Journal of Mathematical Extension
Vol. 15, No. 3, (2021) (11)1-22
URL: https://doi.org/10.30495/JME.2021.1507
ISSN: 1735-8299
Original Research Paper

Fixed Point Results via $\mathcal{L}$-Contractions on Quasi w-Distances

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Abstract. The concept of a w-distance on a metric space has been introduced by Kada et al. [22]. They generalized Caristi fixed point theorem, Ekeland variational principle and the nonconvex minimization theorem according to Takahashi. In the present paper, we first introduce the notion of quasi w-distances in quasi-metric spaces, and then we will prove some fixed point theorems for $\mathcal{L}$-contractive mappings in the class of quasi-metric spaces with w-distances via a control function introduced by Jleli and Samet [20]. These results generalize many fixed point theorems by Kada et al. [22], Suzuki [33], Ćirić [15], Aydi et al. [6], Abbas and Rhoades [1], Kannan [23], Hicks and Rhoades [19], Du [16], Lakzian et al. [26], Lakzian and Rhoades [30] and others. Finally, some examples in support of the concepts and presented results are given.

AMS Subject Classification: 47H10; 54H25; 46J10

Keywords and Phrases: Fixed point, $\mathcal{L}$-contraction, Quasi-metric space

Received: January 2020; Accepted: October 2020
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1 Introduction

The concept of a w-distance on a metric space has been introduced by Kada et al. [22] (see also [11, 24, 26, 28, 29, 27, 25]). They generalized Caristi fixed point theorem, Ekeland variational principle and the nonconvex minimization theorem according to Takahashi. Suzuki [33] extended Kannan fixed point results to metric spaces with w-distances.

The following definition is the concept of w-distance on a metric space (See Kada et al. [22]).

Definition 1.1. [22] Let X be a metric space endowed with a metric d. A function $p : X \times X \longrightarrow [0, \infty)$ is called a w-distance on X if it satisfies the following properties.
(i) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$.
(ii) p is lower semi-continuous in its second variable, i.e., if $x \in X$ and $y_n \to y$ in X, then $p(x, y) \leq \liminf_n p(x, y_n)$.
(iii) For each $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

The following lemma will be used in the next section.

Lemma 1.2. *[22] Let (X, d) be a metric space and p be a w-distance on X.*

(i) Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, y) = 0.$$

Then $x = y$. In particular, if $p(z, x) = p(z, y) = 0$, we have $x = y$.

(ii) If $p(x_n, y_n) \leq \alpha_n$ $p(x_n, y) \leq \beta_n$ for any $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, \infty)$ converging both to 0, then $\{y_n\}$ converges to y.

(iii) Let p be a w-distance on a metric space (X, d). Let $\{x_n\}$ be a sequence in X such that for each $\varepsilon > 0$, there exists $N_\varepsilon \in N$ such that for $m > n > N_\varepsilon$, $p(x_n, x_m) < \varepsilon$ (or $\lim_{n,m} p(x_n, x_m) = 0$). Then $\{x_n\}$ is a Cauchy sequence.

Jleli and Samet [20] introduced the concept of θ-contractions as follows.

Definition 1.3. [20] The mapping θ from $(0,\infty)$ into $(1,\infty)$ is said to be θ-contraction if it satisfies the following conditions.

(θ_1) θ is non-decreasing.

(θ_2) For any $t_n \in (0,\infty)$;

$$\lim_{n\to\infty} \theta(t_n) = 1 \Longleftrightarrow \lim_{n\to\infty} t_n = 0;$$

(θ_3) there exists $(r,l) \in (0,1) \times (0,\infty)$ such that

$$\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l.$$

Ahmed et al. [3] in 2017, replaced the condition (θ_3) by the following condition.

(θ_4)] θ is continuous on $(0,\infty)$.

Cho [14] introduced the concept of $\mathcal{L}$-contraction mappings as follows.

Definition 1.4. [14] Suppose that $\mathcal{L}$ is the family of all mappings $\xi : [1,\infty) \times [1,\infty) \to \mathbb{R}$ such that

(ξ_1) $\xi(1,1) = 1$.

(ξ_2) $\xi(t,s) < \frac{s}{t}$ for all $s,t > 1$.

(ξ_3) For any sequences $\{t_n\}$, $\{s_n\}$ in $(1,\infty)$ with $t_n < s_n$ for $n = 1,2,3,\cdots$

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 1 \Longrightarrow \limsup_{n\to\infty} \xi(t_n, s_n) < 1.$$

Any $\xi \in \mathcal{L}$ is called $\mathcal{L}$-simulation function. Note that $\xi(t,t) < 1$ for each $t > 1$.

In the following, we list some examples of $\mathcal{L}$-simulation functions.

Example 1.5. [14] *(i)* $\xi(t,s) = \frac{s^k}{t}$ for all $s,t \geq 1$, where $k \in (0,1)$.

