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φ-2-Absorbing Submodule

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Abstract. Let R be a commutative ring with identity and M be a unitary R-module. A proper submodule N of M is $2-$ absorbing if $r_1, r_2, r_3 \in R$, $m \in M$ with $r_1r_2r_3m \in M$ implies $r_1r_2m \in N$ or $r_1r_3m \in N$ or $r_2r_3m \in N$. Let $\varphi : S(M) \longrightarrow S(M) \cup \{\o\}$ be a function where $S(M)$ is the set of all submodules of M. We call a proper submodule N of M a φ-2-absorbing submodule if $r_1, r_2, r_3 \in R$, $m \in M$ with $r_1r_2r_3m \in N - \varphi(N)$ implies that $r_1r_2m \in N$ or $r_1r_3m \in N$ or $r_2r_3m \in N$. We want to extend 2-absorbing ideals to φ-2-absorbing submodules and we show that φ-2-absorbing submodules enjoy analogs of many of the properties of 2-absorbing ideals.

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1. Introduction

Throughout this paper R will be a commutative ring and M a unitary $R-$module. A 2-absorbing submodule N of M is a proper submodule with the property that for $r_1, r_2, r_3 \in R$, $m \in M$, $r_1r_2r_3m \in N$ implies $r_1r_2m \in N$ or $r_1r_3m \in N$ or $r_2r_3m \in N$. We can restrict this definition to where $r_1r_2r_3m$ lies. A proper submodule N of M is said to be weakly 2-absorbing submodule if for $r_1, r_2, r_3 \in R$, $m \in M$ with $r_1r_2r_3m \in N - \{0\}$, either $r_1r_2m \in N$ or $r_1r_3m \in N$ or $r_2r_3m \in N$.

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Definition 1.1. ([3]) *A* 2*-absorbing ideal* I *of* R *is a proper ideal with the property that for* $a, b, c \in R$*,* $abc \in I$ *implies that* $ab \in I$ *or* $bc \in I$ *or* $ac \in I$*.*

Definition 1.2. *A proper ideal* I *of* R *is weakly* 2*-absorbing, if* $a, b, c \in R$ *with* $abc \in I - \{0\}$*, implies that* $ab \in I$ *or* $ac \in I$ *or* $bc \in I$*.*

Remark 1.3. *It is clear that if* N *is a* 2*-absorbing submodule of* R*-module* M*, then* $(N : M)$ *is a* 2*-absorbing ideal of* R*. Suppose that* N *is a weakly* 2*-absorbing submodule which is not* 2*-absorbing. Note that the ideal* $(N : M)$ *is not a weakly* 2*-absorbing ideal of* R *generally. For example let* M *denote the cyclic* Z*-module* $\frac{Z}{27Z}$ *and* $N = \{0\}$*. So,* N *is a weakly* 2*-absorbing submodule of* M*, but* $(N : M) = 27Z$ *is not a weakly* 2*-absorbing ideal of* R*.*

Definition 1.4. *A proper submodule* N *of a multiplication* R*-module* M *is said almost* 2*-absorbing submodule if for* $r_1, r_2, r_3 \in R$*,* $m \in M$ *with* $r_1r_2r_3m \in N - N^2$*, either* $r_1r_2m \in N$ *or* $r_1r_3m \in N$ *or* $r_2r_3m \in N$*.*
with weakly 2-absorbing submodules and almost 2*-absorbing submodules in mind, we make the following definition.*
let R *be a commutative ring,* M *a unitary* R*-module and and* $\varphi : S(M) \longrightarrow S(M) \cup \{\o\}$ *be a function , where* $S(M)$ *is the set of all submodules of* M*. A proper submodule* N *of* M *is said to be* φ*-2-absorbing if for* $r_1, r_2, r_3 \in R$*,* $m \in M$ *with* $r_1r_2r_3m \in N - \varphi(N)$ *implies either* $r_1r_3m \in N$ *or* $r_1r_2m \in N$ *or* $r_2r_3m \in N$*. We next give a rather general condition for a* ϕ*-2-absorbing submodule to be 2-absorbing.[theorem 2.12.]. Also we show that, if* S *is a multiplicatively closed subset of* R *and* N *is a* ϕ*-2-absorbing submodule, then* N_S *is* ϕ_S*-2-absorbing submodule.*

2. Main Results

In this section we extend the concept of 2-absorbing submodule and we shall show the extend 2-absorbing submodule enjoy analog many of the properties 2-absorbing submodules.

