F is a continuous single-valued function satisfying Lipschitz condition and preverving bounded sets with compactness of $\{T(t)\}_{t>0}$. While Li [22] acquired existence results regarding nonlocal equations problems under particular restrictions; that are, compactness of the nonlocal term and the semigroup is equicontinuous. Additionally, Ibrahim and Alsarori [19] determined conditions so that the solutions for the problem (P_Ψ) exist in the case when compactness of the semigroup is assumed. Lian et al. [23] recently considered the problem (P_Ψ) and studied the existence results of solutions. They assumed the problem without impulses effect as well as without delay in the case when the multifunction is upper semicontinuous, compact and convex. Very recently, Alsarori et al. [3] investigated the problem (P_Ψ) without delay when the semigroup is not compact and F is upper semicontinuous, compact and convex.

Motivated by the aforementioned papers and work, we consider a case differs from previous cases. In particular, we study the existence of solutions of (P_Ψ) with condition; F is lower semicontinuous with closed values and $\{T(t)\}_{t>0}$ is equicontinuous.
Section 2 consists of some notations and basic materials with respect to NCHM and the set-valued analysis. In Section 3, we go further to achieve the main results of the present article regarding the existence of solutions of (P_Ψ). NCHM and fixed point theorems, among other techniques, are utilized in this research. The applicability of the results is presented through introducing a numerical example in section 4.

2 Preliminaries and Notations

During this section, we state some previous known results so that we can use them later throughout this paper. Let
$C(J,E) = \{\mu : J \to E : \mu \ \ \textit{is continuous} \ \}$,
$L^1(J,E) = \{\mathcal{G} : J \to E : \mathcal{G} \ \textit{is Bochner integrable}\}$,
$P_b(E) = \{X : X \subset E, X \neq \varnothing, X \textit{is bounded}\}$,
$P_{cl}(E) = \{X : X \subset E, X \neq \varnothing, X \textit{is closed}\}$,
$\overline{conv}(B)$ be the closed convex hull in E of subset B.
Let $J_0 = [0, t_1], J_i =]t_i, t_{i+1}], \ i = 1, \cdots, m,$ we consider the sets of functions:

$$\Theta = \{\Psi : [-r, 0] \to E; \ \Psi(s) \textit{is continuous everywhere except for}$$

$$\textit{a finite number of points } s \textit{ at which } \Psi(s^+), \Psi(s^-) \textit{ exist}, \ \Psi(s) = \Psi(s^-)\},$$

$$PC([0,b],E) = \{x : J \to E : \ x_{|_{J_i}} \in C(J_i, E); \ x(t_i^+), \ x(t_i^-) \ exist\},$$

and

$$\Lambda = \{x : [-r,b] \to E : x_{|_{[-r,0]}} \in \Theta, \ x_{|_{J_i}} \in C(J_i,E); \ x(t_i^+), x(t_i^-) \ exist\}.$$

Note that Θ, PC and Λ are Banach spaces with norms:

$$\begin{aligned}\|\Psi\|_{\Theta} &= \max\{\|x(t)\| : \ t \in [-r, 0]\},\\ \|x\|_{PC} &= \max\{\|x(t)\| : \ t \in [0, b]\},\\ \|x\|_{\Lambda} &= \max\{\|x(t)\| : \ t \in [-r, b]\}.\end{aligned}$$

Let $W \subseteq \Lambda$, $\forall i = 0, 1, 2, \cdots, m$, define

$$W_{|\overline{J_i}} = \{x^* : \overline{J_i} \longrightarrow E : \ x^*(t) = x(t), \ t \in J_i, \ x^*(t_i) = x(t_i^+), x \in W\}.$$

Definition 2.1. ([21]). NCHM (noncompactness Hausdorff measure) on E,
$\chi : P_b(E) \to [0, +\infty)$ is defined by

$$\chi(W) = \inf\{\varepsilon > 0 : W \subseteq \cup_{j=1}^{n} W_j \ \ and \ \ radius(W_j) \leq \varepsilon\}.$$