(ii) $\xi(t,s) = \frac{s}{t\varphi(s)}$ for all $s,t \geq 1$ where $\varphi : [1,\infty) \to [1,\infty)$ is nondecreasing and lower semi-continuous such that $\varphi^{-1}(\{1\}) = 1$.
(iii)

$$\xi(t,s) = \begin{cases} 1, & \text{if } (s,t) = (1,1), \\ \frac{s}{2t}, & \text{if } s < t, \\ \frac{s^\lambda}{t}, & \text{otherwise,} \end{cases}$$

for all $s,t \geq 1$ where $\lambda \in (0,1)$.

The concept of quasi-metric spaces obtained by omitting the symmetry condition is used by many authors in [2, 4, 5, 10, 8, 9, 7, 12, 13, 17, 21, 32] for proving fixed point theorems. We recall the following definition.

Definition 1.6. Let X be a non-empty set. Let $q : X \times X \to [0,\infty)$ be a function satisfying
$(q1) q(x,y) = 0$ if and only if $x = y$,
$(q2) q(x,y) \leq q(x,z) + q(z,y)$.
Then q is called a quasi-metric and the pair (X,q) is called a quasi-metric space.

In the present paper, after the definition of the concept of quasi w-distance on a quasi-metric space, we prove some fixed point theorems for $\mathcal{L}$-contraction mappings using θ-functions in the class of quasi-metric spaces with quasi w-distance. Some consequences are also derived. Moreover, we present some examples in support of the given results.

2 Preliminary and Lemmas

In this section, we recall some basic concepts and notations. Some useful lemmas are also included.

First, we give some basic concepts such as convergence and completeness on quasi-metric spaces.

Definition 2.1. [6] Let (X, q) be a quasi-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.
(i) The sequence $\{x_n\}$ converges to x if and only if

$$\lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x, x_n) = 0.$$

(ii) The sequence $\{x_n\}$ is left-Cauchy (resp. right-Cauchy, Cauchy) if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $n \geq m > N$ (resp. $m \geq n > N$, $m, n > N$).

(iii) (X, q) is said left-complete (resp. right-complete, complete) if and only if each left-Cauchy (resp. right-complete, complete) sequence in X is convergent.

Remark 2.2. (i) In a quasi-metric space (X, q), the limit for a convergent sequence is unique. Also, if $x_n \to x$, we have for all $y \in X$

$$\lim_{n\to\infty} q(x_n, y) = q(x, y) \quad \text{and} \quad \lim_{n\to\infty} q(y, x_n) = q(y, x).$$

(i) A sequence $\{x_n\}$ in a quasi-metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.

Lemma 2.3. *[6] Let (X, q) be a quasi-metric space and $T : X \to X$ be a given mapping. Suppose that T is continuous at $u \in X$. Then for all sequences $\{x_n\}$ in X such that $x_n \to u$, we have $Tx_n \to Tu$, that is,*

$$\lim_{n\to\infty} q(Tx_n, Tu) = \lim_{n\to\infty} q(Tu, Tx_n) = 0.$$

In the following, we give the concept of quasi w-distance on a quasi-metric space.

Definition 2.4. Let (X, q) be a quasi-metric space. A function $p : X \times X \to [0, \infty)$ is called a quasi w-distance on X if it satisfies the following properties.

(i) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$.

(ii) p is lower semi-continuous in its second variable; i.e., if $x \in X$ and $y_n \to y$ in X, then $p(x, y) \leq \liminf_n p(x, y_n)$.

(iii) For each $\epsilon > 0$, there exists a $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $q(x, y) \leq \epsilon$ and $q(y, x) \leq \epsilon$.

It is easy to see that each w-distance is a quasi w-distance.

Example 2.5. Let (X, q) be a quasi-metric space. Then each $p : X \times X \to \mathbb{R}^+$ defined by

(i) $p(x, y) = q(y, x_0) + q(x_0, y)$, for some $x_0 \in X$;

(ii) $p(x, y) = \max\{q(y, x_0), q(x_0, y)\}$, for some $x_0 \in X$;

(iii) $p(x, y) = p'(x, y) + \alpha$, for some positive number α and a w-distance p';

(iv) $p(x, y) = \max\{p'(x, y), p''(x, y)\}$, for two w-distances p', p'';

(v) $p(x, y) = p'(x, y) + p''(x, y)$, for two w-distances p', p'';

for each $x, y \in X$, is a quasi w-distance.

In section 3, we give some another examples of quasi w-distances.

The following lemma has an easy proof (following the proof of Lemma 1.2).

Lemma 2.6. *Let (X, q) be a quasi-metric space and p be a quasi w-distance on X.*

(i) If $\{x_n\}$ is a sequence in X such that $lim_n p(x_n, x) = lim_n p(x_n, y) = 0$ then $x = y$. In particular, if $p(z, x) = p(z, y) = 0$ then $x = y$.

