

Journal of Mathematical Extension
Vol. 15, No. 3, (2021) (12)1-19
URL: <https://doi.org/10.30495/JME.2021.1556>
ISSN: 1735-8299
Original Research Paper

Theories and Analytical Solutions for Fractional Differential Equations

A. Khalouta*

Ferhat Abbas Sétif University 1

A. Kadem

Ferhat Abbas Sétif University 1

Abstract. The main objective of this paper is to propose a new analytical method called the inverse fractional Aboodh transform method for solving fractional differential equations. Fractional derivatives are taken in the Riemann-Liouville and Caputo-Liouville sense. The main advantages of this method is that it is direct and concise. Various examples are given to show that the proposed method is very efficient and accurate.

AMS Subject Classification: 34A08; 35A22; 33E12; 35C10

Keywords and Phrases: Fractional differential equations, Riemann-Liouville derivative, Caputo-Liouville derivative, Aboodh transform

1 Introduction

In recent years, interest to fractional differential equations has been increasing considerably by many researchers in mathematics and physics because of its huge application area in various fields such as fluid mechanics, viscoelasticity, control theory, oil industries, relaxation processes, mathematical biology and other fields [2, 10, 11, 12, 14]. Therefore, a

Received: February 2020; Accepted: August 2020

*Corresponding Author

great deal of literature has been provided to create solutions for fractional differential equations as several powerful methods have been proposed to obtain approximate and exact solutions to fractional differential equations such as: Adomian decomposition method [8], variational iteration method [22], fractional difference method [23], differential transform method [3], homotopy analysis method [9], homotopy perturbation method [22], fractional reduced differential transform method (FRDTM) [13], fractional residual power series method (FRPSM) [15].

The integral transform method is an important mathematical technique. With the help of this technique, fractional differential equations which have difficult solution procedure can be converted into well-known algebraic equations which can be easily solved. There are many integral transforms for solving these type of problems are expressed in the literature such as the Laplace transform [19], the Fourier Transform [20], the Mellin transform [7].

In this paper we propose a new analytical method called the inverse fractional Aboodh transform method for solving fractional differential equations. To ensure the accuracy and effectiveness of the proposed method it will be applied to obtain exact solution of Bagley-Torvik equation with Caputo-Liouville fractional derivative of the form

$$y''(t) + {}^C D^{3/2}y(t) + y(t) = f(t),$$

subject to the initial conditions

$$y(0) = y_0, y'(0) = y_1.$$

Here $y(t)$ is the solution of the equation and $f(t)$ is a continuous function.

The Bagley-Torvik equation is a prototype fractional differential equation that was proposed by Bagley and Torvik as an application of fractional calculus to the theory of viscoelasticity [4, 5, 25]. This equation plays an important role in a large number of applied science and engineering problems. More specifically, any linearly damping fractional oscillator with a damping term has a $3/2$ -order fractional derivative can be represented by the Bagley-Torvik equation. Particularly, the equation with $1/2$ -order derivative or $3/2$ -order derivative can predict the models with materials where damping depends on frequency. It can also

describe motion of real physical systems, the modeling of the motion of a rigid plate immersed in a viscous fluid and a gas in a fluid respectively.

The rest of this paper is arranged as follows. In Section 2, we give some definitions and properties of fractional calculus theory. In Section 3, we present the mains results related to the inverse fractional Aboodh transform method. In Section 4, we explain the methodology of the proposed method through some examples of fractional differential equations. Section 5, is for discussion and conclusion of this paper.

2 Preliminaries

In this section, we give some basic definitions and properties of the fractional calculus theory which can be found in [16, 21, 23].

Definition 2.1. The Gamma function is defined as

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt,$$

where $Re(\alpha) > 0$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ for a function $y : \mathbb{R}^+ \rightarrow \mathbb{R}$, denoted by I^α is defined as

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} y(\xi) d\xi \quad , t > 0. \quad (1)$$

For $\alpha = 0$, we set $I^\alpha := I$, the identity operator.

