Journal of Mathematical Extension
Vol. 15, No. 3, (2021) (9)1-11
URL: https://doi.org/10.30495/JME.2021.1516
ISSN: 1735-8299
Original Research Paper

Algebraic Frames and Duality

Sh. Azadi
Kerman Branch, Islamic Azad University

M. Radjabalipour*
Shahid Bahonar University of Kerman

Abstract. The theory of algebraic frames for a Hilbert space H is a generalization of the theory of frames and generalized frames. The paper applies the theory of unbounded operators to define the dual of algebraic frames with densely defined unbounded analysis operators. It is shown that every algebraic frame has an algebraic dual frame, and if an algebraic frame has a nonzero redundancy, then it is not Riesz-type. An example of an algebraic frame with finite redundancy is constructed which is not a Riesz-type algebraic frame. Finally, for a lower bounded analytic frame, the discreteness of its indexing measure space and the uniqueness of its algebraic dual are studied and shown to be interrelated.

AMS Subject Classification: 42C99; 46C99; 47A99
Keywords and Phrases: Unbounded operators, algebraic frames, algebraic dual frame, dual frame, generalized frames.

1 Introduction and preliminaries

In this paper, we will study the algebraic frame theory established in [7, 8]. In fact, we are interested in some applications of unbounded operators to algebraic frames and their duals. Let us first recall few

Received: January 2020; Accepted: May 2020
*Corresponding Author

basic notions needed in the sequel. Throughout the paper, H denotes a Hilbert space. Let X be a linear subspace of H with algebraic dual X'.

Recall that every Hilbert space H with the underlying field $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$ is isometrically isomorphic with $H_{\mathbb{F}} := \ell^2_{\mathbb{F}}(\Lambda)$ for some index set Λ. The space $H_{\mathbb{F}}$ is called a coordinatization of H and, if $\mathbb{F} = \mathbb{R}$, $H_{\mathbb{C}}$ is called a complexification of H. Identifying H and $H_{\mathbb{F}}$, the set $\{e_\alpha : \alpha \in \Lambda\} \subset H_{\mathbb{F}}$ defined by $e_\alpha(\beta) = \delta_{\alpha\beta}$ (the Kroneker delta) will be an orthonormal basis of H. In case H is separable, the index set Λ can be identified as $\mathbb{N}$ or as a finite initial segment $\{1, 2, 3, \cdots, n\}$. If $x \in H_{\mathbb{F}}$, $||x||^2 = \sum_{\alpha \in \Lambda_x} |x(\alpha)|^2$ (independent of the order on α).

We will now summarise some basic definitions from algebraic frame theory. For further details and properties about algebraic frames, generalized frames and unbounded closed operators we refer to [2, 4, 6, 7, 9, 10, 11]. In this paper, to avoid confusion of the notations for "closure of a set" as in $\bar{Z} = Z \cup Z'$ and the "conjugate of a complex number" as in $\bar{z} = \mathfrak{Re}(z) - i\mathfrak{Im}(z)$, the latter will be denoted by z^*. Also, $\oplus$ represents the orthogonal direct sum of Hilbert spaces or operators on Hilbert spaces. The notations $\mathcal{D}(f)$ and $\mathcal{R}(f)$ will denote the domain and the range of a function f, respectively; also, the notations $\mathcal{K}(T)$ and $\mathcal{G}(T)$ will denote the null space and the graph of a linear operator T, respectively.

Definition 1.1. *A preframe on a Hilbert space H is a 4-tuple (Θ, μ, X, H) in which $\Theta = (\theta_z)_{z\in Z} \subseteq X'$ is a family of (not necessarily bounded) linear functionals indexed by a (positive) measure space (Z, M, μ), where X is a linear subspace of H and the functions $z \mapsto \theta_z h$ are μ-measurable for all $h \in X$. The linear operator $t_\Theta : X \to \mathbb{C}^Z$ defined by $(t_\Theta h)(z) = \theta_z h$ is called the analysis operator of Θ. We say that (Θ, μ, X, H) is a pseudo frame in H if $\bar{X} = H$ and its induced operator t_Θ maps X into $L^2(\mu)$; i.e., $\int_Z |\theta_z(h)|^2 d\mu(z) < \infty$, $\forall h \in X$. The measure space Z and the space X are called the index measure space and the signal space of Θ, respectively. The space X, will be also denoted by the notation $\mathcal{D}(\Theta)$.*

