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Study of Subhomomorphic Property to a Ring

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Abstract. Let M and N be two non-zero right R-modules, M is called *subhomomorphic* to N in case there exist $R-$homomorphisms $f : M \to N$, $g : N \to M$ such that gof is non-zero, and M is called *strongly subhomomorphic* to N in case there exist homomorphisms $f : M \to N$, $g : N \to M$ such that both fog and gof are non-zero. After establishing some basic properties of (strongly) subhomomorphic to a ring, it is shown that for a nonsingular ring R the class of injective right R-modules are subhomomorphic to R if and only if R is a semisimple ring.

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1. Introduction

Throughout rings will have unit elements and modules will be right unitary. If M is a module over a ring R, its quasi-injective (injective) hull will be denoted by $\hat{M}_R$ ($\mathrm{E}(M_R)$). For R-modules N and M the submodule $\mathrm{Tr}_R(M,N) := \sum\{\mathrm{Im}h \mid h \in \mathrm{Hom}_R(M,N)\}$ is called the *trace* of M in N, and the submodule $\mathrm{Rej}_R(N,M) := \cap\{\mathrm{Ker}\ f \mid f \in \mathrm{Hom}_R(N,M)\}$ is called the *reject* of M in N. Unexplained terminology and standard results may be found in [1] or [10].

The notions of prime and semiprime for modules have been studies by several authors who have used different definitions [2]-[4], [6] and [9]-[11]. Bican, Jambor, Kepka and Nemec called an R-module M prime

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if $K * L :=\text{Hom}_R(M, L)K \neq 0$ for any non-zero submodules $K, L \leqslant M$ ([2]). This definition of prime called $*$-prime by Lomp ([5]). The notion of primeness had already been extended by Jirasko to semiprimeness for modules ([4]). A semiprime module M (in the sense of Jirasko) is defined by the property that the condition $N * N = 0$ implies $N = 0$, whenever N is a submodule in M. As noted in [5] the notion of semiprime module coincides with that of weakly compressible, a result attributed to Zelmanowitz ([12]). Recall that M_R is called weakly compressible if $\text{Hom}_R(M, N)$ contains an element f with $f \mid_N \neq 0$ whenever N is a non-zero submodule of M. We have the following implications

$$* - prime \Rightarrow weakly\ compressible \Rightarrow retractable,$$

where an R-module M is said to be retractable if $\text{Hom}_R(M, N) \neq 0$ for all non-zero submodules N of M. The reverse implications have been investigated by Lomp who proved that a retractable module with prime endomorphism ring is necessarily $*$-prime, and a retractable module with semiprime endomorphism ring is weakly compressible. Furthermore, for a semi-projective module, it is true that being $*$-prime is the same as being retractable with prime endomorphism ring ([5, Propositions 4.2 and 5.2]).

Wisbauer and Wijayanti called a module M_R *fully prime* if for any non-zero fully invariant submodule K of M, M is $K-$cogenerated. They proved in [9] that M is fully prime if and only if $K * L \neq 0$ for any non-zero fully invariant submodules $K, L \leqslant M$.

The notion of subhomomorphic modules comes from the aforementioned studies. We carry out a thorough investigation of this useful notation. For example by considering the class of simple R-modules which are subhomomorphic to R, some new characterizations of semisimple rings are obtained.

2. Subhomomorphic for Ring

Definition 2.1. *Let M and N be two non-zero R-modules. M is called* subhomomorphic *to N in case there exist $R-$homomorphisms $f : M \to N$, $g : N \to M$ such that gof is non-zero. M is called* strongly subhomomorphic *to* N *in case there exist homomorphisms* $f : M \to N$, $g : N \to M$ *such that* fog *and* gof *are non-zero.*

Proposition 2.2. *Let* M *be an* $*$*-prime* R*-module. Then for every non-zero submodules* K, L *of* M*,* K *is subhomomorphic to* L*.*

Proof. Let K, L be non-zero submodules of M. By $*$-primeness we have $\mathrm{Tr}_R(L, K) \neq 0$ so that $\mathrm{Hom}_R(K, L)\mathrm{Tr}_R(L, K) \neq 0$ and hence there exist $R-$homomorphisms $f : K \to L$, $g : L \to K$ such that gof is non-zero. □

Lemma 2.3. *The following statements for an* R*-module* M *are equivalent.*

(a) M_R *is strongly subhomomorphic to* R_R.
(b) M_R *is subhomomorphic to* R_R.
(c) *There exist* $n, m \in M$ *and a homomorphism* $f : M_R \to R_R$ *such that* $mf(n) \neq 0$.