Definition 2.3. The Riemann-Liouville fractional derivative operator of order $\alpha > 0$, $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, for a function $y : \mathbb{R}^+ \rightarrow \mathbb{R}$, denoted by ${}^R D^\alpha$ is defined as

$$\begin{aligned} {}^R D^\alpha y(t) &= D^n I^{n-\alpha} y(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t - \xi)^{n-\alpha-1} y(\xi) d\xi, \quad t > 0. \quad (2) \end{aligned}$$

Definition 2.4. The Caputo-Liouville fractional derivative operator of order $\alpha > 0, n - 1 < \alpha \leq n, n \in \mathbb{N}$, denoted by ${}^C D^\alpha$ is defined as

$$\begin{aligned} {}^C D^\alpha y(t) &= I^{n-\alpha} D^n y(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} y^{(n)}(\xi) d\xi, \quad t > 0. \end{aligned} \quad (3)$$

Definition 2.5. The Mittag-Leffler function is defined as follows

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}, \quad \alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0. \quad (4)$$

A further generalization of (4) is given in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0.$$

For $\alpha = 1$, $E_\alpha(z)$ reduces to e^z .

3 Theories of the inverse fractional Aboodh transform method

In this section, we prove six theorems related to the inverse fractional Aboodh transform method.

3.1 Aboodh transform

Recently, Aboodh in [1] introduced a new integral transform, called Aboodh transform, which is applied to solve an ordinary and partial differential equations.

Definition 3.1. The Aboodh transform is defined over the set of functions

$$A = \left\{ y(t) \mid \exists M, k_1, k_2 > 0, |y(t)| < M e^{k_j |t|}, \text{ if } t \in (-1)^j \times [0, \infty) \right\},$$

by the following integral

$$\mathcal{A}[y(t)] = K(v) = \frac{1}{v} \int_0^{\infty} y(t) e^{-vt} dt, \quad t \geq 0, \quad k_1 < v < k_2,$$

where v is the factor of the variable t .

Some basic properties of the Aboodh transform are given as follows:

Property 1: The Aboodh transform is a linear operator. That is, if λ and μ are non-zero constants, then

$$\mathcal{A}[\lambda y(t) \pm \mu z(t)] = \lambda \mathcal{A}[y(t)] \pm \mu \mathcal{A}[z(t)].$$

Property 2: If $y^{(n)}(t)$ is the n -th derivative of the function $y(t) \in A$ with respect to " t " then its Aboodh transform is given by

$$\mathcal{A}[y^{(n)}(t)] = v^n K(v) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{v^{2-n+k}}.$$

Property 3: (Convolution property) Suppose $K(v)$ and $G(v)$ are the Aboodh transforms of $y(t)$ and $z(t)$, respectively, both defined in the set A . Then the Aboodh transform of their convolution is given by

$$\mathcal{A}[(y * z)(t)] = vK(v)G(v),$$

where the convolution of two functions is defined by

$$(y * z)(t) = \int_0^t y(\xi) z(t - \xi) d\xi = \int_0^t y(t - \xi) z(\xi) d\xi.$$

Property 4: Some special Aboodh transforms

$$\begin{aligned} \mathcal{A}(1) &= \frac{1}{v^2}, \\ \mathcal{A}(t) &= \frac{1}{v^3}, \\ \mathcal{A}(t^n) &= \frac{n!}{v^{n+2}}, \quad n = 0, 1, 2, \dots \end{aligned}$$

Property 5: The Aboodh transform of t^α is given by

$$\mathcal{A}[t^\alpha] = \frac{\Gamma(\alpha + 1)}{v^{\alpha+2}}, \quad \alpha \geq 0.$$

3.2 Inverse Aboodh transform

Now, we give the proof of Theorems 3.2, 3.3 and 3.4. which are useful for finding the inverse Aboodh transform function

$$y(t) = \mathcal{A}^{-1} [K(v)].$$

Theorem 3.2. *If $\alpha, \beta > 0, a \in \mathbb{R}$, and $|a| < v^\alpha$, then we have the inverse Aboodh transform formula*

$$\mathcal{A}^{-1} \left[\frac{v^{\alpha-\beta-1}}{v^\alpha + a} \right] = t^{\beta-1} E_{\alpha,\beta}(-at^\alpha). \quad (5)$$