Definition 1.2. *A pseudo frame $\Theta = (\theta_z)_{z\in Z}$ is called an algebraic frame if its analysis operator t_Θ has an injective closure T_Θ. The family Θ is an analytic frame, if it is an algebraic frame for which every linear functional θ_z is bounded. For an algebraic frame Θ, its frame operator $S_\Theta : \mathcal{D}(S_\Theta) \subseteq H \to H$ is defined to be the operator $S_\Theta := T^*_\Theta T_\Theta$.*

Lemma 1.3. *If Θ is an algebraic frame, we can assume without loss of generality that $T_\Theta = t_\Theta$.*

Proof. The proof follows from the argument given in the Introduction of the paper by Giv-Radjabalipour [7], showing one can assume without loss of generality that $\mathcal{D}(\Theta) = \mathcal{D}(T_\Theta)$. $\square$

Notes. A generalized (or continuous) frame in H indexed by measure space (Z, M, μ) is a family of vectors $\{f_z\}_{z\in Z}$ such that for all $f \in H$, the function $z \to \langle f, f_z\rangle$ is measurable and, there exist constants $A, B > 0$ such that $A||f||^2 \leqslant \int_Z |\langle f, f_z\rangle|^2 d\mu(z) \leq B||f||^2$ [4].

We will study a class of unbounded operators a special case of which is used in the theory of algebraic frames. For reader's convenience, we conclude the section by certain facts from [3] which will be necessary for the proof of our main results in the next section.

Theorem 1.4. **([3])** *Assume $W \in B(H)$ is such that $0 \leq W \leq I$ and that W and $(I - W)$ are injective. Choose $\alpha \in [0, 1]$. Then $(I - W)^\alpha W^{-\alpha} = W^{-\alpha}(I - W)^\alpha$ and the operator $S := (I - W)^\alpha W^{-\alpha}$ is a selfadjoint operator with $\mathcal{D}(S) = \mathcal{R}(W^\alpha)$. Moreover, $\mathcal{R}(S) = \mathcal{R}(I-W)^\alpha$ and that, if $(I - W)^\alpha x = W^\alpha y$ for some $x, y \in H$, then $x = W^\alpha z$ for some $z \in H$.*

Definition 1.5. **([3])** *Define the von Neumann generator W_T of a linear transformation $T : \mathcal{D}(T) \subset H \to K$ by $W_T := PQ_TP^*$, where $Q_T : H \oplus K \to H \oplus K$ is the orthogonal projection onto (the closed linear space) $\overline{\mathcal{G}(T)}$ and $P : H \oplus K \to H$ is the (bounded linear) operator sending $x \oplus y$ to x. The adjoint $P^* : H \to H \oplus K$ of P sends $x \in H$ to $x \oplus 0 \in H \oplus K$. If there arises no ambiguity, we may drop the subscript T in W_T.*

Theorem 1.6. **([3])** *Let $T : \mathcal{D}(T) \subset H \to K$ be a densely defined closed linear transformation and let $W = W_T$ be its von Neumann generator. The following assertions are true.*

(i) $\ker(W) = \mathcal{D}(T)^\perp = \{0\}$, *and,* $0 \leq W \leq I$.

(ii) $T = V(I - W)^{1/2}W^{-1/2} : \mathcal{R}(W^{1/2}) \subset H \to K$ *, where* $V : H \to K$ *is a partial isometry such that* $V^*V = I$ *and* VV^* *is the orthogonal projection onto* VH.

(iii) $T^*T = (T^*T)^* = (I - W)W^{-1}$, $\mathcal{D}(T^*T) = R(W)$ *and* $\mathcal{R}(T^*T) = R(I - W)$. *In particular,* T^*T *is also densely defined.*

2 Main results

In this section, we will define the dual of an algebraic frame and show that special cases of the algebraic frames have algebraic duals. The section generalizes and improves duality results known for generalized frames.

Definition 2.1. *An algebraic frame* $\Gamma = (\gamma_z)_{z \in Z}$ *with the signal space* $Y \subset H$ *and analysis operator* T_Γ *is called an algebraic dual frame for the algebraic frame* $\Theta = (\theta_z)_{z \in Z}$ *with signal space* $X \subset H$ *and analysis operator* T_Θ *if, for every* $x \in X$ *and* $y \in Y$,

$$\langle x, y \rangle = \langle T_\Theta x, T_\Gamma y \rangle.$$

In this case, (Θ, Γ) *is called an algebraic dual pair in* H. *We call* Θ *a Riesz-type algebraic frame if it has a unique algebraic dual.*

Note that Riesz bases are discrete frames with unique duals. We are following Gabardo and Han [6] who first identified Riesz-type frames as the collection of all generalized frames having unique duals. Our main results are to explore conditions implying algebraic frames to become Riesz-type algebraic frames.