(ii) If $p(x_n, y_n) \leq \alpha_n$ $p(x_n, y) \leq \beta_n$ for any $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, \infty)$ converging to 0, then $\{y_n\}$ converges to y.

(iii) Let p be a quasi w-distance on a quasi-metric space (X, q) and $\{x_n\}$ be a sequence in X such that, for each $\varepsilon > 0$, there exists an $N_\varepsilon \in N$ such that $m > n \geq N_\varepsilon$ (respectively, $n > m > N_\varepsilon$, $m, n > N_\varepsilon$) implies $p(x_n, x_m) < \varepsilon$, then $\{x_n\}$ is a right-Cauchy (respectively, left-Cauchy, Cauchy) sequence.

3 Main results

Suppose that Θ is the class of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying (θ_1) and (θ_2). One of our essential main results is the following theorem.

Theorem 3.1. *Let (X, q) be a complete quasi-metric space with quasi w-distance p and $T : X \to X$ be a given mapping. Suppose that there exist $\xi \in \mathcal{L}$ and $\theta \in \Theta$ such that for all $x, y \in X$ with $q(Tx, Ty) \neq 0$,*

$$\xi\left(\theta(p(Tx, Ty)), \theta(p(x, y))\right) \geq 1. \tag{1}$$

Then T has a unique fixed point. Moreover, if condition (1) is true for each x, y, then $p(u, u) = 0$.

Proof. Consider an $x_0 \in X$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$, for all $n \geq 0$. If $x_n = x_{n+1}$ for some n, then $x_n = x_{n+1} = Tx_n$, that is, x_n is a fixed point of T and so the proof is completed. Suppose from now on that $x_n \neq x_{n+1}$ for all n. Then $q(x_n, x_{n+1}) > 0$.
Now from the condition (1), we have

$$\begin{aligned} 1 &\leq \xi\left(\theta(p(Tx_{n-1}, Tx_n)), \theta(p(x_{n-1}, x_n))\right) \\ &< \frac{\theta(p(x_{n-1}, x_n))}{\theta(p(Tx_{n-1}, Tx_n))} \\ &= \frac{\theta(p(x_{n-1}, x_n))}{\theta(p(x_n, x_{n+1}))}. \end{aligned}$$

Consequently, we obtain

$$\theta(p(x_n, x_{n+1})) < \theta(p(x_{n-1}, x_n)),$$

which implies for all $n = 1, 2, 3, \ldots$,

$$p(x_n, x_{n+1}) < p(x_{n-1}, x_n).$$

Thus, the sequence $\{p(x_n, x_{n+1})\}$ is decreasing and so it is convergent. Let there is an $\varepsilon > 0$ such that

$$\lim_{n \to \infty} p(x_n, x_{n+1}) = \varepsilon.$$

From (θ_2), we have $\lim_{n\to\infty}\theta(p(x_n,x_{n+1}))\neq 1$, and so

$$\lim_{n\to\infty}\theta(p(x_n,x_{n+1}))>1.$$

Using (ξ_3), we get

$$1\leq \limsup_{n\to\infty}\xi(\theta(p(x_n,x_{n+1})),\theta(p(x_{n-1},x_n))<1,$$

a contradiction. This implies that

$$\lim_{n\to\infty}p(x_n,x_{n+1})=0. \tag{2}$$

Similarly,

$$\lim_{n\to\infty}p(x_{n+1},x_n)=0. \tag{3}$$

Now, we shall prove that the sequence $\{x_n\}$ is right-Cauchy. By the triangular inequality,

$$p(x_n,x_{n+2})\leq p(x_n,x_{n+1})+p(x_{n+1},x_{n+2}),$$

and so $\lim_{n\to\infty}p(x_n,x_{n+2})=0$.
Suppose that the induction hypothesis holds; i.e., $\lim_{n\to\infty}p(x_n,x_{n+k})=0$. Then

$$p(x_n,x_{n+k+1})\leq p(x_n,x_{n+k})+p(x_{n+k},x_{n+k+1}),$$

which implies that $\lim_{n\to\infty}p(x_n,x_{n+k+1})=0$. Therefore for each $\varepsilon>0$, there is $N_\varepsilon\in\mathbb{N}$ such that for each $m>n\geq N_\varepsilon$, $p(x_n,x_m)\leq\varepsilon$. So, by Lemma 2.6, the sequence $\{x_n\}$ is right-Cauchy. Similarly, $\{x_n\}$ is a left-Cauchy sequence. Therefore $\{x_n\}$ is a Cauchy sequence. Now since X is complete, there exists $u\in X$ such that

$$\lim_{n\to\infty}q(x_n,u)=\lim_{n\to\infty}q(u,x_n)=0.$$

In the sequel, we shall prove that u is a fixed point of T. Since $x_n\to u$ and $p(x,.)$ is lower semi-continuous, we have

$$p(x_{N_\varepsilon},u)\leq \liminf_{n\to\infty}p(x_{N_\varepsilon},x_n)\leq\varepsilon.$$