Proof. First, we take the Aboodh transform of the right-hand side of Eq. (5) to get

$$\begin{aligned} \mathcal{A} \left[t^{\beta-1} E_{\alpha,\beta}(-at^\alpha) \right] &= \frac{1}{v} \int_0^\infty e^{-vt} t^{\beta-1} E_{\alpha,\beta}(-at^\alpha) dt \\ &= \frac{1}{v} \int_0^\infty e^{-vt} t^{\beta-1} \sum_{k=0}^\infty \frac{(-at^\alpha)^k}{\Gamma(k\alpha + \beta)} dt \\ &= \sum_{k=0}^\infty \frac{1}{v} \frac{(-a)^k}{\Gamma(k\alpha + \beta)} \int_0^\infty e^{-vt} t^{\alpha k + \beta - 1} dt. \end{aligned} \quad (6)$$

Now, by integration by parts we have

$$\int_0^\infty e^{-vt} t^{\alpha k + \beta - 1} dt = \frac{1}{v^{\alpha k + \beta}} \Gamma(k\alpha + \beta). \quad (7)$$

By substituting Eq. (7) into Eq. (6) we get

$$\begin{aligned}
\mathcal{A} \left[t^{\beta-1} E_{\alpha,\beta}(-at^\alpha) \right] &= \sum_{k=0}^{\infty} \frac{1}{v} \frac{(-a)^k}{\Gamma(k\alpha + \beta)} \frac{1}{v^{\alpha k + \beta}} \Gamma(k\alpha + \beta) \\
&= \sum_{k=0}^{\infty} \frac{1}{v^{\alpha k + \beta + 1}} (-a)^k \\
&= \frac{1}{v^{\beta+1}} \sum_{n=0}^{\infty} \left(-\frac{a}{v^\alpha} \right)^k \\
&= \frac{1}{v^{\beta+1}} \frac{v^\alpha}{v^\alpha + a}, \quad \left| \frac{a}{v^\alpha} \right| < 1. \tag{8}
\end{aligned}$$

Then, the inverse Aboodh transform of Eq. (8) is given by

$$\mathcal{A}^{-1} \left[\frac{v^{\alpha-\beta-1}}{v^\alpha + a} \right] = t^{\beta-1} E_{\alpha,\beta}(-at^\alpha).$$

The proof is complete. \square

Theorem 3.3. *If $\alpha \geq \beta > 0, a \in \mathbb{R}$, and $|a| < v^{\alpha-\beta}$, then*

$$\mathcal{A}^{-1} \left[\frac{v^{-1}}{(v^\alpha + av^\beta)^{n+1}} \right] = t^{\alpha(n+1)-1} \sum_{k=0}^{\infty} \frac{(-a)^k \binom{n+k}{k}}{\Gamma(k(\alpha-\beta) + (n+1)\alpha)} t^{k(\alpha-\beta)}. \tag{9}$$

Proof. Similarly to the proof of the Theorem 3.2, we take the Aboodh transform of the right-hand side of Eq. (9), and by integration by parts, we get

$$\begin{aligned}
&\mathcal{A} \left[t^{\alpha(n+1)-1} \sum_{k=0}^{\infty} \frac{(-a)^k \binom{n+k}{k}}{\Gamma(k(\alpha-\beta) + (n+1)\alpha)} t^{k(\alpha-\beta)} \right] \\
&= \frac{1}{v^{\alpha(n+1)+1}} \sum_{k=0}^{\infty} \left(\frac{-a}{v^{\alpha-\beta}} \right)^k \binom{n+k}{k}.
\end{aligned}$$

Using the series expansion of $(1+t)^{-(n+1)}$, of the form

$$\frac{1}{(1+t)^{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k} (-t)^k, \tag{10}$$

we have

$$\begin{aligned} & \mathcal{A} \left[t^{\alpha(n+1)-1} \sum_{k=0}^{\infty} \frac{(-a)^k \binom{n+k}{k}}{\Gamma(k(\alpha-\beta) + (n+1)\alpha)} t^{k(\alpha-\beta)} \right] \\ &= \frac{1}{v(v^\alpha + av^\beta)^{n+1}}, \quad \left| \frac{a}{v^{\alpha-\beta}} \right| < 1. \end{aligned} \quad (11)$$

Then, the inverse Aboodh transform of Eq. (11) is given by

$$\mathcal{A}^{-1} \left[\frac{v^{-1}}{(v^\alpha + av^\beta)^{n+1}} \right] = t^{\alpha(n+1)-1} \sum_{k=0}^{\infty} \frac{(-a)^k \binom{n+k}{k}}{\Gamma(k(\alpha-\beta) + (n+1)\alpha)} t^{k(\alpha-\beta)}.$$