Our first main theorem studies the existence of dual algebraic frames.

Theorem 2.2. *Every algebraic frame has an algebraic dual frame.*

Proof. Let $\Theta = (\theta_z)_{z \in Z}$ be an algebraic frame indexed by the measure space $(Z, \mathcal{M}, \mu)$. Assume $W = W_{T_\Theta}$ is the von Neumann generator of the analysis operator T_Θ with signal space $X = \mathcal{D}(T_\Theta)$. In view of Parts (i)-(ii) of Theorem 1.6, $\ker(W) = \ker(W^{1/2}) = \{0\}$, $T_\Theta = V(I - W)^{1/2}W^{-1/2}$, and $X = \mathcal{R}(W^{1/2})$. Also, $\ker((I-W)^{1/2}) = \ker(I-W) = \ker(T_\Theta) = \{0\}$. Define

$$T_{\Theta'} := V(I - W)^{-1/2}W^{1/2} \ \text{ with } \ \mathcal{D}(T_{\Theta'}) = \mathcal{R}((I - W)^{1/2}).$$

Clearly, $T_{\Theta'}$ is well defined. We claim $T_{\Theta'}$ is the analysis operator of a dual algebraic frame of $(\theta_z)_{z \in Z}$. Since T_Θ is injective, so is $I - W$

and, hence, $(I - W)^{\alpha}$ has a dense range for all $\alpha > 0$. Thus $T_{\Theta'}$ is densely defined. To show $T_{\Theta'}$ is injective, assume $0 = T_{\Theta'}x = V(I - W)^{-1/2}W^{1/2}(I - W)^{1/2}\xi = VW^{1/2}\xi$ for some $x = (I - W)^{1/2}\xi$. Then $W^{1/2}\xi = 0$ and, hence, $\xi = 0$ or, equivalently, $x = 0$.

Next, we show that $T_{\Theta'}$ is closed. Let $x \in H$ be the limit of some sequence $x_n = (I - W)^{1/2}\xi_n \in \mathcal{D}(T_{\Theta'})$, and that y is the limit of the sequence $T_{\Theta'}x_n$. We claim $x \in \mathcal{D}(T_{\Theta'})$ and $T_{\Theta'}x = y$. Since $y = \lim_n VW^{1/2}\xi_n$, it follows that $y \in \overline{VH} = VH$ and $(I - W)^{1/2}V^*y = \lim_n W^{1/2}x_n = W^{1/2}x$. By Part (i) of Theorem 1.4, $x = (I - W)^{1/2}\xi \in \mathcal{D}(T_{\Theta'})$ and $V^*y = W^{1/2}\xi$. Then $T_{\Theta'}x = V(I - W)^{-1/2}W^{1/2}(I - W)^{1/2}\xi = VV^*y = y$ and, thus, $T_{\Theta'}$ is closed.

Now, we show that the preframe $\theta'_z(y) := (T_{\Theta'}y)(z) \ \ \forall\, y \in \mathcal{R}((I - W)^{1/2}$ is a pseudo frame. This easily follows from the fact that, if $y = (I - W)^{1/2}\eta$, then $T_{\Theta'}y = VW^{1/2}\eta \in VH \subset L^2(\mu)$.

Finally, the duality of T_Θ and $T_{\Theta'}$ follows from the following observation. For every $x = W^{1/2}\xi \in \mathcal{D}(T_\Theta)$ and every $y = (I - W)^{1/2}\eta \in \mathcal{D}(T_{\Theta'})$,

$$\begin{aligned}\langle T_{\Theta'}y, T_\Theta x\rangle &= \langle VW^{1/2}\eta, V(I - W)^{1/2}\xi\rangle = \langle W^{1/2}\eta, (I - W)^{1/2}\xi\rangle \\ &= \langle (I - W)^{1/2}\eta\ , W^{1/2}\xi\rangle = \langle y, x\rangle\,.\end{aligned}$$

Therefore, (Θ', Θ) is an algebraic dual pair in H. □

Definition 2.3. The algebraic frame $\Theta' = (\theta'_z)_{z\in Z}$ defined by $\theta'_z(y) = (T_{\Theta'}y)(z)$ with signal space $Y = \mathcal{R}((I-W)^{1/2})$ and the analysis operator $T_{\Theta'} = V(I - W)^{-1/2}W^{1/2}$ is called the *standard algebraic dual* of Θ.

