Journal of Mathematical Extension
Vol. 15, No. 3, (2021) (2)1-8
URL: https://doi.org/10.30495/JME.2021.1545
ISSN: 1735-8299
Original Research Paper

On Square Roots and Quasi-Square Roots of Elements in 2-Normed Algebras

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Abstract. The concept of 2-normed spaces and 2-Banach spaces are considered as a generalization of normed and Banach spaces. In the present paper, we have studied the existence of square roots and quasi square roots of some elements of a 2-Banach algebra. Moreover, the relation between n^{th} roots and quasi n^{th} roots of elements in 2-Banach algebras are considered.

AMS Subject Classification: Primary 64A Secondary 64H
Keywords and Phrases: 2-normed algebra, 2-normed space, 2-Banach algebra, square root, quasi square root.

1 Introduction

Study of square roots and quasi square roots of elements of topological algebras has started in 1966 by Gardner's and Ford's papers in Banach and Banach *- algebras ([6] and [4]).

Sterbova has studied the subject with considering the quasi square roots in locally multiplicatively convex topological algebras ([12] and [13]). After then the author generalized the existence of n^{th} roots of elements of topological algebra to a more general and non normable topological algebras [3].

Received: February 2020; Accepted: June 2020

An extensive study about Ford's lemma (related to this subject) has been done by Abel in 2011 [1].

In this article, we will study the subject for 2-normed algebras.

The notion of 2-metric spaces and 2-normed spaces was introduced by Gahler in 1960. This subject as a generalization of metric spaces and normed spaces was studied by many authors such as A. White , Gunawan and Mashadi. They are obtained various results about 2-normed algebras[1] [2][8]. By [10] there exist 2-normed algebras (with or without unity) which are not normable, so the study of this kinds of algebras may be considered as generalization of normed algebras.

Considering the concept of 2-normed algebras; Noor Mohammad and siddiqi , Lal, have shown that the class of 2-normed algebras with unity as defined in [8] is either void or contains only trivial algebras [8].

Also sum aspects of 2-normed and 2-Banach algebras are studied in [15], [14], [7]. A new definition of 2-normed algebras and an example satisfying this definition is given in [10], which in a subsequent work they were trying to show that there exists 2-normable algebra(with or without unity) which are normable.

Gahler in his first paper [5] mentioned the real motivation for studying 2-norm structure, and also he asked that if there is a physical situation or an abstract concept where norm topology does not work but 2-norm topology does work?

The theory of 2-normed spaces and their structure and difference of this structure with the normed spaces one is considered in [2].

An embedding of a generalized 2-normed space into the space of all bounded linear mappings on the set of all bounded 2-linear mappings are investigated in [9].

In this article we have studied the n^{th} roots and quasi n^{th} roots of elements of 2-normed algebras. The similar theorems which are proved for Banach algebras and for LMC algebras, are generalized for fundamental topological algebras, by the author [3].

2 Definitions and preliminary remarks

In this section, we give the basic definitions and properties of 2-normed spaces and algebras.

Definition 2.1. [10] *Let E be a linear space of dimension greater than one over the field $\mathbb{K}$ where $\mathbb{K}$ is the field of real or complex numbers. The real valued function $\|.,.\|$ on $E \times E$ is said to be a 2-norm if it satisfies the following axioms:*

(i) $\|x, y\| = 0$, *if and only if x and y are linearly dependent in E;*

(ii) $\|x, y\| = \|y, x\|$ *for all $x, y \in E$;*

(iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$ *for all $\alpha \in \mathbb{K}$ and for all $x, y \in E$;*

(iv) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ *for all $x, y, z \in E$.*

The pair $(E, \|.,.\|)$ is said to be a 2-normed linear space over the field $\mathbb{K}$.