The proof is complete. \square

Theorem 3.4. *If $\alpha \geq \beta, \alpha > \gamma, a \in \mathbb{R}, |a| < v^{\alpha-\beta}$, and $|b| < v^\alpha + av^\beta$, then*

$$\begin{aligned} & \mathcal{A}^{-1} \left[\frac{v^{\gamma-1}}{v^\alpha + av^\beta + b} \right] \\ &= t^{\alpha-\gamma-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-b)^n (-a)^k \binom{n+k}{k}}{\Gamma(k(\alpha-\beta) + (n+1)\alpha - \gamma)} t^{k(\alpha-\beta)+n\alpha}. \end{aligned} \quad (12)$$

Proof. We take the Aboodh transform of the right-hand side of Eq. (12), by integration by parts and using the series expansion (10), we get

$$\begin{aligned} & \mathcal{A} \left[t^{\alpha-\gamma-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-b)^n (-a)^k \binom{n+k}{k}}{\Gamma(k(\alpha-\beta) + (n+1)\alpha - \gamma)} t^{k(\alpha-\beta)+n\alpha} \right] \\ &= \sum_{n=0}^{\infty} \frac{(-b)^n}{v^{1-\gamma}} \frac{1}{(v^\alpha + av^\beta)^{n+1}}, \quad \left| \frac{a}{v^{\alpha-\beta}} \right| < 1 \\ &= \frac{v^{\gamma-1}}{v^\alpha + av^\beta} \sum_{n=0}^{\infty} \left(\frac{-b}{v^\alpha + av^\beta} \right)^n \\ &= \frac{v^{\gamma-1}}{v^\alpha + av^\beta + b}, \quad \left| \frac{b}{v^\alpha + av^\beta} \right| < 1. \end{aligned} \quad (13)$$

Then, the inverse Aboodh transform of Eq. (13) is given by

$$\begin{aligned} & \mathcal{A}^{-1} \left[\frac{v^{\gamma-1}}{v^\alpha + av^\beta + b} \right] \\ &= t^{\alpha-\gamma-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-b)^n (-a)^k \binom{n+k}{k}}{\Gamma(k(\alpha-\beta) + (n+1)\alpha - \gamma)} t^{k(\alpha-\beta)+n\alpha}. \end{aligned}$$

The proof is complete. \square

3.3 Aboodh transform for fractional derivatives

Theorem 3.5. *If $K(v)$ is the Aboodh transform of $y(t)$, then the Aboodh transform of the Riemann-Liouville fractional integral for the function $y(t)$ of order α , is given by*

$$\mathcal{A}[I^\alpha y(t)] = \frac{1}{v^\alpha} K(v).$$

Proof. The Riemann-Liouville fractional integral for the function $y(t)$, as in (1), can be expressed as the convolution

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * y(t). \quad (14)$$

Applying the Aboodh transform in the Eq. (14) and using the properties 3 and 5, we have

$$\begin{aligned} \mathcal{A}[I^\alpha y(t)] &= \mathcal{A} \left[\frac{1}{\Gamma(\alpha)} t^{\alpha-1} * y(t) \right] = v \mathcal{A} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} \right] \mathcal{A}[y(t)] \\ &= v \frac{1}{v^{\alpha+1}} K(v) = \frac{1}{v^\alpha} K(v). \end{aligned}$$

The proof is complete. \square

Theorem 3.6. *Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n-1 < \alpha \leq n$ and $K(v)$ be the Aboodh transform of the function $y(t)$, then the Aboodh transform denoted by $K_\alpha^R(v)$ of the Riemann-Liouville fractional derivative of $y(t)$ of order α , is given by*

$$\mathcal{A}[{}^R D^\alpha y(t)] = K_\alpha^R(v) = v^\alpha K(v) - \sum_{k=0}^{n-1} \frac{1}{v^{1-k}} \left[{}^R D^{\alpha-k-1} y(t) \right]_{t=0}.$$

Proof. Since

$${}^R D^\alpha y(t) = D^n I^{n-\alpha} y(t) = \frac{d^n}{dt^n} I^{n-\alpha} y(t).$$