Proposition 2.4. *If $\Theta = (\theta_z)_{z\in Z}$ is an algebraic frame and if $\mathcal{D}(T_\Theta) = \mathcal{D}(S_\Theta)$, then T_Θ is a bounded operator.*

Proof. By theorem 1.6, $\mathcal{R}(W^{1/2}) = \mathcal{D}(T_\Theta) = \mathcal{D}(S_\Theta) = \mathcal{R}(W)$. Thus, for every $x \in H$ there exists $y \in H$ such that $W^{1/2}x = Wy$. Therefore, $x = W^{1/2}y$ and, so, $H = \mathcal{R}(W^{1/2})$. This implies that $W^{1/2}$ and $W^{-1/2}$ are bounded operators and, so, $T_\Theta = V(I - W)^{1/2}W^{-1/2} : H \to L^2(\mu)$ is a bounded operator. □

Definition 2.5. *Let Θ be an algebraic frame in H. Define its redundancy as*

$$\mathrm{red}(\Theta) := \mathrm{codim}(\overline{\mathcal{R}}(T_\Theta)) = \dim(\ker(T_\Theta^*)).$$

Theorem 2.6. *Suppose $\Theta = (\theta_z)_{z\in Z}$ is an algebraic frame with signal space $X = \mathcal{D}(T_\Theta)$ and let $\Theta' = (\theta'_z)_{z\in Z}$ be its standard algebraic dual. Assume $\Gamma = (\gamma_z)_{z\in Z}$ is a pseudo frame whose analysis operator t_Γ is a closable operator with $\mathcal{D}(t_\Gamma) = \mathcal{D}(\Theta')$ and $\mathcal{R}(T_\Theta) \perp \mathcal{R}(t_\Gamma)$. Then the family $\Delta := \Theta' + \Gamma$ is an algebraic dual frame for Θ in H. In particular, if* $\operatorname{red}(\Theta) \neq 0$*, then Θ is not a Riesz type algebraic frame.*

Proof. Assume $x_n = (I-W)^{1/2}\xi_n \in \mathcal{D}(T_{\Theta'})$ converges to $x \in H$ and $y_n = T_{\Theta'}x_n \oplus t_\Gamma x_n$ converges to $y = y_1 \oplus y_2$, where $y_1 \in \overline{\mathcal{R}}(T_{\Theta'})$ and $y_2 \in \overline{\mathcal{R}}(t_\Gamma)$. Since $T_{\Theta'}$ is a closed operator, then $x \in \mathcal{D}(\Theta')$ and $y_1 = T_{\Theta'}x$. Also, $y_2 = \bar{t}_\Gamma x$ and, since, $x \in \mathcal{D}(t_\Gamma)$, $y_2 = t_\Gamma x$. Hence, $y_1 \oplus y_2 = (T_{\Theta'} + t_\Gamma)x$ and, thus, $T_{\Theta'} + t_\Gamma$ is a closed operator with domain $\mathcal{R}((I-W)^{1/2})$. To show its injectivity, suppose $(T_{\Theta'} + t_\Gamma)x = 0$. It follows that $T_{\Theta'}x = 0$ and, thus, $x = 0$. Therefore, $T_{\Theta'} + t_\Gamma$ is injective and, hence, $\Delta = \Theta' + \Gamma$ is an algebraic frame in H. Moreover, for $x \in X$ and $y \in \mathcal{D}(T_{\Theta'})$,

$$\langle T_\Theta x, T_\Delta y\rangle = \langle T_\Theta x, T_{\Theta'}y\rangle + \langle T_\Theta x, t_\Gamma y\rangle = \langle T_\Theta x, T_{\Theta'}y\rangle = \langle x, y\rangle.$$

This shows that (Θ, Δ) is an algebraic dual pair in H.

