On Square Roots and Quasi-Square Roots of Elements in 2-Normed Algebras

Ali Zohri
Payame Noor University

Abstract. The concept of 2-normed spaces and 2-Banach spaces are considered as a generalization of normed and Banach spaces. In the present paper, we have studied the existence of square roots and quasi square roots of some elements of a 2-Banach algebra. Moreover, the relation between \( n^{th} \) roots and quasi \( n^{th} \) roots of elements in 2-Banach algebras are considered.

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1 Introduction

Study of square roots and quasi square roots of elements of topological algebras has started in 1966 by Gardner’s and Ford’s papers in Banach and Banach *- algebras ([6] and [4]).

Sterbova has studied the subject with considering the quasi square roots in locally multiplicatively convex topological algebras ([12] and [13]). After then the author generalized the existence of \( n^{th} \) roots of elements of topological algebra to a more general and non normable topological algebras [3].
An extensive study about Ford’s lemma (related to this subject) has been done by Abel in 2011 [1].

In this article, we will study the subject for 2-normed algebras.

The notion of 2-metric spaces and 2-normed spaces was introduced by Gahler in 1960. This subject as a generalization of metric spaces and normed spaces was studied by many authors such as A. White, Gunawan and Mashadi. They are obtained various results about 2-normed algebras [1], [2], [8]. By [10] there exist 2-normed algebras (with or without unity) which are not normable, so the study of this kinds of algebras may be considered as generalization of normed algebras.

Considering the concept of 2-normed algebras; Noor Mohammad and Siddiqi, Lal, have shown that the class of 2-normed algebras with unity as defined in [8] is either void or contains only trivial algebras [8].

Also sum aspects of 2-normed and 2-Banach algebras are studied in [15], [14], [7]. A new definition of 2-normed algebras and an example satisfying this definition is given in [10], which in a subsequent work they were trying to show that there exists 2-normable algebra (with or without unity) which are normable.

Gahler in his first paper [5] mentioned the real motivation for studying 2-norm structure, and also he asked that if there is a physical situation or an abstract concept where norm topology does not work but 2-norm topology does work?

The theory of 2-normed spaces and their structure and difference of this structure with the normed spaces one is considered in [2].

An embedding of a generalized 2-normed space into the space of all bounded linear mappings on the set of all bounded 2-linear mappings are investigated in [9].

In this article we have studied the \( n^{th} \) roots and quasi \( n^{th} \) roots of elements of 2-normed algebras. The similar theorems which are proved for Banach algebras and for LMC algebras, are generalized for fundamental topological algebras, by the author [3].

## 2 Definitions and preliminary remarks

In this section, we give the basic definitions and properties of 2-normed spaces and algebras.
Definition 2.1. [10] Let $E$ be a linear space of dimension greater than one over the field $\mathbb{K}$ where $\mathbb{K}$ is the field of real or complex numbers. The real valued function $\|.,.\|$ on $E \times E$ is said to be a 2-norm if it satisfies the following axioms:

(i) $\|x, y\| = 0$, if and only if $x$ and $y$ are linearly dependent in $E$;
(ii) $\|x, y\| = \|y, x\|$ for all $x, y \in E$;
(iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$ for all $\alpha \in \mathbb{K}$ and for all $x, y \in E$;
(iv) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $x, y, z \in E$.

The pair $(E, \|.,.\|)$ is said to be a 2-normed linear space over the field $\mathbb{K}$.

Definition 2.2. [10] Let $E$ be a real algebra of dim$\geq 2$ with the 2-norm $\|.,.\|$. $E$ is said to be a 2-normed algebra if there is some $k > 0$ such that $\|xy, z\| \leq k\|x, z\||y, z\|$ for all $x, y, z \in E$.

There are several examples in the literature of this subject such as [10] and [11]. Also there are many definitions for 2-normed algebras which some of them are not useful [10]. The following interesting definition is given in [10].