Definition 2.1. *An* R*-module* M *is called a multiplication module pro-*

vided that for every submodule N of M there exists an ideal I of R such that $N = IM$.

Definition 2.2. *([1]) Let M be a multiplication R-module and $N_1 = I_1M$, $N_2 = I_2M$. Then we define the product of N_1 and N_2 by $N_1N_2 = I_1I_2M$.*

Lemma 2.3. *The product of two submodules in a multiplication module is well-defined and it is a submodule of M.*

Proof: See [5], Lemma 1.3. □

Note 2.4. If M is a multiplication R-module and $N = IM$, then $N = IM = (N : M)M$.

Lemma 2.5. *Let M be a faithful multiplication R-module and $\{I_\lambda\}_{\lambda\in\Lambda}$ be a non-empty collection of ideals of R for $\lambda \in \Lambda$. If $I_\lambda \supset$ ann(M) for all $\lambda \in \Lambda$, then $\cap(I_\lambda M) = (\cap I_\lambda)M$.*

Proof: See [4], Corollary 1.7,P.759. □

Definition 2.6. *Let R be a commutative ring, M a unitary R-module, and $\varphi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ be a function , where $S(M)$ is the set of all submodules of M. A proper submodule N of M is said to be φ-2-absorbing if for $r_1, r_2, r_3 \in R$, $m \in M$ with $r_1r_2r_3m \in N - \varphi(N)$ implies either $r_1r_3m \in N$ or $r_1r_2m \in N$ or $r_2r_3m \in N$.*

Note 2.7. Since $N - \varphi(N) = N - (N \cap \varphi(N))$, without loss of generality in one can assume that $\varphi(N) \subseteq N$.

Definition 2.8. *Let M be an R-module and $S(M)$ is the set of all submodules of M. For two functions $\psi_1, \psi_2 : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ We define $\psi_1 \leqslant \psi_2$, if $\psi_1(N) \subseteq \psi_2(N)$ for all $N \in S(M)$.*

Definition 2.9. *Let R be a commutative ring and M be a multiplication R-module. Define the following functions $\varphi_\alpha : S(M) \longrightarrow S(M) \cup \{\emptyset\}$*

and the corresponding φ_α-2-absorbing submodules:

$\varphi_\emptyset$	$\varphi(N)=\emptyset$	*2-absorbing submodule*
φ_0	$\varphi(N)=0$	*weakly 2-absorbing submodule*
φ_2	$\varphi(N)=N^2$	*almost 2-absorbing submodule*
φ_n	$\varphi(N)=N^n$	*n-almost 2-absorbing submodule*
φ_ω	$\varphi(N)=\cap_{n=1}^{\infty}N^n$	*ω- 2-absorbing submodule*
φ_1	$\varphi(N)=N$	*any submodule*

Observe that $\varphi_\emptyset \leqslant \varphi_0 \leqslant \varphi_w \leqslant \cdots \leqslant \varphi_{n+1} \leqslant \varphi_n \leqslant \cdots \leqslant \varphi_2 \leqslant \varphi_1$.