To prove the last conclusion, assume $N : \mathcal{R}((I-W)^{1/2}) \to \mathcal{R}(T_\Theta)^\perp$ is any (nonzero) isometry or coisometry between H and $\mathcal{R}(T_\Theta)^\perp$. Define $\Gamma = (\gamma_z)_{z\in Z}$ by $\gamma_z(x) = (Nx)(z)$ for all $x \in \mathcal{R}((I-W)^{1/2})$ and observe that the algebraic frame Δ specified by $T_\Delta := T_{\Theta'} + t_\Gamma$ is an algebraic dual of T_Θ different from $T_{\Theta'}$. □

Remark 2.7. (1) In view of [6], if Θ is a generalized frame, then $T_\Theta S_\Theta^{-1}$ is the analysis operator of its dual frame. In the following, we show that $T_\Theta S_\Theta^{-1} \subset T_{\Theta'}$ which turns into an equality for the generalized frames.

(2) In view of Theorem 2.6, to have a Riesz type algebraic frame it is necessary to have $\operatorname{red}(T_\Theta) = 0$ or, equivalently, to have $\overline{\mathcal{R}}(T_\Theta) = L^2(\mu)$. In the following theorem we show that Θ is a Riesz type algebraic frame if $\mathcal{R}(T_\Theta) = L^2(\mu)$. We answer the case $L^2(\mu) = \mathcal{R}(T_\Theta)$ in Part (γ) of the following theorem and leave the case $L^2(\mu) = \overline{\mathcal{R}}(T_\Theta) \neq \mathcal{R}(T_\Theta)$ as an open question.

Theorem 2.8. *Let Θ be an algebraic frame. Then the following assertions are true.*

(α) $T_\Theta S_\Theta^{-1} \subset T_{\Theta'}$.

(β) $(\mathcal{R}(T_\Theta))^\perp$ *is the linear span* V_Θ *of all vectors* $(T_\Gamma - T_K)x$ *as* x *runs in* $\mathcal{D}(T_\Gamma) \cap \mathcal{D}(T_K)$ *when* Γ *and* K *run in the collection of all algebraic duals of* Θ.

(γ) *If* Θ *has zero redundancy and if* Γ *is any algebraic dual of* Θ*, then* $T_\Gamma \subset (T_\Theta^*)^{-1} = T_{\Theta'}$*. In particular, if* $\mathcal{R}(T_\Theta) = L^2(\mu)$*, then* Θ *is a Riesz type algebraic frame.*

Proof. Part(α): By Part (iii) of Theorem 1.6, $S_\Theta = T_\Theta^* T_\Theta = (I - W)W^{-1}$ with $\mathcal{D}(S_\Theta) = \mathcal{R}(W)$. We claim that $T_\Theta S_\Theta^{-1} \subset T_{\Theta'}$. Since T_Θ is injective, $I - W$ is injective and has a dense range. So, S_Θ is injective and $S_\Theta^{-1} = W(I - W)^{-1}$ with $\mathcal{D}(S_\Theta^{-1}) = \mathcal{R}(I - W)$. Let $x \in \mathcal{D}(T_\Theta S_\Theta^{-1})$. Then $x = (I - W)\xi \in \mathcal{D}(S_\Theta^{-1})$ and $W(I - W)^{-1}(I - W)\xi = W^{1/2}(W^{1/2}\xi) \in \mathcal{D}(T_\Theta)$ for some $\xi \in H$. Also, $x = (I - W)^{1/2}[(I - W)^{1/2}\xi] \in \mathcal{D}(T_{\Theta'})$ and, hence, $\mathcal{D}(T_\Theta S_\Theta^{-1}) \subset \mathcal{D}(T_{\Theta'})$. Therefore,

$$\begin{aligned} T_\Theta S_\Theta^{-1} x &= V(I - W)^{1/2} W^{-1/2} W (I - W)^{-1} (I - W)\xi \\ &= V(I - W)^{1/2} W^{1/2} \xi = V(I - W)^{-1/2}(I - W) W^{1/2} \xi \\ &= V(I - W)^{-1/2} W^{1/2} (I - W)\xi = T_{\Theta'} x. \end{aligned}$$

This shows that $T_\Theta S_\Theta^{-1} \subset T_{\Theta'}$.