Proof. (a)⇒(b). By definition.
(b)⇒(c). Let $f : M_R \to R_R$, $g : R_R \to M_R$ such that fog is a non-zero $R-$homomorphism so $gf(m) \neq 0$ for some $m \in M$. Then $gf(m) = g(1f(m)) = g(1)f(m) \neq 0$.
(c)⇒(a). If $xf(x) \neq 0$ for some $x \in M$ then we can define $g : R_R \to M_R$ via $r \mapsto xr$ then $gof(x) \neq 0$, $fog(1) \neq 0$. Otherwise we may suppose that $m \neq n$, $f(m) \neq f(n)$, $mf(n) \neq 0$ and $mf(m) = 0$. Then we define $R-$homomorphism $h : R_R \to M_R$ such that $r \mapsto (m - n)r$ then $fh(1) = f(m - n) \neq 0$ and $hf(m) = h(1)f(m) = (m - n)f(m) = mf(m) - nf(m) = -nf(m) \neq 0$. □

Corollary 2.4. *The following statements are equivalent for an* R*-module* M*.*

(a) *The module* M *is subhomomorphic to* R*-module* $R/\mathrm{ann}(M)$*.*
(b) $Hom_R(M, R/\mathrm{ann}(M)) \neq 0$*.*

Proof. (a)⇒ (b)By definition.

(b)⇒(a). Let $f : M \to (R/\text{ann}(M))$ be a non-zero homomorphism then there exist $x \in M$ such that $f(x) \neq 0$. If $mf(x) = 0$ for all $m \in M$ we have $f(x) \in \text{ann}(M)$ and $f(x)$ must be zero, contradiction, then $mf(x) \neq 0$ for some $x, m \in M$ and by Lemma 2.3 the proof is completed. □

Examples 2.5. (a) By Lemma 2.3 if M is a semiprime R-module in the sense of Zelmanowitz, [i.e. for each $0 \neq m \in M$ there exists $f \in \text{Hom}(M, R)$ with $mf(m) \neq 0$] then M is strongly subhomomorphic to R.
(b) Let M be a faithful R-module and R be an injective R-module, then R is strongly subhomomorphic to M^I for some set I (R can be embedded in M^I and by injectivity it is a direct summand of R).
(c) Let R be a semiprime ring. Then every non-zero ideal of R is strongly subhomomorphic to R (by Lemma 2.3).
(d) Let $R = \mathbb{Z}$, $M = \mathbb{Z}_4$ and $M = \mathbb{Z}_2$. Then M is subhomomorphic to N, but it is not strongly subhomomorphic to N.
Let M be an R-module. Then R is subhomomorphic to M if and only if $M^* =\text{Hom}(M, R) \neq 0$. So that R is subhomomorphic to I for every right ideal I of R.

Lemma 2.6. *Let I, J be non-zero two sided ideals of R. If $IJI \neq 0$ then I is subhomomorphic to J.*

Proof. If $IJI \neq 0$, there exist $x \in I$, $b \in J$ such that $xbI \neq 0$ then we can define $R-$homomorphisms $f_b : I_R \to J_R$ and $g_x : J_R \to I_R$ via $f_b(i) = bi$, $g_x(j) = xj$, so that $g_x f_b(I) = g_x(bI) = xbI \neq 0$, thus I is subhomomorphic to J. □

Corollary 2.7. *If I is an ideal of R with, $I^2 \neq 0$ then I is subhomomorphic to R.*

Proof. Apply Lemma 2.6 for $J = R$. □

Lemma 2.8. *Let I be a right ideal of R. Then R/I is subhomomorphic to R if and only if there exist $x, y \in R$ such that $xI = 0$ and $yx \notin I$.*

Proof. Let R/I be subhomomorphic to R so there exist $f : R/I \to R$

and $\bar{y}, \bar{t} \in R/I$ such that $\bar{y}f(\bar{t}) = \bar{y}f(\bar{1})t = \bar{y}xt = (y+I)xt \neq 0$ so that $xI = f(\bar{1})I = f(I) = 0$ and $yx \notin I$. Conversely if we define $f : R/I \to R$ and $g : R \to R/I$ via $f(r+I) = xr$, $g(r) = yr + I$ for $r \in R$ then $gf(1+I) = g(x) = yx + I \neq 0$, and the proof is completed. □

Note: Let I be a two sided ideal of R then I is not subhomomorphic to R if and only if $I.\mathrm{Tr}(I, R) = 0$ if and only if $\mathrm{Tr}(I, R) \subseteq \mathrm{r.ann}(I)$.

Let I be a proper ideal of R. Then R/I is subhomomorphic to R if and only if there exists x, y in R such that $xI = 0$ and $yx \notin I$ if and only if $l(I) \nsubseteq I$.