Definition 2.2. [10] *Let E be a real algebra of dim≥ 2 with the 2-norm $\|.,.\|$. E is said to be a 2-normed algebra if there is some $k > 0$ such that $\|xy, z\| \leq k\|x, z\|\|y, z\|$ for all $x, y, z \in E$.*

There are several examples in the literature of this subject such as [10] and [11]. Also there are many definitions for 2-normed algebras which some of them are not usefull [10]. The following interesting definition is given in [10].

Definition 2.3. . *Let E be a subalgebra of dimension ≥ 2 of an algebra B, and $\|.,.\|$ be a 2-norm in B and $a_1, a_2 \in B$ be linearly independent, non-invertible and be such that*

$$\forall x, y \in E, \|xy, a_i\| \leq \|x, a_i\|\|y, a_i\|, i = 1, 2.$$

Then E is called a 2-normed algebra with respect to a_1, a_2.

Definition 2.4. *Let E be an algebra and $x, y \in E$. Then the quasi product of x, y is defined as $x \circ y = x + y - xy$, and we denote*

$$x \circ x \circ \cdots \circ x = x^{\circ n}.$$

Definition 2.5. *Let x be an element of a 2-normed algebra E. The element $y \in E$ is said to be quasi-inverse of x if*

$$y \circ x = 0, x \circ y = 0$$

An element that has a quasi-inverses said to be quasi-invertible (or quasi-regular), all other elements are said to be quasi-singular.

Remark 2.6. *The quasi-inverse of a quasi-invertible element x in a topological algebra is denoted by x^0 , and the set of all quasi-invertible elements of E by q-Inv(E), and the set of all quasi-singular elements of E by q-Sing(E).*

Definition 2.7. *Let x be an element of a 2-normed algebra E. The element $y \in E$ is said to be quasi square root of x if*

$$y \circ y = x.$$

By above definition and definition of quasi product it is eassily seen that y is a quasi square root of x if and only if $x = 2y - y^2$.

Definition 2.8. *We say that a 2-Banach algebra E has an identity element e if for every $a \in E$, $e.a = a.e = a$ and $\|a, e\| \neq 0$.*

3 New Results

Let E be a 2-Banach algebra with unit element e, in this section we give a condition that any $x \in E$ have a square root and in general n^{th} root. Also conditions for existence of quasi square roots and quasi n^{th} roots will be given.

Theorem 3.1. *Let E be a 2-Banach algebra. If $\|e - x, z\| < 1$ for all $z \in E$, then x has an square root in E.*

Proof. Let $\|e - x, z\| < \eta < 1$ now suppose that

$$y_m = \sum_{k=0}^{m} \binom{\frac{1}{2}}{k} (x - e)^k e^{m-k}.$$

We have

$$\|(x - e)^k, z\| \leq \|(x - e)^{k-1}, z\|\|e - x, z\|$$
$$\leq ... \leq \|(x - e), z\|^k < \eta^n,$$

which tends to zero when k tends infinity. Also since

$$\sum_{k=0}^{m} \binom{\frac{1}{2}}{k} \eta^n < \sum_{k=0}^{m} \binom{\frac{1}{2}}{k} = \sqrt{2} - 1,$$

implies that $\sum_{k=0}^{m}\binom{\frac{1}{2}}{k}(x-e)^k e^{m-k}$ is a convergent series. As we know

$$\lim_{m\to\infty}\sum_{k=0}^{m}\binom{\frac{1}{2}}{k}(x-e)^k e^{m-k} = x^{\frac{1}{2}}.$$

then x has an square root in E. □

By replacing $\frac{1}{n}$ instead of $\frac{1}{2}$ in the previous theorem we conclude that x has nth root.

Corollary 3.2. *Let A be a 2-Banach algebra. If $\|e-x,z\|<1$ for all $z\in E$ then x has an n^{th} root in E.*

Proof. In this case we will have

$$\lim_{m\to\infty}\sum_{k=0}^{m}\binom{\frac{1}{n}}{k}(x-e)^k e^{m-k} = x^{\frac{1}{n}}.$$

since E is Banach, it is clear that $x^{\frac{1}{n}}\in E$. □

The important results on inverses regarding to the above theorem proved an interesting data for algebras without a unit element. It is worth noting that this notion have been extended to such algebras in ordinary case in two ways:

(1) by the adjunction of a unit element,

(2) by using the concept of quasi-inverse.