Part (β): Let $y \in V_\Theta$. There exist algebraic duals G and Γ such that for some $x \in H$, $y = (T_G - T_\Gamma)x$. Thus, for every $\xi \in \mathcal{D}(T_\Theta)$, $\langle y, T_\Theta \xi \rangle = \langle (T_G - T_\Gamma)x, T_\Theta \xi \rangle = \langle T_G x, T_\Theta \xi \rangle - \langle T_\Gamma x, T_\Theta \xi \rangle = \langle x, \xi \rangle - \langle x, \xi \rangle = 0$. Hence, $\mathcal{V}_\Theta \subset (\mathcal{R}(T_\Theta))^\perp$. For the converse, assume without loss of generality that $(\mathcal{R}(T_\Theta))^\perp \neq \{0\}$. Let $g \in (\mathcal{R}(T_\Theta))^\perp$ be arbitrary and choose the algebraic dual Δ of Θ as in the last paragraph of the proof of Theorem 2.6 in which the (isometry or coisometry) N contains g in its range; i.e., $g = Nx$ for some $x \in \mathcal{D}(T_{\Theta'})$. Then $g = Nx = (T_\Delta - T_{\Theta'})x \in V_\Theta$.

Part (γ): Assume Θ has zero redundancy and Γ is any algebraic dual of Θ. Then, for all $x \in \mathcal{D}(T_\Theta)$ and all $y \in \mathcal{D}(T_\Gamma)$, $\langle T_\Theta x, T_\Gamma y \rangle = \langle x, y \rangle$. By definition of the adjoint, $T_\Gamma y \in \mathcal{D}(T_\Theta^*)$ and $T_\Theta^* T_\Gamma y = y$. Therefore, $T_\Theta^* T_\Gamma y = y$ for all $y \in \mathcal{D}(T_\Gamma)$ and, thus, $T_\Theta^* T_\Gamma = I|_{\mathcal{D}(T_\Gamma)}$. Hence, $\mathcal{R}(T_\Gamma) \subset \mathcal{D}(T_\Theta^*)$ and $\mathcal{D}(T_\Gamma) \subset \mathcal{R}(T_\Theta^*)$. Since $\ker(T_\Theta^*) = \mathcal{R}(T_\Theta)^\perp = \{0\}$, it follows that $(T_\Theta^*)^{-1}$ exists and $T_\Gamma = (T_\Theta^*)^{-1}|_{\mathcal{D}(T_\Gamma)} = (T_\Theta^{-1})^*|_{\mathcal{D}(T_\Gamma)}$. But

$(T_\Theta^*)^{-1} = [(W^{-1/2}(I-W)^{1/2}V^*]^{-1} = V(I-W)^{-1/2}W^{1/2} = T_{\Theta'}$. This means that the standard algebraic dual is the largest algebraic dual.

Finally, if $\mathcal{R}(T_\Theta) = L^2(\mu)$, we claim Θ is a Riesz type frame. In fact, it follows from the closed graph theorem that T_Θ^{-1} is a bounded operator. Therefore, $(T_\Theta^*)^{-1}$ is bounded and so is every algebraic dual frame $T_\Gamma(\subset (T_\Theta^*)^{-1})$. Therefore, $T_\Gamma = (T_\Theta^*)^{-1} = T_{\Theta'}$. $\square$

Open question. What happens in Part (γ) of Theorem 2.8 in case $L^2(\mu) = \overline{\mathcal{R}}(T_\Theta) \neq \mathcal{R}(T_\Theta)$?

Remark 2.9. Let Θ be a lower bounded algebraic frame; i.e., $c||x|| \leq ||T_\Theta x||$ for all $x \in \mathcal{D}(T_\Theta)$ and some $c > 0$. Then it follows that $T_\Theta^{-1} : \mathcal{R}(T_\Theta) \to \mathcal{D}(T_\Theta)$ is a surjective bounded operator; since T_Θ^{-1} is a closed operator, its domain $\mathcal{R}(T_\Theta)$ must be closed. Now, if Θ has zero redundancy, then $\mathcal{R}(T_\Theta) = L^2(\mu)$ and, hence, $T_\Theta^{-1} : L^2(\mu) \to H$ and $(T_\Theta^*)^{-1} : H \to L^2(\mu)$ are bounded operators. Therefore, in view of Part (γ) of Theorem 2.8, all algebraic duals are bounded operators with the same domain H and, hence, they are all equal. That is Θ is a Riesz type algebraic frame (cf. Proposition 2.2 of [6]).