Let

$$g(t) = I^{n-\alpha} y(t), \quad (15)$$

then

$${}^R D^\alpha y(t) = \frac{d^n}{dt^n} g(t) = g^{(n)}(t).$$

Applying the Aboodh transform on both sides of (15) using the Theorem 3.5, we get

$$G(v) = \mathcal{A}[g(t)] = \mathcal{A}[I^{n-\alpha} y(t)] = \frac{1}{v^{n-\alpha}} K(v). \quad (16)$$

Also, we have from the Property 2

$$\begin{aligned} \mathcal{A}[{}^R D^\alpha f(t)] &= \mathcal{A}\left[\frac{d^n}{dt^n} g(t)\right] = v^n G(v) - \sum_{k=0}^{n-1} \frac{[g^{(k)}(t)]_{t=0}}{v^{2-n+k}} \\ &= v^n G(v) - \sum_{k=0}^{n-1} \frac{1}{v^{1-k}} [g^{(n-k-1)}(t)]_{t=0}. \end{aligned} \quad (17)$$

From the Definition of the Riemann-Liouville fractional derivative as in (2), we obtain

$$\begin{aligned} [g^{(n-k-1)}(t)]_{t=0} &= \left[\frac{d^{n-k-1}}{dt^{n-k-1}} g(t) \right]_{t=0} \\ &= \left[D^{n-k-1} I^{n-\alpha} y(t) \right]_{t=0} \\ &= \left[{}^R D^{\alpha-k-1} y(t) \right]_{t=0}. \end{aligned} \quad (18)$$

Hence, by using Eqs. (18) and (16) in (17), we get

$$\mathcal{A}[{}^R D^\alpha y(t)] = K_\alpha^R(v) = v^\alpha K(v) - \sum_{k=0}^{n-1} \frac{1}{v^{1-k}} [{}^R D^{\alpha-k-1} y(t)]_{t=0}.$$

The proof is complete. \square

Theorem 3.7. Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n - 1 < \alpha \leq n$ and $K(v)$ be the Aboodh transform of the function $y(t)$, then the Aboodh transform denoted by $K_\alpha(v)$ of the Caputo-Liouville fractional derivative of $y(t)$ of order α , is given by

$$\mathcal{A}[{}^C D^\alpha f(t)] = K_\alpha(v) = v^\alpha K(v) - \sum_{k=0}^{n-1} \frac{[D^k y(t)]_{t=0}}{v^{2-\alpha+k}}.$$

Proof. Let

$$g(t) = y^{(n)}(t),$$

then by the Definition of the Caputo-Liouville fractional derivative as in (3), we obtain

$$\begin{aligned} {}^C D^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} y^{(n)}(\xi) d\xi \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} g(\xi) d\xi = I^{n-\alpha} g(t). \end{aligned} \quad (19)$$

Applying the Aboodh transform on both sides of (19) using the Theorem 3.5, we get

$$\mathcal{A}[{}^C D^\alpha y(t)] = \mathcal{A}[I^{n-\alpha} g(t)] = \frac{1}{v^{n-\alpha}} G(v). \quad (20)$$

Also, we have from the Property 2

$$\begin{aligned} \mathcal{A}[g(t)] &= \mathcal{A}[y^{(n)}(t)], \\ G(v) &= v^n K(v) - \sum_{k=0}^{n-1} \frac{[y^{(k)}(t)]_{t=0}}{v^{2-n+k}}. \end{aligned}$$

Hence, (20) becomes

$$\begin{aligned} \mathcal{A}[{}^C D^\alpha y(t)] &= \frac{1}{v^{n-\alpha}} \left(v^n K(v) - \sum_{k=0}^{n-1} \frac{[y^{(k)}(t)]_{t=0}}{v^{2-n+k}} \right) \\ &= v^\alpha K(v) - \sum_{k=0}^{n-1} \frac{[D^k y(t)]_{t=0}}{v^{2-\alpha+k}} = K_\alpha(v). \end{aligned}$$

The proof is complete. \square

4 Examples

In this section, six numerical examples are presented to illustrate the accuracy and effectiveness of the proposed method.

Example 4.1. Consider the following linear fractional initial value problem [18]

$${}^R D^{1/2} y(t) + y(t) = 0, \quad (21)$$

subject to the initial condition

$$\left[{}^R D^{-1/2} y(t) \right]_{t=0} = 2, \quad (22)$$

where ${}^R D^{1/2}$ is the Riemann-Liouville fractional derivative operator of order $1/2$.