In next theorem we would like to consider the subject in 2-normed algebras by using the concept of quasi-inverse.

Theorem 3.3. *Let A be a 2-Banach algebra. If $\|x,z\|<1$ for all $z\in E$, then there is a $y\in E$ such that $2y-y^2=x$.*

Proof. Let

$$y_m=\sum_{k=0}^{m}\binom{\frac{1}{2}}{k}(x)^k,$$

similar to above theorem, the series $\sum_{k=0}^{m}\binom{\frac{1}{2}}{k}(-1)^k\|x,z\|^k$ is absolutely convergent for every $z\in E$. Now since the sum $u(t)$ of the series

$$u(t)=-\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}(-t)^k,$$

satisfies the equation $2u(t) - [u(t)]^2 = t$ in which by replacing t by x we get the conclusion.That is with above condition x has quasi square root. $\square$

3.1 Quasi invetibility.

In this section we consider quasi-invertibility of elements of 2-Banach algebras.

Theorem 3.4. *Let E be a $2-$ Banach algebra. If $\|x, z\| < 1$ for all $z \in E$, then x is quasi invertible and*

$$x^0 = -\sum_{n=1}^{\infty} y^n.$$

Proof. Let t be positive real number and $\|x, z\| < t < 1$. Then we have $\|(x, z)^n\| < t^n < 1$. So the series $\sum_{n=1}^{\infty} \|(x, z)^n\|$ is convergent.

Since E is Banach then it converges to an element of E such as y. Let $s_n = 1 + a + \cdots + a_{n-1}$. Tehen $s_n \to s$ and $\|x, z\| \Rightarrow 0$ as $n \to \infty$, and we have

$$(1 - y)s_n = s_n(1 - y) = 1 - y_n.$$

Therefore, by continuity of multiplication, we have , $(1-y)s = s(1-y) = 1$. $\square$

Theorem 3.5. *If E has a unit element e, then an element x of E has the quasi-inverse y if and only if $e - x$ has the inverse $e - y$.*

Proof. It is obviously seen that we have $(e - x)(e - y) = e - (x \circ y)$.

Then $(x \circ y) = 0$ if and only if $(e - x)(e - y) = e$. $\square$

Similar to above theorem we can state it for n^{th} and quasi n^{th} roots.

Theorem 3.6. *Let E be a 2-Banach algebra with unit element e. Then z is quasi n^{th} root of y if and only if $e - z$ is n^{th} root of $e - y$*

Proof. For $n = 2$ we have $z^{\circ 2} = z \circ z = 2z - z^2 = y$ if and only if $(e - z)^2 = e - y$.

Now we have to prove the theorem by induction. Let $z^{\circ n} = e-(e-z)^n$ we have to prove that $z^{\circ(n+1)} = e-(e-z)^{n+1}$, to do it, we have

$$\begin{aligned} z^{\circ(n+1)} &= z \circ z^{\circ n} = z \circ (e-(e-z)^n) \\ &= z + (e-(e-z)^n) - z(e-(e-z)^n) = e-(e-z)^n(e-z)) \\ &= e-(e-z)^{n+1}. \end{aligned}$$

That is $y = z^{\circ n} = e-(e-z)^n$ if and only if $(e-z)^n = e-y$. □

4 Conclusion

In this article we considered the notion of square roots and quasi square roots also n^{th} roots and quasi n^{th} roots of elements of a $2-$normed algebra. We proved that every x with condition $\|e-x,z\| < 1$ for all$x \in E$ has a square root and an n^{th} root in E. Also every element x of E has the quasi-inverse y if and only if $e-x$ has the inverse $e-y$.

Acknowledgements
I would like to thank the referees for carefully reading the manuscript, and usefull comments.

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