Definition 2.3. Let $E$ be a subalgebra of dimension $\geq 2$ of an algebra $B$, and $\|.,.\|$ be a 2-norm in $B$ and $a_1, a_2 \in B$ be linearly independent, non-invertible and be such that

$$\forall x, y \in E, \|xy, a_i\| \leq \|x, a_i\||y, a_i\|, i = 1, 2.$$ 

Then $E$ is called a 2-normed algebra with respect to $a_1, a_2$.

Definition 2.4. Let $E$ be an algebra and $x, y \in E$. Then the quasi product of $x, y$ is defined as $x \circ y = x + y - xy$, and we denote $x \circ x \circ \cdots \circ x = x^{on}$.

Definition 2.5. Let $x$ be an element of a 2-normed algebra $E$. The element $y \in E$ is said to be quasi-inverse of $x$ if

$$y \circ x = 0, x \circ y = 0$$

An element that has a quasi-inverses said to be quasi-invertible (or quasi-regular), all other elements are said to be quasi-singular.
Remark 2.6. The quasi-inverse of a quasi-invertible element $x$ in a topological algebra is denoted by $x^0$, and the set of all quasi-invertible elements of $E$ by $q$-$\text{Inv}(E)$, and the set of all quasi-singular elements of $E$ by $q$-$\text{Sing}(E)$.

Definition 2.7. Let $x$ be an element of a 2-normed algebra $E$. The element $y \in E$ is said to be quasi square root of $x$ if

$$y \circ y = x.$$ 

By above definition and definition of quasi product it is easily seen that $y$ is a quasi square root of $x$ if and only if $x = 2y - y^2$.

Definition 2.8. We say that a 2-Banach algebra $E$ has an identity element $e$ if for every $a \in E$, $e.a = a.e = a$ and $\|a, e\| \neq 0$.

3 New Results

Let $E$ be a 2-Banach algebra with unit element $e$, in this section we give a condition that any $x \in E$ have a square root and in general $n^{th}$ root. Also conditions for existence of quasi square roots and quasi $n^{th}$ roots will be given.

Theorem 3.1. Let $E$ be a 2-Banach algebra. If $\|e - x, z\| < 1$ for all $z \in E$, then $x$ has an square root in $E$.

Proof. Let $\|e - x, z\| < \eta < 1$ now suppose that

$$y_m = \sum_{k=0}^{m} \left( \frac{1}{2} \right)^k (x - e)^k e^{m-k}.$$ 

We have

$$\|(x - e)^k, z\| \leq \|(x - e)^{k-1}, z\| \|e - x, z\|$$

$$\leq \ldots \leq \|(x - e), z\|^k < \eta^n,$$

which tends to zero when $k$ tends infinity. Also since

$$\sum_{k=0}^{m} \left( \frac{1}{2} \right)^k \eta^n < \sum_{k=0}^{m} \left( \frac{1}{2} \right) = \sqrt{2} - 1,$$
implies that $\sum_{k=0}^{m} \left( \frac{1}{k} \right) (x-e)^k e^{m-k}$ is a convergent series. As we know

$$\lim_{m \to \infty} \sum_{k=0}^{m} \left( \frac{1}{2} \right) (x-e)^k e^{m-k} = x^{1/2}.$$ 

then $x$ has an square root in $E$. □

By replacing $\frac{1}{n}$ instead of $\frac{1}{2}$ in the previous theorem we conclude that $x$ has $n^{th}$ root.

**Corollary 3.2.** Let $A$ be a 2-Banach algebra. If $\|e - x, z\| < 1$ for all $z \in E$ then $x$ has an $n^{th}$ root in $E$.

**Proof.** In this case we will have

$$\lim_{m \to \infty} \sum_{k=0}^{m} \left( \frac{1}{n} \right) (x-e)^k e^{m-k} = x^{1/n}.$$ 

since $E$ is Banach, it is clear that $x^{1/n} \in E$. □

The important results on inverses regarding to the above theorem proved an interesting data for algebras without a unit element. It is worth noting that this notion have been extended to such algebras in ordinary case in two ways:

1. by the adjunction of a unit element,
2. by using the concept of quasi-inverse.