Lemma 2.10. *Every 2-absorbing submodule is a φ-2-absorbing submodule.*

Proof: It is clear by definition 2.9. □

Theorem 2.11. *Let M be a multiplication$R$-module and N be a proper submodule of M.*
(1) Suppose $\psi_1,\psi_2 : S(M) \longrightarrow S(M)\cup\{\emptyset\}$ are functions with $\psi_1 \leqslant \psi_2$. Then N is ψ_2-2-absorbing if N is ψ_1-2-absorbing.
(2) N, 2-absorbing $\Longrightarrow$ N weakly 2-absorbing $\Longrightarrow$ N, ω-2-absorbing $\Longrightarrow$ N, (n+1)-2-
absorbing $\Longrightarrow$ N, n-almost 2-absorbing ($n \geqslant 2$).
(3) N is ω-2-absorbing if and only if N is n-almost 2-absorbing for all $n \geqslant 2$.

Proof: Assume that N is ψ_1-2-absorbing. Let $r_1r_2r_3m \in N-\psi_2(N)$ for $r_1,r_2,r_3 \in R$, and $m \in M$. Then, $r_1r_2r_3m \in N-\psi_1(N)$. Since N is ψ_1- 2-absorbing, $r_1r_2m \in N$ or $r_1r_3m \in N$ or $r_2r_3m \in N$. Hence N is ψ_2-2-absorbing.
(2) and (3) are clear by definition. □

Theorem 2.12. *Let R be a commutative ring, M be a multiplication R-module, $\varphi : S(M) \longrightarrow S(M)\cup\{\emptyset\}$ be a function, and N be a φ-2-absorbing submodule of M such that $(N:M)^3N \nsubseteq \varphi(N)$. Then N is a 2-absorbing submodule.*

Proof: Let $a_1,a_2,a_3 \in R$, $x \in M$ and $a_1a_2a_3x \in N$. If $a_1a_2a_3x \notin \varphi(N)$, then we have $a_1a_2x \in N$ or $a_2a_3x \in N$ or $a_1a_3x \in N$ since

N is φ-2-absorbing. So assume that $a_1a_2a_3x \in \varphi(N)$. First, suppose that $a_1a_2a_3N \nsubseteq \varphi(N)$; say $a_1a_2a_3n \nsubseteq \varphi(N)$ where $n \in N$. Then $a_1a_2a_3(x+n) \in N - \varphi(N)$. So $a_1a_2(x+n) \in N$ or $a_2a_3(x+n) \in N$ or $a_1a_3(x+n) \in N$. Hence $a_1a_2x \in N$ or $a_2a_3x \in N$ or $a_1a_3x \in N$. So we can assume that $a_1a_2a_3N \subseteq \varphi(N)$. We can also assume that $(N:M)^3x \subseteq \varphi(N)$. If this is not true, then there exist $u, v, w \in (N:M)$ such that $uvwx \notin \varphi(N)$. So $(a_1+u)(a_2+v)(a_3+w)x \in N - \varphi(N)$. Since N is a φ-2-absorbing submodule we have $a_1a_2x \in N$ or $a_1a_3x \in N$ or $a_2a_3x \in N$. Now there exist $r_1, r_2, r_3 \in (N:M)$ and $n \in N$ such that $r_1r_2r_3n \notin \varphi(N)$, since $(N:M)^3N \nsubseteq \varphi(N)$. So, $(a_1+r_1)(a_2+r_2)(a_3+r_3)(x+n) \in N - \varphi(N)$. Hence $(a_1+r_1)(a_2+r_2)(x+n) \in N$ or $(a_2+r_2)(a_3+r_3)(x+n) \in N$ or $(a_1+r_1)(a_3+r_3)(x+n) \in N$. Therefore $a_1a_2x \in N$ or $a_2a_3x \in N$ or $a_1a_3x \in N$. $\square$

Theorem 2.13. *Let R be a commutative ring, M be a multiplication R-module, $\varphi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ be a function, and N be a proper submodule of M. If $N = IM$ for some ideal I of R, is a φ-2-absorbing submodule which is not 2-absorbing, then $N^4 \subseteq \varphi(N)$.*