Proposition 2.10. *Assume that $\Theta = (\theta_z)_{z\in Z}$ is an algebraic frame in H with signal space $X = \mathcal{D}(T_\Theta)$. If K is an algebraic dual for Θ, then for all $x \in \mathcal{D}(T_{\Theta'}) \cap \mathcal{D}(T_K)$, $||T_{\Theta'}x|| \leq ||T_K x||$.*

Proof. Let $x \in \mathcal{D}(T_{\Theta'}) \cap \mathcal{D}(T_K)$. Since $\mathcal{R}(T'_\Theta) \subset VH = \overline{\mathcal{R}}(T_\Theta)$, we have $T_K x = (T_K - T'_\Theta)x \oplus T_{\Theta'}x \in V_\Theta \oplus \overline{\mathcal{R}}(T_\Theta) = (\mathcal{R}(T_\Theta))^\perp \oplus \overline{\mathcal{R}}(T_\Theta)$. Thus, $||T_K x||^2 = ||(T_K - T_{\Theta'})x||^2 + ||T_{\Theta'}x||^2$ and, so, $||T_{\Theta'}x|| \leq ||T_K x||$. $\square$

Theorem 2.11. *Assume $\Theta = (\theta_z)_{z\in Z}$ is a lower bounded algebraic frame in H with finite redundancy. There exists an analytic frame $\Gamma = (\gamma_z)_{z\in Z}$ in the Hilbert space $K := \mathcal{R}(T_\Theta)^\perp$ such that $\Theta\oplus\Gamma = (\theta_z\oplus\gamma_z)_{z\in Z}$ is a Riesz-type algebraic frame in $H \oplus K$.*

Proof. We follow a modification of the argument given at the end of the proof of Theorem 2.6. Let $N : K \to K$ be any unitary operator. Define $\Gamma = (\gamma_z)_{z\in Z}$ by $\gamma_z(x) = (Nx)(z)$ for all $x \in K$. If $\delta_z = \theta_z \oplus \gamma_z$, then the family $\Delta = (\delta_z)_{z\in Z}$ is an algebraic frame in $H \oplus K$ with signal space $\mathcal{D}(T_\Theta) \oplus K$ and closed analysis operator $T_\Delta = T_\Theta \oplus T_\Gamma$. Since T_Θ is lower bounded, $\mathcal{R}(T_\Delta) = \mathcal{R}(T_\Theta) \oplus K = \overline{\mathcal{R}}(T_\Theta) \oplus \mathcal{R}(T_\Theta)^\perp = L^2(\mu)$.

Now, Theorem 2.8 implies that Δ is a Riesz-type algebraic frame in the Hilbert space $H \oplus K$. $\square$

The following example from [8], is a lower bounded algebraic frame whose redundancy is finite. Here, we study some duals of this algebraic frame.

Example 2.12. Let

$$X = \{f \in L^2([0,1]) : \text{f is abs. cont.},\ f' \in L^2,\ f(0) = f(1) = 0\}.$$

Define the densely defined unbounded operator $T : X \subset L^2([0,1]) \to L^2([0,1])$ by $Tf = f'$ for all $f \in X$. Thus, T is a closed injective operator with a bounded inverse. Moreover, $T^*g = -g'$ and $\mathcal{D}(T_\Theta{}^*) = \{g \in L^2([0,1]) :\ g \text{ is abs. cont. and } g' \in L^2([0,1])\}$ [5]. Next, for all $z \in Z = [0,1] (\subset \mathbb{R})$, define the unbounded linear functional $\theta_z : X \to \mathbb{C}$ by $\theta_z(f) = (Tf)(z) = f'(z)$ for all $f \in X$. Therefore, the family $\Theta = (\theta_z)_{z \in Z}$ is a lower bounded algebraic frame with the analysis operator $T_\Theta = T$. Since $\mathcal{R}(T_\Theta) = \{f' : f \in X\} = \{g \in L^2([0,1]) : \int_0^1 g(t)dt = 0\} = \{1\}^\perp$, it follows that $\text{red}(\Theta) = 1$.