Putting $\varepsilon = 1/k$ and $N_\varepsilon = n_k$, we have

$$\lim_{k\to\infty} p(x_{n_k}, u) = 0. \tag{4}$$

Again by condition (1) with $x = x_{n_k+1}$ and $y = u$, we get

$$\begin{aligned} 1 &\leq \xi(\theta(p(Tx_{n_k+1}, Tu)), \theta(p(x_{n_k+1}, u))) \\ &< \frac{\theta(p(x_{n_k+1}, u))}{\theta(p(x_{n_k+2}, Tu))}. \end{aligned}$$

Therefore,

$$\theta(p(x_{n_k+2}, Tu)) < \theta(p(x_{n_k+1}, u)).$$

Since θ is non-decreasing, we obtain

$$p(x_{n_k+2}, Tu) < p(x_{n_k+1}, u). \tag{5}$$

By triangular inequality we have

$$p(x_{n_k+1}, u) \leq p(x_{n_k+1}, x_{n_k}) + p(x_{n_k}, u).$$

From the relations (3) and (4), we conclude that

$$\lim_{k\to\infty} p(x_{n_k+1}, u) = 0.$$

Similarly

$$\lim_{k\to\infty} p(x_{n_k+2}, u) = 0. \tag{6}$$

Now, by inequality (5), we obtain

$$\lim_{k\to\infty} p(x_{n_k+2}, Tu) = 0. \tag{7}$$

Therefore by Lemma 2.6 and the equalities (6) and (7), we get $u = Tu$.

Now, we show that u is unique. Let $x_1, x_2 \in X$ be two distinct fixed points (i.e., $x_1 \neq x_2$). So $q(Tx_1, Tx_2) > 0$ and from condition (1),

$$\begin{aligned} 1 &\leq \xi(\theta(p(Tx_1, T^2x_1)), \theta(p(x_1, Tx_1))) \\ &< \frac{\theta(p(x_1, x_1))}{\theta(p(x_1, x_1))} \\ &= 1. \end{aligned}$$

It is a contradiction, so $x_1 = x_2$.

Moreover, if condition (1) is true for each x, y, without the condition $q(Tx, Ty) \neq 0$, then

$$\begin{aligned} 1 &\leq \xi(\theta(p(Tx, T^2x)), \theta(p(x, Tx))) \\ &< \frac{\theta(p(x, Tx))}{\theta(p(Tx, T^2x))}. \end{aligned}$$

Then we conclude that there is $t_x \in (0, 1)$ such that $p(Tx, T^2x) \leq t_x p(x, Tx)$, for each $x \in X$. In particular

$$\begin{aligned} p(u, u) &= p(Tu, T^2u) \\ &\leq t_u p(u, Tu) \\ &= t_u p(u, u). \end{aligned}$$

Which implies that $p(u, u) = 0$.

□

Corollary 3.2. *Let (X, q) be a complete quasi-metric space with quasi w-distance p and $T : X \to X$ be a given mapping such that for all $x, y \in X$ with $q(Tx, Ty) \neq 0$,*

$$p(Tx, Ty) \leq p(x, y)) - \varphi(p(x, y)) \tag{8}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is nondecreasing and lower semi-continuous such that $\varphi^{-1}(\{0\}) = \{0\}$. Then T has a unique fixed point.

Proof. From condition (8), we have

$$e^{p(Tx,Ty)} \leq e^{p(x,y) - \varphi(p(x,y))}$$

Put $\theta(t) = e^t$, we get

$$\theta(p(Tx, Ty)) \leq \frac{\theta(p(x, y))}{e^{\varphi(p(x,y))}}.$$

Now consider the nondecreasing and lower semi-continuous map $\psi : [1, \infty) \to [1, \infty)$ such that $\psi \circ \theta(t) = e^{\varphi(t)}$ and $\psi^{-1}(\{1\}) = \{1\}$. Then

$$\theta(p(Tx, Ty)) \leq \frac{\theta(p(x, y))}{\psi(\theta(p(x, y)))}.$$

By putting $\xi(t,s) = \frac{s}{t\psi(s)}$, we get

$$\begin{aligned} 1 &\le \frac{\theta(p(x,y))}{\theta(p(Tx,Ty))\psi(\theta(p(x,y)))} \\ &= \xi(\theta(p(Tx,Ty)), \theta(p(x,y))). \end{aligned}$$

Then by Theorem 3.1, T has a unique fixed point. □ Following Example 1.5 and taking in Theorem 3.1, $\xi(t,s) = \frac{s^k}{t}$ for all $s,t \ge 1$, where $k \in (0,1)$, we have the following corollary.