Proof: Since M is a multiplication module, $N = (N:M)M$. Therefore $N^4 = (N:M)^4M = (N:M)^3N \subseteq \varphi(N)$. $\square$

Corollary 2.14. *Let M be an R-module and N be a φ-2-absorbing submodule of M, where $\varphi \leqslant \varphi_4$. Then N is ω- 2-absorbing.*

Theorem 2.15. *Let $R = R_1 \times R_2$ and each R_i is a commutative ring with identity. Let M_i be an R_i-module and $M = M_1 \times M_2$ with $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$, be an R-module, where $r_i \in R_i$, $m_i \in M_i$. Then we have:*
(1) If N_1 is a 2-absorbing submodule of M_1, then $N_1 \times M_2$ is a 2-absorbing submodule of M.
(2) If N_2 is a 2-absorbing submodule of M_2, then $M_1 \times N_2$ is a 2-absorbing submodule of M.

Proof: (1) Suppose that N_1 is a 2-absorbing submodule of M_1 and let $(acem_1, bdfm_2) = (a,b)(c,d)(e,f)(m_1,m_2) \in N_1 \times M_2$, where $(a,b), (c,d)$, $(e,f) \in R_1 \times R_2$, $(m_1, m_2) \in M_1 \times M_2$. Then $acem_1 \in N_1$. Since

N_1 is 2-absorbing, $acm_1 \in N_1$ or $aem_1 \in N_1$ or $cem_1 \in N_1$. Hence $(a,b)(c,d)(m_1,m_2) \in N_1 \times M_2$ or $(a,b)(e,f)(m_1,m_2) \in N_1 \times M_2$ or $(c,d)(e,f)(m_1,m_2) \in N_1 \times M_2$. Hence $N_1 \times M_2$ is 2-absorbing.
(2) It is proved similarly. □

Theorem 2.16. *Let $R = R_1 \times R_2$ where each R_i is a commutative ring with identity. Let M_i be an R_i-module and $M = M_1 \times M_2$ with $(r_1,r_2)(m_1,m_2) = (r_1m_1, r_2m_2)$, be an R-module, where $r_i \in R_i$, $m_i \in M_i$, $\psi_i : S(M_i) \longrightarrow S(M_i) \cup \{\emptyset\}$ be functions, and $\varphi = \psi_1 \times \psi_2$. Then we have:*
(1) $N_1 \times N_2$ is a φ-2-absorbing submodule, where N_i is a proper submodule of M_i with $\psi_i(N_i) = N_i$;
(2) $N_1 \times M_2$ is a φ-2-absorbing submodule of M, where N_1 is a ψ_1-2-absorbing of M_1 which must be 2-absorbing if $\psi_2(M_2) \neq M_2$.
(3) $M_1 \times N_2$ is a φ-2-absorbing submodule of M, where N_2 is a ψ_2-2 absorbing of M_2 which must be 2-absorbing if $\psi_1(M_1) \neq M_1$.

Proof: (1) It is Clear, since $N_1 \times N_2 = \varphi(N_1 \times N_2)$.
(2) If N_1 is 2-absorbing, then $N_1 \times M_2$ is a 2-absorbing and hence φ-2-absorbing. So, suppose that N_1 is ψ_1-2-absorbing and $\psi_2(M_2) = M_2$. Also Suppose that $(a,b)(c,d)(e,f)$
$(m_1,m_2) \in N_1 \times M_2 - \varphi(N_1 \times M_2)$, where $(a,b),(c,d),(e,f) \in R$,$(m_1,m_2) \in M$. Then $(a,b)(c,d)(e,f)(m_1,m_2) = (acem_1, bdfm_2) \in N_1 \times M_2 - \psi_1(N_1) \times \psi_2(M_2) = (N_1 - \psi_1(N_1)) \times M_2$. So $acem_1 \in N_1 - \psi_1(N_1)$. Since N_1 is ψ_1-2- absorbing, $acm_1 \in N_1$ or $aem_1 \in N_1$ or $cem_1 \in N_1$. Hence $(a,b)(c,d)(m_1,m_2) \in N_1 \times M_2$ or $(a,b)(e,f)(m_1,m_2) \in N_1 \times M_2$ or $(c,d)(e,f)(m_1,m_2) \in N_1 \times M_2$. Thus $N_1 \times M_2$ is φ-2-absorbing.
(3) The proof is similar to (2). □