Applying the Aboodh transform on both sides of Eq. (21) and using the Theorem 3.6, we get

$$v^{1/2} K(v) + \sum_{k=0}^{n-1} \frac{1}{v^{1-k}} \left[{}^R D^{1/2-k-1} f(t) \right]_{t=0} + K(v) = 0. \quad (23)$$

Substituting Eq. (22) into Eq. (23), we get

$$\left(v^{1/2} + 1 \right) K(v) - \frac{2}{v} = 0.$$

So

$$K(v) = \mathcal{A}[y(t)] = \frac{2v^{-1}}{v^{1/2} + 1}.$$

Using the Theorem 3.2, the exact solution of this problem can be obtained as

$$y(t) = 2t^{-1/2} E_{\frac{1}{2}, \frac{1}{2}}(-t^{1/2}).$$

Example 4.2. Consider the initial value problem for a non-homogeneous fractional differential equation [18]

$${}^R D^\alpha y(t) - \lambda y(t) = h(t), \quad t > 0, n-1 < \alpha \leq n, \quad (24)$$

subject to the initial conditions

$$\left[{}^R D^{\alpha-k-1} y(t) \right]_{t=0} = b_k, \quad k = 0, 1, 2, \dots, \quad (25)$$

where λ and b_k are constants and fractional derivative ${}^R D^\alpha$ is the Riemann-Liouville fractional derivative operator of order α , $n - 1 < \alpha \leq n$.

Applying the Aboodh transform on both sides of Eq. (24) and using the Theorem 3.6, we get

$$v^\alpha K(v) - \sum_{k=0}^{n-1} \frac{1}{v^{1-k}} \left[D^{\alpha-k-1} f(t) \right]_{t=0} - \lambda K(v) = H(v). \quad (26)$$

Substituting Eq. (25) into Eq. (26), we get

$$(v^\alpha - \lambda) K(v) - \sum_{k=0}^{n-1} \frac{1}{v^{1-k}} b_k = H(v).$$

So

$$K(v) = \mathcal{A}[y(t)] = \frac{1}{v^\alpha - \lambda} H(v) + \sum_{k=0}^{n-1} \frac{v^{k-1}}{v^\alpha - \lambda} b_k.$$

Using the Theorem 3.2, and the Convolution property, we get

$$\begin{aligned} y(t) &= \left[t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha) * h(t) \right] + \sum_{k=0}^{n-1} b_k t^{\alpha-k-1} E_{\alpha,\alpha-k}(\lambda t^\alpha) \\ &= \int_0^\infty (t - \xi)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t - \xi)^\alpha) h(\xi) d\xi + \sum_{k=0}^{n-1} b_k t^{\alpha-k-1} E_{\alpha,\alpha-k}(\lambda t^\alpha). \end{aligned}$$

This is the exact solution of this problem.

Example 4.3. Consider the initial value problem of non-homogeneous Bagley-Torvik equation [6]

$$y''(t) + {}^C D^{3/2} y(t) + y(t) = 1 + t, \quad (27)$$

subject to the initial conditions

$$y(0) = y'(0) = 1, \quad (28)$$

where ${}^C D^{3/2}$ is the Caputo-Liouville fractional derivative operator of order 3/2.

Applying the Aboodh transform on both sides of Eq. (27), and using Theorem 3.7, we get

$$\begin{aligned} & v^2 K(v) - y(0) - \frac{1}{v} y'(0) + v^{3/2} K(v) - \frac{1}{v^{1/2}} y(0) - \frac{1}{v^{3/2}} y'(0) \\ & + K(v) = \frac{1}{v^2} + \frac{1}{v^3}. \end{aligned} \quad (29)$$

Substituting Eq. (28) into Eq. (29), we get

$$K(v) \left[v^2 + v^{3/2} + 1 \right] = \frac{1}{v^2} + \frac{1}{v^3} + 1 + \frac{1}{v} + \frac{1}{v^{1/2}} + \frac{1}{v^{3/2}}. \quad (30)$$

Then Eq. (30) becomes

$$K(v) \left[v^2 + v^{3/2} + 1 \right] = \left(\frac{1}{v^2} + \frac{1}{v^3} \right) \left(v^2 + v^{3/2} + 1 \right).$$

So

$$K(v) = \mathcal{A}[y(t)] = \frac{1}{v^2} + \frac{1}{v^3}. \quad (31)$$

Taking the inverse Aboodh transform of Eq. (31), we have

$$y(t) = 1 + t.$$

This is the exact solution of this problem.