Corollary 3.3. *Let (X,q) be a complete quasi-metric space and $T : X \to X$ be a given mapping such that for all $x,y \in X$ with $q(Tx,Ty) \neq 0$, $\theta(p(Tx,Ty)) \le [\theta(p(x,y))]^k$. Then T has a unique fixed point.*

Remark 3.4. Corollary 3.2 improves Theorem 3.2 in [21], where φ is lower semi-continuous, not necessary continuous. Corollary 3.3 is the quasi-metric part of Theorem 2.1 in [14], Theorem 2.1 in [20] without the condition (θ_3) and Theorem 2.2 in [3] without condition (θ_4).

Theorem 3.5. *Let (X,q) be a complete quasi-metric space with quasi w-distance p and $T : X \to X$ be a given mapping. Suppose that there exist $\xi \in \mathcal{L}$ and $\theta \in \Theta$ such that for all $x \in X$ with $q(Tx,T^2x) \neq 0$,*

$$\xi\left(\theta(p(Tx,T^2x)), \theta(p(x,Tx))\right) \ge 1. \tag{9}$$

Suppose that one of the following conditions hold.

- *(i)* $\inf\{p(x,y) + p(x,Tx) : x \in X\} > 0$ *for every $y \in X$ with $y \neq Ty$,*
- *(ii) The mapping T is continuous,*
- *(iii) If for some sequence $\{x_n\}$, $\lim_{n\to\infty} p(x_n,x) = \lim_{n\to\infty} p(Tx_n,x)$, then $Tx = x$.*

Then T has a fixed point u. Moreover, if inequality (9) is true for each x, then $p(u,u) = 0$.

Proof. Fix $x_0 \in X$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$, for all $n \ge 0$. Then similar to Theorem 1 we can show that there exists $u \in X$ such that

$$\lim_{n\to\infty} q(x_n,u) = \lim_{n\to\infty} q(u,x_n) = 0.$$

Now, we prove u is a fixed point of T.

Case (i). If $\inf\{p(x,y)+p(x,Tx) : x \in X\} > 0$ for every $y \in X$ with $y \neq Ty$, then for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for $n > N_\varepsilon$, $p(x_{N_\varepsilon}, x_n) < \varepsilon$. But, $x_n \to u$ and $p(x,.)$ is lower semi-continuous, so we have

$$\begin{aligned} p(x_{N_\varepsilon}, u) &\leq \liminf_{n\to\infty} p(x_{N_\varepsilon}, x_n) \\ &\leq \varepsilon. \end{aligned}$$

Putting $\varepsilon = 1/k$ and $N_\varepsilon = n_k$, we have

$$\lim_{k\to\infty} p(x_{n_k}, u) = 0. \tag{10}$$

Assume that $u \neq Tu$. Then

$$0 < \inf\{p(z,u)+p(z,Tz) : z \in X\} \leq \inf\{p(x_{n_k}, u)+p(x_{n_k}, x_{n_k+1}) : k \in \mathbb{N}\}.$$

Using Cauchiness and equality (10), we get

$$\inf\{p(x_{n_k}, u) + p(x_{n_k}, x_{n_k+1}) : k \in \mathbb{N}\} = 0.$$

Which is a contradiction. Thus, $Tu = u$.

Case (ii). Now if T is continuous, we have

$$q(u, Tu) = \lim_{n\to\infty} q(x_n, Tu) = \lim_{n\to\infty} q(Tx_{n-1}, Tu) = q(Tu, Tu) = 0.$$

Hence, $u = Tu$.

Case (iii). We have

$$\lim_{n\to\infty} p(Tx_n, u) = \lim_{n\to\infty} p(x_{n+1}, u) = \lim_{n\to\infty} p(x_n, u).$$

Hence $Tu = u$

Moreover, if inequality (9) is true for each x, without the condition $q(Tx, T^2x) \neq 0$, similar to Theorem 1 we can prove $p(u,u) = 0$. □

If $f : X \to X$ and $F(f)$ is the set of all fixed points of f, then in a general case $F(f) \neq F(f^n)$. Abbas and Rhoades [1] studied cases when $F(f) = F(f^n)$ for each $n \in \mathbb{N}$, that is, when f has a property P. The following theorem extends and improves Theorem 3.1 of [1].

Theorem 3.6. *Let (X, q) be a complete quasi-metric space with quasi w-distance p on X. Let $T : X \to X$ be a given mapping. Suppose that there exist $\xi \in \mathcal{L}$ and $\theta \in \Theta$ such that for all $x \in X$ with $q(Tx, T^2x) \neq 0$,*

$$\xi\left(\theta(p(Tx, T^2x)), \theta(p(x, Tx))\right) \geq 1. \tag{11}$$

Then T has property P.