Lemma 2.2. *([21]). Let χ as defined above and $W_1, W_2 \in P_b(E)$, then*

1. *If $W_1 \subset W_2$, then $\chi(W_1) \leq \chi(W_2)$;*
2. *$\chi(\{c\} \cup W_1) = \chi(W_1)$, $\forall c \in E$;*
3. *If $Y \subset E$ with Y is a compact, then $\chi(W_1 \cup Y) = \chi(W_1)$;*
4. *$\chi(W_1 + W_2) \leq \chi(W_1) + \chi(W_2)$;*
5. *$\chi(W_1) = 0$ iff W_1 is relatively compact;*
6. *$\chi(tW_1) = \mid t \mid \chi(W_1)$, $t \in \mathbb{R}$;*
7. *$\chi(\mathcal{L}(W_1)) \leq \|\mathcal{L}\| \chi(W_1)$, where $\mathcal{L}$ is a linear bounded operator on E.*

Let us consider the map $\chi_\Lambda : P_b(\Lambda) \to [0, \infty[$, such that for every $W \in P_b(\Lambda)$,

$$\begin{aligned}\chi_\Lambda(W) &= \chi_\Theta(W_{|_{[-r,0]}}) + \chi_{PC}(W)\\ &= \chi_\Theta(W_{|_{[-r,0]}}) + \max_{i=0,1,\cdots,m} \chi_i(W_{|\overline{J_i}}),\end{aligned}$$

where χ_i is the NCHM on $C(\overline{J_i}, E)$.

Definition 2.3. A function $x \in \Lambda$ is a mild solution for (P_Ψ) if

$$x(t) = \begin{cases} \Psi(t) - g(x), & t \in [-r, 0] \\ \mathcal{T}_\alpha(t)(\Psi(0) - g(x)) & \\ + \int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s) f(s) ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(\Psi(0) - g(x)) + \sum_{i=1}^{i=m} \mathcal{T}_\alpha(t-t_i) I_i(x(t_i^-)) & \\ + \int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s) f(s) ds, & t \in J_i, \end{cases}$$

where $i = 1, \cdots, m$, $f \in S^1_{F(\cdot, \tau(\cdot)x)}$,

$$\mathcal{T}_\alpha(t) = \int_0^\infty \xi_\alpha(\theta) T(t^\alpha \theta) d\theta,$$
$$\mathcal{S}_\alpha(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T(t^\alpha \theta) d\theta,$$

where ξ is a probability density function on $(0, \infty)$ defined as

$$\xi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \varpi_\alpha(\theta^{\frac{-1}{\alpha}}) \geq 0,$$

such that $\varpi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-\alpha n - 1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha)$, $\theta \in (0, \infty)$.

Next, we restate some results regarding of $\mathcal{T}_\alpha(\cdot)$ and $\mathcal{S}_\alpha(\cdot)$.

Lemma 2.4. *([29]).*

1. *If* $\|T(t)\| \leq M$, $\forall\ t \geq 0$, *then* $\forall\ x \in E$, $\|\mathcal{T}_\alpha(t)x\| \leq M\|x\|$ *and* $\|\mathcal{S}_\alpha(t)x\| \leq \dfrac{M}{\Gamma(\alpha)}\|x\|$.
2. *If* $\{T(t)\}_{t \geq 0}$ *is equicontinuous, then* $\mathcal{T}_\alpha(t)$ *and* $\mathcal{S}_\alpha(t)$ *are equicontinuous.*

Lemma 2.5. *([11]). Assume that* $(W_n)_{n \geq 1}$ *is a decreasing sequence of nonempty, closed and bounded subsets of* E, *with* $\chi(W_n) \to 0$ *as* $n \to \infty$, *then* $W = \cap_{n=1}^\infty W_n$ *is nonempty and compact in* E.

Lemma 2.6. *([6]). If* $W \subset C(J, E)$ *is bounded and equicontinuous, then* $\chi(W(t))$ *is continuous on* J *and* $\chi(W) = \sup_{t \in J} \chi(W(t))$.