Let us first construct the algebraic dual $\tilde{\Theta}$ with analysis operator $T_{\tilde{\Theta}} = T_\Theta S_\Theta^{-1} \subset T_{\Theta'}$ and $\mathcal{D}(T_{\tilde{\Theta}}) = \mathcal{R}(S_\Theta) = \mathcal{R}(I - W)$. Now, we can write $S_\Theta f = T_\Theta{}^* T_\Theta f = T_\Theta{}^* f' = -f''$ for all $f \in \mathcal{D}(S_\Theta)$, where $\mathcal{D}(S_\Theta) := \{f \in X : f' \text{ is abs.cont.}, f'' \in L^2([0,1])\}$. Let $Y = \mathcal{R}(S_\Theta)$. If $g \in Y$, there exists $f \in X$ such that $g = S_\Theta f = -f''$. Define the linear functional $\tilde{\theta}_z : Y \subset L^2(0,1) \to \mathbb{C}$ by $\tilde{\theta}_z = \theta_z S_\Theta{}^{-1}$, for all $z \in Z$. Therefore, for all $g \in Y$, $\tilde{\theta}_z g = \theta_z S_\Theta{}^{-1}(S_\Theta f) = \theta_z f = f'(z) = -\int_0^z g(t)dt$. Therefore, we have $\tilde{\Theta} = (-\int_0^z g(t)dt)_{z \in Z}$. Since Θ is lower bounded, the operator $T_{\tilde{\Theta}}$ is closed. Thus, part (α) of theorem 2.8 implies that $T_{\tilde{\Theta}} \subset T_{\Theta'}$ and, so, $\tilde{\Theta}$ is an algebraic dual frame for Θ.

Similar to what we did at the end of the proof of Theorem 2.6, assume $N : Y \to \mathcal{R}(T_\Theta)^\perp$ is any (nonzero) isometry or coisometry between H and $\mathcal{R}(T_\Theta)^\perp$. Define $\Gamma = (\gamma_z)_{z \in Z}$ by $\gamma_z(x) = (Nx)(z)$ for all $x \in Y$ and observe that the algebraic frame $\Delta = \tilde{\Theta} + \Gamma$ specified by $T_\Delta := T_{\tilde{\Theta}} + T_\Gamma$ is an algebraic dual of Θ different from $\tilde{\Theta}$.

The following Proposition is a consequence of Part (2) of Theorem 2 of [7] relates the discreteness of the indexing measure and the zero redundancy of the algebraic frame. Note that a measurable subset E of

$\mathcal{M}$ is called an atom if $0 < \mu(E) < \infty$ and E contains no measurable subset F such that $0 < \mu(F) < \mu(E)$ [1].

Proposition 2.13. *Assume that* $\Theta = (\theta_z)_{z \in Z}$ *is a lower bounded analytic frame indexed by the measure space* $(Z, \mathcal{M}, \mu)$ *and that* $\theta_z \neq 0$ *for almost all* z*. If* Θ *is a Riesz-type algebraic frame, then the measure* μ *is discrete in the sense that* Z *is the union of (countably many) atoms.*

Proof. Theorem 2.6 implies that red(Θ) is zero. Thus the proof follows from the Part (2) of Theorem 2 of [7]. □

References

[1] C.D. Aliprantis and O.B. Shaw, *Principles of Real Analysis*, 2nd Edition, Academic Press, New York(1990).

[2] A. Askari Hemmat, M.A. Dehghan and M. Radjabalipour, Generalized frames and their redundancy, *Procedings of the American Mathematical Society*, 129(4)(2000), 1143-1147.

[3] S. Azadi and M. Radjabalipour, On the structure of unbounded operators, *Iran. J. Sci. Technol. Trans. Sci.* 44 (2020), 1711-1719 .

[4] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhauser, Boston(2002).

[5] J.B. Conway, *A Course in Functional Analysis*, 2nd Edition, Springer-Verlag, New York(1997).

[6] J.P. Gabardo, and M.D. Han, Frames associated with measurable spaces, *Advances in Computational Mathematics*, 18(2003), 127-147.

[7] H. Giv and M. Radjabalipour, On the structure and properties of lower bounded analytic frames, *Iran. J. Sci. Technol. Trans. Sci.*, 37A3(2013), 227-230.

[8] H. Giv, M. Radjabalipour and A. Askari Hemmat, Unbounded analysis operators, *Bull. Belg. Math. Soc. Simon Stevin*, 20(2013), 123-132.

[9] W. Groetsch, *Stable Approximate Evaluation of Unbounded Operators*, Springer, New York(2006).

[10] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 2nd Edition(1980).

[11] M. Kebryaee and M. Radjabalipour, Homeomorphic images of orthogonal bases, *Electronic Journal of Linear Algebra*, 31(2016), 485-491.

Shahrzad Azadi
Department of Mathematics
PHD. student of Mathematics
Kerman Branch, Islamic Azad University
Kerman, Iran
E-mail: azadi.sd@gmail.com

Mehdi Radjabalipour
Department of Mathematics
Professor of Mathematics
Shahid Bahonar University of Kerman
Kerman, Iran
E-mail: mradjabalipour@gmail.com