Proposition 2.9. *If I is a non-zero two sided ideal of R and $I = \mathrm{l.ann}(I)$, then I is not subhomomorphic to R and R/I is not subhomomorphic to R.*

Proof. By Lemma 2.8, R/I is subhomomorphic to R if and only if $l(I) \nsubseteq I$ for proper ideal I of R so this trivial that R/I is not subhomomorphic to R.
If $f : I \to R$ and $m, n \in I$ with $mf(n) \neq 0$ then $f(I^2) = f(I)I = 0$ so $f(I) \subseteq I$ and $f(n) \in I$ thus $mf(n) \in I^2$ and $mf(n) = 0$, this is a contradiction, so that I is not subhomomorphic to R. □

Proposition 2.10. *Let I be an ideal of R. If for every right ideal K of R we have R/I subhomomorphic to R/K, then I is $T-$nilpotent.*

Proof. Let $a_1, a_2, a_3, ...$ be elements in I such that $a_1a_2...a_n \neq 0$ for all n. Then $S = \{K \leqslant R_R \mid a_1a_2...a_t \notin K$ for all t$\}$ is a non-empty set. By Zorn's lemma has a maximal element, say B. Because R/I is subhomomorphic to R/B, then there exists $r + B \neq B$ such that $(r+B)I = 0$. Thus $rI \subseteq B$ and $a_1a_2...a_n \in rR + B$ for some n. Then $a_1a_2...a_n = rt + b$ for some $t \in R$ and $b \in B$ and hence $a_1a_2...a_nI \subseteq (rt+b)I \subseteq B$ and $a_1a_2....a_na_{n+1} \in B$ and this is a contradiction, and I is $T-$ nilpotent. □

Proposition 2.11. *Let R be a commutative ring and N be an R-module. Then the following statements are equivalent.*

(1) *N is semiprime in the sense of Zelmanowitz.*
(2) *The class of submodules of N are subhomomorphic to R.*
(3) *The class of cyclic submodules of N are subhomomorphic to R.*

Proof. (1)⇒(2). For $0 \neq k \in K$ there exists $f : N \to R$ such that $kf(k) \neq 0$ then the restriction of f to K is in K^* and $kf(k) \neq 0$ (by Lemma 2.3) the proof is completed.
(2)⇒(3). Trivial.
(3)⇒(1). For $0 \neq n \in N$ there exist $f : Rn \to R$ and $g : R \to Rn$ such that $gf(n) \neq 0$ then $gf(n) = g(1)f(n) = nrf(n) = nf(n)r \neq 0$ and $nf(n) \neq 0$. Then N is semiprime in the sense of Zelmanowitz. □

Let $E(S)$ be an injective hull of simple R-module S. If $E(S)$ is subhomomorphic to R with $R-$homomorphisms $f : E(S) \to R$ and $g : R \to E(S)$ then we can deduce that either S is subhomomorphic to R or $gf \in J(End(E(S)))$.
A ring R is called *right (left) hereditary* if every right (left) ideal of R is projective.

Theorem 2.12. *Let R be a nonsingular ring. Then the following statements are equivalent.*

(1) *The class of injective R-modules is subhomomorphic to R.*
(2) *For any injective R-module E there exists a nonsingular projective R-module P such that E is subhomomorphic to P.*
(3) *R is a semisimple ring.*

Proof. (1)⇒(2) Let $P = R$.
(2)⇒(3) Let S be a simple R-module and $E(S)$ be subhomomorphic to P for some projective R-module P and $f : E(S) \to P$. If $f(S) = 0$, then $\text{Ker}f \leqslant_e E(S)$, and $E/\text{Ker}f$ is a singular module embedded in P, a contradiction. Then $f(S) \neq 0$ and hence S is nonsingular. It follows that S is a projective R-module. The proof is now completed by [10, 20.3(i)].
(3)⇒(1) This is routine. □

Proposition 2.13. *Let R be a ring with a unique simple R-module (up* *to isomorphism). Then the class of projective modules is subhomomorphic to itself.*

Proof. Let P_1, P_2 be two projective modules. Then by [1, 17.14] there exist maximal submodules $M_1 \leqslant P_1$ and $M_2 \leqslant P_2$. So by our assumption $P_1/M_1 \cong P_2/M_2$ with isomorphism $\psi : P_1/M_1 \to P_2/M_2$. Let $\pi_2 : P_2 \to P_2/M_2$ and $\pi_1 : P_1 \to P_1/M_1$ be the natural projections then by projectivity of P_1, P_2 there exist $f_1 : P_1 \to P_2$ and $f_2 : P_2 \to P_1$ such that $\pi_2 f_1 = \psi\pi_1$ and $\psi\pi_1 f_2 = \pi_2$. Now let $x \in P_2 \setminus M_2$ if $f_1 f_2(x) = f_1(f_2(x)) = 0$ then $\pi_2(x) = \psi\pi_1(f_2(x)) = 0$ this is a contradiction. □

Corollary 2.14. *Let R be a ring with Jacobson radical J such that R/J is a simple Artinian ring. Then the class of projective modules is subhomomorphic to itself.*

Proof. By Proposition 2.13. □

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