Lemma 2.7. *([16]). Let $\{u_n\}_{n=1}^{\infty} \subset L^1(J,E)$ be a sequence of uniformly integrable functions, then $\chi(\{\int_0^t u_n(s)ds\}_{n=1}^{\infty}) \leq 2\int_0^t \chi(\{u_n(s)\}_{n=1}^{\infty})ds$ and $\chi(\{u_n(t)\}_{n=1}^{\infty})$ is measurable.*

Lemma 2.8. *([8]). If $W \in P_b(E)$, then $\forall\, \epsilon > 0$, $\exists\, \{\mu_n\}_{n=1}^{\infty} \subset W$ such that $\chi(W) \leq 2\chi(\{\mu_n\}_{n=1}^{\infty}) + \epsilon$.*

Definition 2.9. ([16], [21]). If X , Y are two topological spaces. A multifunction $F : X \to P(Y)$ is called:

1. Upper semicontinuous ($u.s.c$) if $F^{-1}(W) \subset X$ is an open for every open subset W of Y.
2. Lower semicontinuous ($l.s.c$) when $F^{+1}(W) = \{x \in X : F(x) \cap W \neq \emptyset\}$ is an open for every open subset W of Y.
3. Closed in case when its graph is closed in the topological space $X \times Y$.
4. F has fixed point if there is $x \in X$, such that $x \in F(x)$.

Remark 2.10. Let X , Y be two topological spaces and $F : X \to P(Y)$.

1. For any closed subset D in X, if $F(x)$ is closed $\forall\, x \in D$, and $\overline{F(D)}$ is compact, then F is $u.s.c.$ iff F is closed.
2. If $F : X \to P(Y) - \{\emptyset\}$. Then $d(y, F(\cdot))$ is u.s.c. iff F is l.s.c. for every $y \in Y$, where X, Y are Banach spaces.

Definition 2.11. If W is a nonempty subset of $L^1(J,E)$, we call W is decomposable if for every $f, g \in W$ and for all Lebesgue measurable set $M \subset J$, $f\beta_M + g\beta_{(J-M)} \in W$, where β_M is the characteristic function of M.

Lemma 2.12. *(Theorem 3,[9]). If $F : J \times X \to P(L^1(J,X))$ multifunction with closed decomposable values, Then F has a continuous selection, where X is separable metric space.*

Theorem 2.13. *([15]). If E is Banach space, assume that $W \subset E$ which is convex, closed, bounded and nonempty and $\mathcal{G} : W \to W$ is continuous function. If either $\mathcal{G}$ or W is compact, then $\mathcal{G}$ has a fixed point.*

3 Main results

By using NCHM with fixed point theorems, we will show that our problem (P_Ψ) has mild solutions.

Theorem 3.1. *Suppose the following hypotheses:*

HA: C_0-*semigroup* $\{T(t) : t \geq 0\}$ *is equicontinuous and there is a positive constant* M *such that* $\sup_{t\in J} \|T(t)\| \leq M$.

HF: Let $F : J \times \Theta \to P_{cl}(E)$ *be a multifunction such that:*

1. $x \to F(t,x)$ *is lower semicontinuous and* $(t,x) \to F(t,x)$ *is graph measurable.*
2. *There is a function* $\vartheta \in L^{\frac{1}{q}}(J,\mathbb{R}^+)$, $q \in (0,\alpha)$ *with* $\forall x \in \Theta$, $\|F(t,x)\| \leq \vartheta(t)$ *for a.e.* $t \in J$.
3. *There is a function* $\mu \in L^{\frac{1}{q}}(J,\mathbb{R}^+)$, $q \in (0,\alpha)$ *such that* $4L\|\mu\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} < 1$, *and if* $W \subset \Theta$ *is bounded and* χ *is NCHM in* E, *then we have*
$$\chi(F(t,W)) \leq \mu(t)\chi(W), \qquad a.e. t \in J,$$
where, $L = \dfrac{b^{\alpha-q}}{\Gamma(\alpha)(\omega+1)^{1-q}}, \omega = \dfrac{\alpha-1}{1-q}$.

Hg: $g : \Lambda \to E$ *is compact and continuous with* $\|g(x)\| \leq N$, $\forall\, x \in \Lambda$, *where* N *is a positive constant.*

HI: $\forall i = 1,2\cdots,m$, $I_i : E \to E$ *is compact and continuous with* $\|I_i(x)\| \leq h_i\|x\|$, $x \in E$, *where* h_i *is positive constant.*

Hr: There is a positive constant r *such that*

$$M(\|\Psi\| + N) + M\left[\sum_{i=1}^{m} h_i(r + \|\Psi\|) + L\|\vartheta\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}\right] \leq r. \qquad (1)$$

Then the problem (P_Ψ) *has a mild solution on* $[-r,b]$.