Theorem 2.17. *Let R be a commutative ring, M be a unitary R-module, $\varphi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ be a function and N be a proper submodule of M. Then the following are equivalent:*
(1) N is φ-2-absorbing;
(2) For $m \in M$, $r_1, r_2 \in R$, which $r_1r_2m \in M - N$, we have $(N : r_1r_2m) = (N : r_1m) \cup (N : r_2m) \cup (\varphi(N) : r_1r_2m)$;
(3) For $r_1, r_2 \in R$,$m \in M$ which $r_1r_2m \in M - N$, we have $(N : r_1r_2m) =$

$(N : r_1m)$ or $(N : r_1r_2m) = (N : r_2m)$ or $(N : r_1r_2m) = (\varphi(N) : r_1r_2m)$;
(4) For ideals I,J,K of R and submodule D of M, if $IJKD \subseteq N$, $IJKD \nsubseteq \varphi(N)$, then $IJD \subseteq N$ or $IKD \subseteq N$ or $JKD \subseteq N$.

Proof: (1) $\Rightarrow$ (2) Let $x \in (N : r_1r_2m)$, then $xr_1r_2m \in N$. If $xr_1r_2m \notin \varphi(N)$, since N is φ-2-absorbing submodule and $r_1r_2m \notin N$, then $xr_1m \in N$ or $xr_2m \in N$. So $x \in (N : r_1m)$ or $x \in (N : r_2m)$. If $xr_1r_2m \in \varphi(N)$, then $x \in (\varphi(N) : r_1r_2m)$. the other inclusion always holds Since $\varphi(N) \subseteq N$.
(2) $\Rightarrow$ (3) If an ideal is a union of two ideals, it is equal to one of them.
(3) $\Rightarrow$ (4) Let I,J,K be ideals of R, and D be a submodule of R-module M with $IJKD \subseteq N$. Suppose that $IJD \nsubseteq N$, $IKD \nsubseteq N$ and $JKD \nsubseteq N$. We show that $IJKD \subseteq \varphi(N)$. For this let $r_1r_2m \in D$ but $r_1r_2m \notin N$, where $r_1r_2 \in IJ$. Since $IJKD \subseteq N$, we have $r_1r_2Km \subseteq N$. Thus $K \subseteq (N : r_1r_2m) = (\varphi(N) : r_1r_2m)$ and so $KIJD \subseteq \varphi(N)$. Let $r_1r_2m \in N$ and choose $r_1^{'}r_2^{'}m^{'} \in D\setminus N$. Then $(r_1r_2m+r_1^{'}r_2^{'}m^{'}) \in D\setminus N$. So by the first case, $Kr_1^{'}r_2^{'}m^{'} \subseteq \varphi(N)$ and $K(r_1r_2m + r_1^{'}r_2^{'}m^{'}) \subseteq \varphi(N)$. Now for $k \in K$ we have $r_1r_2mk = (r_1r_2m+r_1^{'}r_2^{'}m^{'})k - r_1^{'}r_2^{'}m^{'}k \in \varphi(N)$. Then $IJKD \subseteq \varphi(N)$.
(4) $\Rightarrow$ (1) Let $r_1r_2r_3Rm \in N - \varphi(N)$. Then $< r_1 >< r_2 >< r_3 > Rm \subseteq N$, $< r_1 >< r_2 >< r_3 > Rm \nsubseteq \varphi(N)$. Then $< r_1 >< r_2 > Rm \subseteq N$ or $< r_1 >< r_3 > Rm \subseteq N$ or $< r_2 >< r_3 > Rm \subseteq N$. Then $r_1r_2m \in N$ or $r_1r_3m \in N$ or $r_2r_3m \in N$. □