Example 4.4. Consider the following linear fractional initial value problem [9, 17, 24]

$${}^C D^\alpha y(t) + y(t) = 0, \quad (32)$$

subject to the initial conditions

$$y(0) = 1, y'(0) = 0, \quad (33)$$

where ${}^C D^\alpha$ is the Caputo-Liouville fractional derivative operator of order $\alpha, 0 < \alpha \leq 2$.

The second initial condition in (33) is for $\alpha > 1$ only. In two cases of α , $\mathcal{A} [{}^C D^\alpha y(t)]$ is obtained as

1- For $\alpha < 1$

$$\mathcal{A} [{}^C D^\alpha y(t)] = \frac{v^2 K(v) - 1}{v^{2-\alpha}} = v^\alpha K(v) - v^{\alpha-2}.$$

2- For $\alpha > 1$

$$\mathcal{A}[{}^C D^\alpha y(t)] = \frac{vK(v) - v^{-1}}{v^{1-\alpha}} = v^\alpha K(v) - v^{\alpha-2}.$$

Which are the same.

Applying the Aboodh transform to both sides of Eq. (32) and using the Theorem 3.7, we get

$$v^\alpha K(v) - v^{\alpha-2} + K(v) = 0.$$

So

$$K(v) = \mathcal{A}[y(t)] = \frac{v^{\alpha-2}}{v^\alpha + 1}.$$

Using the Theorem 3.2, the exact solution of this problem can be obtained as

$$y(t) = E_\alpha(-t^\alpha).$$

Example 4.5. Consider the following linear fractional initial value problem [22]

$${}^C D^\alpha y(t) = y(t) + 1, \quad (34)$$

subject to the initial condition

$$y(0) = 0, \quad (35)$$

where ${}^C D^\alpha$ is the Caputo-Liouville fractional derivative operator of order $\alpha, 0 < \alpha \leq 1$.

Applying the Aboodh transform to both sides of Eq. (34) and using the Theorem 3.7 with the initial condition (35), we get

$$v^\alpha K(v) = K(v) + \frac{1}{v^2}.$$

So

$$K(v) = \mathcal{A}[y(t)] = \frac{v^{-2}}{v^\alpha - 1}.$$

Using the Theorem 3.2, the exact solution of this problem can be obtained as

$$y(t) = t^\alpha E_{\alpha, \alpha+1}(t^\alpha).$$

Example 4.6. Consider the composite fractional oscillation equation [22]

$$y''(t) - a {}^C D^\alpha y(t) - by(t) = 8, \quad (36)$$

subject to the initial conditions

$$y(0) = y'(0) = 0, \quad (37)$$

where ${}^C D^\alpha$ is the Caputo-Liouville fractional derivative operator of order α , $1 < \alpha \leq 2$.

Applying the Aboodh transform to both sides of Eq. (36) and using the Theorem 3.7 with the initial conditions (37), we get

$$v^2 K(v) - av^\alpha K(v) - bK(v) = \frac{8}{v^2}.$$

So

$$K(v) = \mathcal{A}[y(t)] = \frac{8v^{-2}}{v^2 - av^\alpha - b}.$$

Using the Theorem 3.4, the exact solution of this problem can be obtained as

$$y(t) = 8t^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{b^n a^k \binom{n+k}{k}}{\Gamma(k(2-\alpha) + 2(n+1) + 1)} t^{k(2-\alpha) + 2n}.$$

5 Conclusion

In this paper, a new technique called the inverse fractional Aboodh transform method have been successfully applied to homogenous and non-homogenous linear fractional differential equations. We proved six theorems related to this method. The solutions obtained by our technique were in excellent agreement with those obtained via previous works and also conformed with the exact solution to confirm the effectiveness and accuracy of this technique. It is concluded that this technique is very powerful mathematical tool for solving different kinds linear fractional differential equations.

Acknowledgements

The authors would like to thank Professor Mazyar Zarepour (Managing Editor) as well as the anonymous referees who have made valuable and careful comments, which improved the paper considerably.