Proof. Let $\Pi : \Lambda \to 2^{L^1(J,E)}$, defined by

$$\Pi(x) = S^1_{F(\cdot,\tau(\cdot)x)} = \{f \in L^1(J,E) : f(t) \in F(t,\tau(t)x), a.e.t \in J\}.$$

We prove that Π has a nonempty closed, lower semicontinuous and decombsable values. S^1_F is closed because F has closed value. From (HF)(2), F is integrably bounded. So, S^1_F is nonempty (Theorem 3.2 [18]). One can easily check that S^1_F is decomposable. Now, we prove that Π is lower semicontinuous. To do so, we need to show that $x \to d(u,\Pi(x))$ is u.s.c. for every $u \in L^1(J,E)$. From Theorem 2.2 in [18],

$$\begin{aligned} d(u,\Pi(x)) &= \inf_{f\in\Pi(x)} \|u-f\|_{L^1} \\ &= \inf_{f(t)\in F(t,\tau(t)x)} \int_0^b \|u(t)-f(t)\|dt \\ &= \int_0^b \inf_{f(t)\in F(t,\tau(t)x)} \|u(t)-f(t)\|dt \\ &= \int_0^b d(u(t),F(t,\tau(t)x))dt. \end{aligned} \tag{2}$$

For any $\delta \geq 0$, we show that the set $u_\delta = \{x \in \Lambda : d(u,\Pi(x)) \geq \delta\}$ is closed. To this end, let $\{x_n\}_{n\geq 1} \subseteq u_\delta$ and $x_n \to x$ in Λ. So, for every $x_n(t) \to x(t)$ in E. From (HF)(1), F is l.s.c.. By Remark 2.10, $z \to d(u(t),F(t,z))$ is u.s.c. and then by Fatou Lemma with (2),

$$\begin{aligned} \delta &\leq \lim_{n\to\infty} \sup d(u,\Pi(x_n)) \\ &= \lim_{n\to\infty} \sup \int_0^b d(u(t),F(t,\tau(t)x_n)dt \\ &\leq \int_0^b \lim_{n\to\infty} \sup d(u(t),F(t,\tau(t)x_n)dt \\ &\leq \int_0^b d(u(t),F(t,\tau(t)x)dt = d(u,\Pi(x)). \end{aligned}$$

Therefore, $x \in u_\delta$. This means $d(u,\Pi(x))$ is u.s.c.. So, by Remark 2.10, the multifunction Π is l.s.c. and by Lemma 2.12, Π has a continuous selection $f : \Lambda \to L^1(J,E)$ such that $f(x) \in \Pi(x)$, for every $x \in \Lambda$. So,

$f(x)(s) \in F(s, \tau(s)x), a.e. s \in J$. Now, let us define the map $G : \Lambda \to \Lambda$, such that

$$G(x)(t) = \begin{cases} \Psi(t) - g(x), & t \in [-r, 0], \\ \mathcal{T}_\alpha(t)(\Psi(0) - g(x)) & \\ + \int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s) f(s) ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(\Psi(0) - g(x)) & \\ + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(x(t_k^-)) & \\ + \int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s) f(s) ds, & t \in J_i, \end{cases} \tag{3}$$

where $i = 1, \cdots, m,\ f \in S^1_{F(\cdot,\tau(\cdot)x)}$. Thus, if G has fixed point, then the problem (P_Ψ) has a mild solution. So, we prove that G satisfies all the hypothesis of Theorem 2.13. We give our proof in several steps. in the first let us define the set $W_0 = \{x \in \Lambda : \|x - x_0\| \leq r\}$, where

$$x_0(t) = \begin{cases} \Psi(t), & t \in [-r, 0], \\ \Psi(0), & t \in J. \end{cases}$$

Clearly, W_0 is bounded, convex and closed subset of Λ.
Step 1. We prove that $G(W_0) \subseteq W_0$. Let $x \in W_0$, if $t \in [-r, 0]$, then from (3), (Hg) and (1) we have

$$\|G(x)(t) - x_0(t)\| \leq \|g(x)\| \leq N \leq r.$$

If $t \in J$, then, by using Holder's inequality, (3), (HF)(2), (Hg), with Lemma 2.4 and (1), we get $\forall\ t \in J_0$,

$$\begin{aligned} \|G(x)(t) - x_0(t)\| &\leq \|\mathcal{T}_\alpha(t)(\Psi(0) + g(x))\| \\ &\quad + \|\int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s) f(s) ds\| \\ &\leq M(\|\Psi\| + N) + \frac{M}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \vartheta(s) ds \\ &\leq M(\|\Psi\| + N) + ML\|\vartheta\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} \\ &\leq r. \end{aligned}$$