Remark 2.18. *Let $\varphi : S(M) \longrightarrow S(M) \cup \{\o\}$ be a function, and S be a multiplicatively closed subset of R. We define $\varphi_S : S(M_S) \longrightarrow S(M_S) \cup \{\o\}$ by $\varphi_S(N) = (\varphi(N \cap M))_S$. and $\varphi_S(N) = \emptyset$, if $\varphi(N \cap M) = \emptyset$. Note that $\varphi_S(N) \subseteq N$. and $(\varphi_\alpha)_S = \varphi_\alpha$ for $\alpha \in \{\o\} \cup \{0\} \cup \mathbb{N}$. We show that if $(\varphi(N))_S \subseteq \varphi_S(N_S)$, and N is a φ-2-absorbing submodule, then N_S is φ_S-2-absorbing submodule. We define $\varphi_N : S(M) \longrightarrow S(M) \cup \{\o\}$ by $\varphi_N(\frac{K}{N}) = \frac{\varphi(K)+N}{N}$ for $K \supseteq N$. Note that $\varphi_N(\frac{K}{N}) = \o$ if $\varphi(K) = \o$, and $\varphi_N(\frac{K}{N}) \subseteq \frac{K}{N}$.*

Theorem 2.19. *Let R be a commutative ring and M be a multiplication R-module, and $\varphi : S(M) \longrightarrow S(M) \cup \{\o\}$ be a function. Let P be a φ-*

2-absorbing submodule of M.
(1) If N is a submodule of R-module M with $N \subseteq P$, then $\frac{P}{N}$ is a φ_N-2-absorbing submodule of $\frac{M}{N}$.
(2) Suppose that S is a multiplicatively closed subset of R with $(\varphi(P))_S \subseteq \varphi_S(P_S)$. Then P_S is a φ_S-2-absorbing submodule of M_S.

Proof: (1) Let $r_1, r_2, r_3 \in R$, $m \in M$. Suppose that $\overline{m} = m + N$,$r_1r_2r_3m \in \frac{P}{N} - \varphi_N(\frac{P}{N}) = \frac{P}{N} - \frac{\varphi(P)+N}{N}$. Hence $r_1r_2r_3m \in P - (\varphi(P) + N)$. Thus $r_1r_2r_3m \in P - \varphi(P)$. So $r_1r_2m \in P$ or $r_1r_3m \in P$ or $r_2r_3m \in P$. Therefore $(r_1r_2m + N) \in \frac{P}{N}$ or $(r_1r_3m + N) \in \frac{P}{N}$ or $(r_2r_3m + N) \in \frac{P}{N}$. Hence $r_1r_2\overline{m} \in P$ or $r_1r_3\overline{m} \in P$ or $r_2r_3\overline{m} \in P$.
(2) Let $r_1, r_2, r_3 \in S^{-1}R, m \in M$. Suppose that $r_1r_2r_3m \in P_S - \phi_S(P_S)$. Then there exist $s_1, s_2, s_3 \in S$, $x, y, z \in R$, such that $r_1 = \frac{x}{s_1}$, $r_2 = \frac{y}{s_2}$, $r_3 = \frac{z}{s_3}$, $(\frac{x}{s_1})(\frac{y}{s_2})(\frac{z}{s_3})m \in P_S$. Then $xyzm \in P$, $xyzm \notin \varphi(P)$. Since P is a φ-2-absorbing submodule, $xym \in P$ or $xzm \in P$ or $yzm \in P$. Hence $r_1r_2m \in P_S$ or $r_1r_3m \in P_S$ or $r_2r_3m \in P_S$. So P_S is φ_S-2-absorbing submodule. $\square$

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