In next theorem we would like to consider the subject in 2-normed algebras by using the concept of quasi-inverse.

**Theorem 3.3.** Let $A$ be a 2-Banach algebra. If $\|x, z\| < 1$ for all $z \in E$, then there is a $y \in E$ such that $2y - y^2 = x$.

**Proof.** Let

$$y_m = \sum_{k=0}^{m} \left( \frac{1}{k} \right) (x)^k,$$

similar to above theorem, the series $\sum_{k=0}^{m} \left( \frac{1}{k} \right) (-1)^k \|x, z\|^k$ is absolutely convergent for every $z \in E$. Now since the sum $u(t)$ of the series

$$u(t) = -\sum_{k=1}^{\infty} \left( \frac{1}{k} \right) (-t)^k,$$
satisfies the equation $2u(t) - [u(t)]^2 = t$ in which by replacing $t$ by $x$ we get the conclusion. That is with above condition $x$ has quasi square root.

\[\square\]

### 3.1 Quasi invertibility.

In this section we consider quasi-invertibility of elements of 2-Banach algebras.

**Theorem 3.4.** Let $E$ be a 2-Banach algebra. If $\|x, z\| < 1$ for all $z \in E$, then $x$ is quasi invertible and

$$x^0 = -\sum_{n=1}^{\infty} y^n.$$

**Proof.** Let $t$ be positive real number and $\|x, z\| < t < 1$. Then we have $\|(x, z)^n\| < t^n < 1$. So the series $\sum_{n=1}^{\infty} \|(x, z)^n\|$ is convergent.

Since $E$ is Banach then it converges to an element of $E$ such as $y$. Let $s_n = 1 + a + \cdots + a_{n-1}$. Tehen $s_n \to s$ and $\|x, z\| \Rightarrow 0$ as $n \to \infty$, and we have

$$(1 - y)s_n = s_n(1 - y) = 1 - y_n.$$

Therefore, by continuity of multiplication, we have $(1 - y)s = s(1 - y) = 1$. $\square$

**Theorem 3.5.** If $E$ has a unit element $e$, then an element $x$ of $E$ has the quasi-inverse $y$ if and only if $e - x$ has the inverse $e - y$.

**Proof.** It is obviously seen that we have $(e - x)(e - y) = e - (x \circ y)$. Then $(x \circ y) = 0$ if and only if $(e - x)(e - y) = e$. $\square$

Similar to above theorem we can state it for $n^{th}$ and quasi $n^{th}$ roots.

**Theorem 3.6.** Let $E$ be a 2-Banach algebra with unit element $e$. Then $z$ is quasi $n^{th}$ root of $y$ if and only if $e - z$ is $n^{th}$ root of $e - y$.

**Proof.** For $n = 2$ we have $z^2 = z \circ z = 2z - z^2 = y$ if and only if $(e - z)^2 = e - y$. 

Now we have to prove the theorem by induction. Let $z^{on} = e-(e-z)^n$ we have to prove that $z^{o(n+1)} = e - (e - z)^{n+1}$, to do it, we have
\[
z^{o(n+1)} = z \circ z^{on} = z \circ (e - (e - z)^n)
\]
\[= z + (e - (e - z)^n) - z(e - (e - z)^n) = e - (e - z)^n(e - z)
\]
\[= e - (e - z)^{n+1}.
\]
That is $y = z^{on} = e - (e - z)^n$ if and only if $(e - z)^n = e - y$. □

4 Conclusion

In this article we considered the notion of square roots and quasi square roots also $n^{th}$ roots and quasi $n^{th}$ roots of elements of a 2–normed algebra. We proved that every $x$ with condition $\|e - x, z\| < 1$ for all $x \in E$ has a square root and an $n^{th}$ root in $E$. Also every element $x$ of $E$ has the quasi-inverse $y$ if and only if $e - x$ has the inverse $e - y$.

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References


**Ali Zohri**

Department of Mathematics,  
Assistant Professor of Mathematics  
Payame Noor University  
Tehran, Iran E-mail: zohri_a@pmu.ac.ir