For $t \in J_i$, $i = 1, \cdots, m$, in the same way, with the condition (HI), we get

$$\begin{aligned}\|G(x)(t) - x_0(t)\| &\leq M(\|\Psi\| + N) + M\sum_{k=1}^{k=i} h_k\ (\|\Psi\| + r) \\ &\quad + \frac{M}{\Gamma(1+\alpha)} \frac{t^{(1+\omega)(1-q)}}{(1+\omega)^{(1-q)}} \|\vartheta\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} \\ &\leq r.\end{aligned}$$

Then, $G(W_0) \subseteq W_0$.
Let $W_n = \overline{conv} G(W_{n-1})$, $n \geq 1$. Clearly, W_n is closed, convex and nonempty subset of Λ. Moreover, $W_1 = \overline{conv} G(W_0) \subseteq W_0$ and $W_2 = \overline{conv} G(W_1) \subseteq \overline{conv} G(W_0) \subseteq W_1$. It can easily be proven that the sequence $(W_n)_{n=1}^{\infty}$ is decreasing of bounded, convex and closed subsets of Λ. By Lemma 2.5, we only need to show that $W = \cap_{n=1}^{\infty} W_n$ is compact and nonempty set. To do that, we shall prove

$$\lim_{n\to\infty} \chi_\Lambda(W_n) = 0. \tag{4}$$

where χ_Λ is defined in the previous section. Now, we will prove (4) by step 2 and step 3.

Step 2. For every $n \in \mathbb{N}$ and $\forall i = 0, 1, \cdots, m$, let

$$W_{n|_{\overline{J_i}}} = \{x^* \in C(\overline{J_i}, E) : \ x^*(t) = x(t),\ x^*(t_i) = x(t_i^+), t \in J_i, x \in W_n\}.$$

Without loss of generality, we show that $W_{1|_{\overline{J_i}}}$ is equicontinuous. Since, $W_1 = \overline{conv} G(W_0)$, so we only need to prove that $G(W_0)_{|\overline{J_i}}$ is equicontinuous. Let $x \in W_0$ and $y = G(x)$. Form (3), we have

$$y(t) = \begin{cases} \Psi(t) - g(x), & t \in [-r, 0], \\ \mathcal{T}_\alpha(t)(\Psi(0) - g(x)) + \int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s) f(s) ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(\Psi(0) - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t - t_k) I_k(x(t_k^-)) & \\ + \int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s) f(s) ds, & t \in J_i, \end{cases}$$

where $i = 1, \cdots, m$. By the continuity of Ψ, one can easily see that if $t, t + \upsilon \in [-r, 0]$, then

$$\lim_{\upsilon\to 0} \|y^*(t+\upsilon) - y^*(t)\| = 0,$$

not dependent on x.
For $t \in J_i$, $\forall\ i = 0, 1, \cdots, m$, by the same way in Step 3 and Step 4 in the proof of Theorem 2 of [3] and Theorem 4 of [19] respectively, we obtain

$$\lim_{\upsilon \to 0} \|y^*(t+\upsilon) - y^*(t)\| = 0,$$

not dependent on x. Therefore, $W_{1|_{\overline{J_i}}}$ is equicontinuous for all i.

Step 3. Set $W = \cap_{n=1}^{\infty} W_n$. Our goal to show that $W \subset \Lambda$ is nonempty and compact in Λ. To this end, by Lemma 2.5, we need only to prove that $\lim_{n\to\infty} \chi_\Lambda(W_n) = 0$. From Lemma 2.8, $\forall\ \varepsilon > 0$, $\exists\ \{u_k\}_{k=1}^{\infty} \subset\ G(W_{n-1})$, such that

$$\begin{aligned} \chi_\Lambda(W_n) = \chi_\Lambda G(W_{n-1}) &\le 2\chi_\Lambda\{u_k : k \ge 1\} + \epsilon \\ &\le 2\chi_\Theta\{u_k : k \ge 1\} + 2\chi_{PC}\{u_k : k \ge 1\} + \epsilon. \end{aligned}$$