Proof. From Theorem 3.5, $F(T) \neq \emptyset$. Now we prove that T has property P. Obviously $F(T) \subseteq F(T^n)$, for each natural number n. We prove by induction $F(T^n) \subseteq F(T)$. First for $n = 2$, if $F(T^2) \neq F(T)$, then there is $x \in X$ such that $T^2x = x \neq Tx = T^3x$. Therefore $q(Tx, T^2x), q(T^2x, T^3x) > 0$ and by inequality (11) we conclude that

$$1 \leq \xi(\theta(p(T^2x, T^3x)), \theta(p(Tx, T^2x)) < \frac{\theta(p(Tx, T^2x))}{\theta(p(T^2x, T^3x))} = \frac{\theta(p(Tx, x))}{\theta(p(x, Tx))}$$

and

$$1 \leq \xi(\theta(p(Tx, T^2x)), \theta(p(x, T^x)) < \frac{\theta(p(x, Tx))}{\theta(p(Tx, T^2x))} = \frac{\theta(p(x, Tx))}{\theta(p(Tx, x))}$$

or equivalently

$$\theta(p(x, Tx)) < \theta(p(Tx, x)) \quad \text{and} \quad \theta(p(Tx, x)) < \theta(p(x, Tx)),$$

which is a contradiction. So $F(T) = F(T^2)$.

Now let $F(T) = F(T^2) = \ldots = F(T^{n-1})$ but $F(T) \neq F(T^n)$. Then there is $x \in X$ such that $T^nx = x$ but $x \neq T^ix$ for each $1 \leq i < n$. Then obviously $T^ix \neq T^jx$, for each distinct $1 \leq i, j \leq n$ and similar to case $n = 2$ we conclude that $\theta(p(T^jx, T^{j+1}x)) < \theta(p(T^{j-1}x, T^jx))$, for each $1 \leq j \leq n$, where $T^0x = x$. Now we have

$$\begin{aligned}\theta(p(x, Tx) = \theta(p(T^nx, T^{n+1}x)) &< \theta(p(T^{n-1}x, T^nx)) \\ &< \ldots < \theta(p(Tx, T^2x)) < \theta(p(x, Tx));\end{aligned}$$

which is a contradiction. Therefore for each natural number n we have $F(T) = F(T^n)$. $\square$

For $x \in X$, $T : X \to X$, $O(x, \infty) = \{x, Tx, T^2x, ...\}$ is called the orbit of x. The mapping $G : X \to [0, \infty)$ is T-orbitally lower semicontinuous at x if for any sequence $\{x_n\}$ in $O(x; \infty)$ which is convergent to x, we have $G(x) \leq \liminf G(x_n)$. The following theorem extends Theorem 2.1 of ([19]) and implies Theorem 3.7 of [31].

Theorem 3.7. *Let (X, q) be a complete quasi-metric space with quasi w-distance p on X. Let $T : X \to X$ be a given mapping. Suppose that there exist $\xi \in \mathcal{L}$, $\theta \in \Theta$, and $x \in X$ such that for all $y \in O(x; \infty)$ with $p(Ty, T^2y) \neq 0$, $\xi\big(\theta(p(Ty, T^2y)), \theta(p(y, Ty))\big) \geq 1$. Then*

(i) $\lim T^n x = z$ exists and for some $0 \leq k < 1$

$$p(T^n x, z) \leq \frac{k^n}{1-k} p(x, Tx) \quad (n \geq 1).$$

(ii) $p(z, Tz) = 0$ if and only if $G(x) = p(x, Tx)$ is T-orbitally lower semicontinuous at z.

Proof. (i) The proof of Theorem 3.5 implies the existence of the limit. Now by lower semi continuity of p, we have

$$p(T^n x, z) \leq \liminf p(T^n x, T^m x),$$

and similar to end of the of proof of Theorem 3.1 we can show that

$$\begin{aligned} p(T^n x, z) &\leq \liminf_{m\to\infty} p(T^n x, T^m x) \\ &\leq \liminf_{m\to\infty} \frac{k^n - k^m}{1-k} p(x, Tx) \\ &\leq \frac{k^n}{1-k} p(x, Tx), \end{aligned}$$

for some $k \in [0, 1)$.

(ii) If $p(z, Tz) = 0$, then obviously $p(z, Tz) = 0 < p(x_n, x_{n+1}) + \epsilon$, for each $\epsilon > 0$ and each natural number n. Conversly, if $G(x) = p(x, Tx)$ is T-orbitally lower semicontinuous at z, then for each $\epsilon > 0$, there is n_0, such that

$$p(z, Tz) < p(x_n, x_{n+1}) + \epsilon, \quad (n \geq n_0).$$

But in proof of Theorem 3.5, we show that $\{x_n\}$ is right Cauchy sequence and so $\lim p(x_n, x_{n+1}) = 0$. Therefore $p(z, Tz) \leq \epsilon$, for each ϵ. Hence $p(z, Tz) = 0$. $\square$

4 Examples

In this section, we will give some examples to illustrate our results.