References

- [1] K. S. Aboodh, The new integral transform "Aboodh transform", *Global Journal of Pure and Applied Mathematics*, 9 (2013), 35-43.
- [2] O. P. Agrawal and D. Baleanu, Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems, *J. Vib. Control.*, 13 (2007), 1269-1281.
- [3] A. Arikoglu and I. Ozkol, Solution of fractional differential equations by using differential transform method, *Chaos Solitons Fractals*, 34 (2007), 1473-1481.
- [4] R. L. Bagley and P. J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, *Journal of Rheology*, 27 (1983), 201-210.
- [5] R. L. Bagley and P. J. Torvik, Fractional calculus in the transient analysis of viscoelastically damped structures, *AIAA Journal*, 23 (1985), 918-925.
- [6] M. K. Bansal and R. Jain, Analytical solution of Bagley-Torvik equation by generalize differential transform, *International Journal of Pure and Applied Mathematics*, 110 (2016), 265-273.
- [7] S. Butera and M. Di Paola, Fractional differential equations solved by using Mellin transform, *Commun. Nonlinear Sci. Numer. Simul.*, 19 (2014), 2220-2227.
- [8] J. F Cheng and Y. M Chu, Solution to the linear fractional differential equation using Adomian decomposition method, *Math. Probl. Eng.*, 2011, Article ID 587068, (2011).
- [9] I. Hashim, O. Abdulaziz, and S. Momani, Homotopy analysis method for fractional IVPs, *Commun. Nonlinear Sci. Numer. Simul.*, 14 (2009), 674-684.
- [10] R. Herrmann, *Fractional Calculus: An Introduction for Physicists*, World Scientific, New Jersey, (2014).

- [11] I. S. Jesus and J. A. Tenreiro Machado, Fractional control of heat diffusion systems, *Nonlinear Dyn.*, 54 (2008), 263-282.
- [12] A. Khalouta, A. Kadem, A new numerical technique for solving Caputo time-fractional biological population equation, *AIMS Mathematics*, 4 (2019), 1307-1319.
- [13] A. Khalouta and A. Kadem, A new representation of exact solutions for nonlinear time-fractional wave-like equations with variable coefficients, *Nonlinear Dyn. Syst. Theory*, 19 (2019), 319-330.
- [14] A. Khalouta, A. Kadem, New analytical method for solving nonlinear time-fractional reaction-diffusion-convection problems, *Rev. Colombiana Mat.*, 54 (2020), 1-11.
- [15] A. Khalouta and A. Kadem, Solution of the fractional Bratu-type equation via fractional residual power series method, *Tatra Mt. Math. Publ.*, 76 (2020), 1-16.
- [16] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, (2006).
- [17] P. Kumar and O. P. Agrawal, An approximate method for numerical solution of fractional differential equations, *Signal Processing*, 86 (2006), 2602-2610.
- [18] Z. B. Li, An extended fractional complex transform, *Int. J. Nonlinear Sci. Numer. Simul.*, 11 (2010), 335-337
- [19] S. Lin and C. Lu, Laplace transform for solving some families of fractional differential equations and its applications, *Adv Differ Equ.*, 2013, 137 (2013). <https://doi.org/10.1186/1687-1847-2013-137>.
- [20] Y. F. Luchko, H. Matrínez, and J. J. Trujillo, Fractional Fourier transform and some of its applications, *Fract. Calc. Appl. Anal.*, 11 (2008), 457-470.

- [21] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, New York, (1993).
- [22] Z. Odibat and S. Momani, Analytical comparison between the homotopy perturbation method and variational iteration method for differential equations of fractional order, *International Journal of Modern Physics B*, 22 (2008), 4041-4058.
- [23] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, (1999).
- [24] A. Saadatmandi and M. Dehghan, A new operational matrix for solving fractional order differential equations, *Comput. Math. Appl.*, 59 (2010), 1326-1336.
- [25] P. J. Torvik and R. L. Bagley, On the appearance of the fractional derivative in the behavior of real materials. *J. Appl. Mech.*, 51 (1984), 294-298.

Ali Khalouta

Department of Mathematics, Faculty of Sciences
Associate Professor of Mathematics
Ferhat Abbas Sétif University 1
Sétif, Algeria
E-mail: nadjibkh@yahoo.fr

Abdelouahab Kadem

Department of Mathematics, Faculty of Sciences
Professor of Mathematics
Ferhat Abbas Sétif University 1
Sétif, Algeria
E-mail: abdelouahabk@yahoo.fr