It follows from definition of χ_Λ that

$$\chi_\Lambda(W_n) \le 2\chi_\Theta(z_{|_{[-r,0]}}) + 2 \max_{0\le i\le m} \chi_i(z_{|_{\overline{J_i}}}) + \varepsilon,$$

where $z = \{u_k : k \ge 1\}$. From Lemma 2.6,

$$\chi_i(z_{|_{\overline{J_i}}}) = \sup_{t\in\overline{J_i}} \chi(z(t)).$$

Henece, using the nonsinglarity of χ we get

$$\begin{aligned} \chi_\Lambda(W_n) &\le 2\chi_\Theta(z_{|_{[-r,0]}}) + 2 \max_{i=0,1,\cdots,m} [\sup_{t\in\overline{J_i}} \chi(z(t))] + \varepsilon \\ &= 2 \sup_{t\in[-r,0]} \chi(z(t)) + 2 \sup_{t\in J} \chi(z(t)) + \epsilon. \end{aligned}$$

Then,

$$\chi_\Lambda(W_n) \le 2 \sup_{t\in[-r,0]} \chi(z(t)) + 2 \sup_{t\in J} \chi\{u_k : k \ge 1\} + \epsilon. \tag{5}$$

Since $u_k \in G(W_{n-1}), k \ge 1\ \exists\ x_k \in W_{n-1}$ with $u_k \in G(x_k)$. So, (5) can be rewrite as

$$\chi_\Lambda(W_n) \le \begin{cases} \chi(\Psi(t) - g(x_k)), & t \in [-r,0], \\ \chi(\mathcal{T}_\alpha(t)(\Psi(0) - g(x_k))) & \\ +\chi(\int_0^t (t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s) f_k(s)ds), & t \in J_0, \\ \chi(\mathcal{T}_\alpha(t)(\Psi(0) - g(x_k))) & \\ +\sum_{j=1}^{j=i} \chi(\mathcal{T}_\alpha(t-t_j) I_j(x_k(t_j^-))) & \\ +\chi(\int_0^t (t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s) f_k(s)ds), & t \in J_i. \end{cases}$$

Since, g and I_i are compact for all i, then by Lemma 2.2, $\forall\, t \in [-r, b]$,

$$\begin{aligned} &\chi\{\mathcal{T}_\alpha(t)(\Psi(t) - g(x_k)) : k \ge 1\} = 0, \\ &\chi\{\mathcal{T}_\alpha(t)(\Psi(0) - g(x_k)) : k \ge 1\} = 0, \\ &\chi\{\mathcal{T}_\alpha(t-t_j) I_j(x_k(t_j^-)) : k \ge 1\} = 0. \end{aligned}$$

Hence, for every $t \in [-r, b]$ we have

$$\chi_\Lambda(W_n) \le \varepsilon + 2\sup_{t\in J} \chi\left\{\int_0^t (t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s) f_k(s)ds : k \ge 1\right\}.$$

From [3],

$$0 \le \chi_\Lambda(W_n) \le \left(4L\|\mu\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}\right)^{n-1} \chi_{PC}(W_1).$$

If we take the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \chi_\Lambda(W_n) = 0.$$

Thus, $W = \cap_{n=1}^{\infty} W_n$ is nonempty and compact.

Step 4. We prove that G is continuous on W.
Let (x_n) be a sequence in W with $x_n \to x$ in $W \subset \Lambda$. From the uniform convergence of x_n towards x, for any $t \in J$,

$$\lim_{n\to\infty} \|\tau(t)x_n - \tau(t)x\| = 0.$$

As consequence, for every $t \in J$,

$$\lim_{n\to\infty} \|F(t,\tau(t)x_n) - F(t,\tau(t)x)\| = 0.$$

For every $t, s \in J$,

$$\|(t-s)^{\alpha-1} f(x_n)(s)\| \leq (t-s)^{\alpha-1}\vartheta(s) \in L^1(J,\mathbb{R}^+),$$

and

$$\|(t-s)^{\alpha-1} f(x)(s)\| \leq (t-s)^{\alpha-1}\vartheta(s) \in L^1(J,\mathbb{R}^+).$$

Then, from the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_0^t (t-s)^{\alpha-1} \|f(x_n)(s) - f(x)(s)\| ds = 0.$$