Example 4.1. Let $k \in (0,1)$, G be a locally compact group and $X = L^1(G)$. Then $L^1(G)$ with the following (quasi) metric is a complete (quasi) metric space.

$$q(f,g) = \|f-g\|_1, \quad (f,g \in L^1(G)).$$

Where $\|f\|_1 = \int_G |f(x)| d\lambda(x)$, for each $f \in L^1(G)$, and λ is the Haar measure on G. Consider the modular function $\Delta : G \to (0,1)$ and $x \in G$ such that $0 < \Delta(x^{-1}) \leq k^{\frac{1}{k}}$ (for more details about these concepts see [18]). For an arbitrary $h \in L^1(G)$ define

$$\begin{array}{rcl} T : L^1(G) & \to & L^1(G) \\ f & \mapsto & R_x(f-h) \end{array}$$

where $R_x f(y) = f(yx)$, for each $f \in L^1(G)$ and $y \in G$.

Now for the quasi w-distance $p(f,g) = \|g\|_1$ on the quasi metric space $L^1(G)$ and $\theta : [0,\infty) \to [1,\infty)$, defined by $\theta(t) = e^{t^k}$ and

$$\xi(t,s) = \frac{s^k}{t} \quad \text{or} \quad \xi(t,s) = \begin{cases} 1, & (t,s) = (1,1), \\ \frac{s}{2t}, & t > s, \\ \frac{s^k}{t}, & otherwise, \end{cases}$$

we can see that $\xi\left(\theta(p(Tf, T^2 f)), \theta(p(f, Tf))\right) \geq 1$, for each f with $Tf \neq T^2 f$. Therefore since T is continuous, it has a fixed point. Note that if $h = 0$, then 0 is the fixed point of T.
For example if $k = \frac{1}{2}$, then $\Delta(x^{-1}) \leq \frac{1}{4}$, $\theta(t) = e^{\sqrt{t}}$ and $\xi(t,s) = \frac{\sqrt{s}}{t}$.

Example 4.2. Let $(X, \leq)$ be a an order set with a norm $\|.\|$ (such as $\mathbb{R}$) and let $T : X \to X$ be a map such that for some $k \in (0,1)$,

$$k^{\frac{1}{k}}(1 + \|Tx\|) \geq 1 + \|T^2 x\|,$$

for each x with $Tx \neq T^2 x$. Define a quasi metric q and a w-distance p on X as follows.

$$q(x,y) = \begin{cases} 0, & x = y, \\ 1, & x < y, \\ 2, & y < x, \end{cases} \quad p(x,y) = 1 + \|y\|, \quad (x, y \in X).$$

Obviously a sequence $\{x_n\}$ in (X,q) is a Cauchy sequence if there is $N \in \mathbb{N}$ such that for every $n \geq N$, $x_n = x_N$. Therefore each map on X is continuous . Also

$$\inf\{p(x,w) + p(x,Tx) : x \in X\} \geq 1,$$

for every $w \in X$.
Note that $k^{\frac{1}{k}}(1 + \|Tx\|) \geq 1 + \|T^2x\|$ is equivalent to $k(1+\|Tx\|)^k \geq (1+\|T^2x\|)^k$ and so $e^{k(1+\|Tx\|)^k} \geq e^{(1+\|T^2x\|)^k}$.
Consider θ and ξ as the latter example. Then we have for each x, with $Tx \neq T^2x$

$$\begin{aligned}
\xi\left(\theta(p(Tx,T^2x)), \theta(p(x,Tx))\right) &= \frac{\theta(p(x,Tx))^k}{\theta(p(Tx,T^2x))} \\
&= \frac{e^{kp(x,Tx)^k}}{e^{kp(Tx,T^2x)}} \\
&= \frac{e^{k(1+\|Tx\|)^k}}{e^{(1+\|T^2x\|)^k}} \\
&\geq 1
\end{aligned}$$

Therefore T has a fixed point.

Example 4.3. Consider a set $X = \{x_n; n = 0,1,2,\cdots\}$ and a map $T : X \to X$ as $Tx_n = x_{10n}$. Define a quasi metric q and a w-distance p on X as follows.

$$q(x_n,x_m) = \begin{cases} 0, & n = m, \\ \frac{1}{n} - \frac{1}{m}, & m > n, m,n \neq 0, \\ \frac{2}{m} - \frac{1}{n}, & m < n, m,n \neq 0, \\ \frac{1}{n}, & m = 0, n \neq 0, \\ \frac{1}{m}, & n = 0, m \neq 0, \end{cases}$$

$$p(x_n,x_m) = \begin{cases} \frac{1}{n} + \frac{1}{m}, & m,n \neq 0, \\ 1, & n = 0 \text{ or } m = 0. \end{cases}$$

Note that with this quasi metric, each sequence is a Cauchy sequence and convergent to x_0. Let $\theta(t) = e^{\sqrt{t}}$ and $\xi(t,s) = \frac{s}{t\varphi(s)}$ for all $s,t \geq 1$ where $\varphi : [1,+\infty) \to [1,\infty)$ is nondecreasing and lower semi-continuous such that $\varphi^{-1}(\{1\}) = 1$. Then similar to latter we can show that for each $x \neq x_0$, $\xi\left(\theta(p(Tx,T^2x)), \theta(p(x,Tx))\right) \geq 1$.

5 Application

Consider the following integrals.
(I) (non linear) Feredholm integral (second Kind):

$$f(x) = \phi(x) + \lambda \int_a^b K(x,t)\psi(f(t))dt.$$

(II) (non linear) Volterra integral (second Kind):

$$\phi(x) = f(x) - \int_a^x K(x,t)\psi(f(t))dt$$

.

Example 5.1. Let $X = C^+([a,b])$ be the set of all upper semi-continuous functions on $[a,b]$ or $X = C([a,b])$. Then

$$q(f,g) = \begin{cases} \min\{\sup\{g(x) - f(x); x \in [a,b]\}, 1\}, & f \leq g, \\ 1, & otherwise, \end{cases}$$

is a quasi metric on X which is not a metric. Also $q(f,g) = \|f - g\|$, where $\|f\| = \sup\{f(x); x \in [a,b]\}$, is a quasi metric which is a metric.

For finding a solution in $X = C([a,b])$ or $X = C^+([a,b])$ (with each of the latter quasi metrics) for (I) or (II) define $T : C^+([a,b]) \to C^+([a,b])$ with
(I) $Tf(x) = \phi(x) + \lambda \int_a^b K(x,t)\psi(f(t))dt$.
(II) $Tf(x) = \phi(x) + \lambda \int_a^x K(x,t)\psi(f(t))dt$.
Then we have the following theorem.

Theorem 5.2. *If there is $k \in (0,1)$ such that for each $f \in X$ with $Tf \neq f$ we have $k^{\frac{1}{k}}\|Tf\| \geq \|T^2 f\|$, then under each of the following conditions (I) (similarly (II)) has a solution. That is there is $f \in X$ which satisfies in (I) (similarly (II)).*

(i) ψ is continuous.

(ii) $\phi = 0$ and $\psi(0) = 0$.

(iii) $\|g\| \neq 0$ for every $g \in X$ with $g \neq Tg$ or $\|Tf\| \neq 0$ for each $f \in X$

Proof. We only prove for (I). For (II) we should only put x insteed of b in the upper bound of integral. Put $p(f,g)=\|g\|$, $\theta(t)=e^{t^k}$, $\xi(t,s)=\frac{s^k}{t}$. Then for each $f\in X$ with $q(Tf,T^2f)\neq 0$ we have $Tf\neq T^2f$. Therefore

$$\phi(x)+\lambda\int_a^b K(x,t)\psi(f(t))dt \neq \phi(x)+\lambda\int_a^b K(x,t)\psi(Tf(t))dt \quad (x\in[a,b]).$$

So $f\neq Tf$. This implies that $k^{\frac{1}{k}}\|Tf\|\geq\|T^2f\|$. i. e. $k^{\frac{1}{k}}(p(f,Tf))\geq p(Tf,T^2f)$ or equivalently $k(p(f,Tf))^k\geq p(Tf,T^2f)^k$. Then since e^t is increasing we have $e^{k(p(f,Tf))^k}\geq e^{p(Tf,T^2f)^k}$ That is

$$\begin{aligned}\xi(\theta(p(Tf,T^2f)),\theta(p(f,Tf)) &= \frac{\theta(p(f,Tf))^k}{\theta(p(Tf,T^2f))}\\ &= \frac{e^{kp(f,Tf)^k}}{e^{p(Tf,T^2f)^k}}\\ &\geq 1\end{aligned}$$

Now if ψ is continuous, then T is continuous and if $\phi=0$ and $\psi(0)=0$, then $\inf\{p(f,g)+p(f,Tf):f\in X\}\geq\|g\|>0$ for every $g\in X$ with $g\neq Tg$. Also if (iii) is true, then $\inf\{p(f,g)+p(f,Tf):f\in X\}\geq\|g\|>0$ for every $g\in X$ with $g\neq Tg$. That is in each case the conditions of Theorem 3.5 are valid. Therefore T has a fixed point $u\in X$. Obviously this u is the solution of the integral equation(I) or similarly (II). □

Acknowledgements
This research was supported by a grant from University of Bojnord (No 98/ 367/6169).

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