Therefore, if $t \in J_0$, by the continuity of g, we get

$$\begin{aligned}
\lim_{n\to\infty} & \|G(x_n)(t) - G(x)(t)\| \\
&\leq \lim_{n\to\infty} M\|g(x_n) - g(x)\| \\
&\quad + \lim_{n\to\infty} \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(x_n)(s) - f(x)(s)\| ds \\
&= 0.
\end{aligned}$$

Similarly, if $t \in J_i$, then by the continuity of I_i for all i, we get

$$\begin{aligned}
\lim_{n\to\infty} & \|G(x_n)(t) - G(x)(t)\| \\
&\leq \lim_{n\to\infty} M\|g(x_n) - g(x)\| \\
&\quad + M \sum_{k=1}^{k=i} \lim_{n\to\infty} \|I_k(x_n(t_k)) - I_k(x(t_k))\| \\
&\quad + \lim_{n\to\infty} \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(x_n)(s) - f(x)(s)\| ds \\
&= 0.
\end{aligned}$$

This shows the continuity of G. Thus, by Theorem 2.13 G has a fixed point $x \in W \subset \Lambda$. So, (P_Ψ) has a mild solution on $[-r, b]$.

4 Example

For all $z \in [0,1]$, $0 < \alpha < 1$ and $i = 1, 2, \cdots, m$, consider the problem:

$$\begin{cases} \partial_t^\alpha u(t,z) \in \partial_z^2 u(t,z) + R(t, \tau(t,z)u), & t \in [0,1],\, t \neq t_i, \\ u(t,0) = u(t,1) = 0, \\ u((\dfrac{i}{m+1})^+, z) = u(\dfrac{i}{m+1}, z) + \dfrac{1}{2^i}, \\ u(t,z) = \sum_{j=0}^{j=q} \int_0^1 k_j(z,v) tan^{-1}(u(p_j,v))dv \\ +u_0(v,z), & -1 \leq v \leq 0, \end{cases} \tag{6}$$

where ∂_t^α is the Caputo fractional partial derivative, $0 < p_0 < p_1 < \cdots < p_q < 1$, $k_j \in C([0,1]\times[0,1], \mathbb{R})$, $j = 0, 1, \cdots, q$ and $R : [0,1]\times E \to P(E)$. Put $E = L^2([-1,1], \mathbb{R})$, and $A = \dfrac{\partial^2}{\partial z^2}$ on $D(A) = \{y \in E : y, y'$ are absolutely continuous, $y'' \in E, y(0) = y(1) = 0\}$. From [26], A is the infinitesimal generator of compact and analytic semigroup $\{T(t)\}_{t\geq 0}$ in E. This implies that A satisfies the assumption (HA). For every $i = 1, \cdots, m$ define $I_i : E \to E$ by

$$I_i(y)(z) = \frac{1}{2^i}, z \in [0,1].$$

The functions I_i satisfy (HI).
$\forall j = 0, 1, \cdots, q$, let $H_j : E \to E$ such that

$$(H_j(y))(z) = \int_0^1 k_j(z,v) tan^{-1}(y(v))dv, z \in [0,1].$$

Now take $g : \Lambda \to E$ as

$$g(y) = \sum_{j=0}^{j=q} H_j(y(p_j)).$$

Also, we define $\Psi : [-1,0] \to E$ by

$$\Psi(t) = u_0(t,z), \quad z \in [0,1].$$

Finally, let $F(t,\tau(t)y) = R(t,\tau(t,z)u)$ where $z \in [0,1]$. Then,we can rewrite (6) as

$$\begin{cases} {}^cD^\alpha y(t) \in Ay(t) + F(t,\tau(t)y),\ t \in J = [0,1], t \neq t_i, \\ y(t) = \Psi(t) - g(y), \\ y(t_i^+) - y(t_i) = I_i(y(t_i^-)). \end{cases}$$

If we put conditions on F as in Theorem 3.1, then (6) has solution on $[-1,1]$.

Conclusion

This study argued about the existence results of Nonlocal functional fractional differential inclusions with impulses effect in Banach spaces. We investigated the situation when F is lower semicontinuous, nonconvex and $\{T(t)\}_{t>0}$ is not essentially compact. Various techniques were utilized such as NCHM and theorems of fixed point by which the existence of solutions to (P_Ψ) was established. In the essence, the results given in this work widened and advanced some preceding results. A numerical system was demonstrated in Section 4 to strengthen our results.

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