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Original Research Paper

Translation Hypersurfaces in Lorentz-Minkowski Spaces Satisfying $L_{n-1}G = AG$

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Abstract. In this paper, we give a complete classification of the translation hypersurfaces in the $(n+1)$-dimensional Lorentz-Minkowski space whose Gauss map G satisfies the condition $L_{n-1}G = AG$ where L_{n-1} is the linearized operator of the first variation of the the Gauss-Kronecker curvature of the hypersurface and $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is a constant matrix.

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1 Introduction

Inspired by finite type theory, was first proposed by Bang-Yen Chen in [2], S. M. B. Kashani naturally introduced the notion of L_k-finite type hypersurface in [7]. The L_k operator of an Euclidean hypersurface M^n , $0 \leq k \leq n-1$, is a linearized operator of the $(k+1)$th mean curvature (see [26]), which is a generalization of the Laplace operator $\Delta = L_0$ and $L_1 = \Box$ is called the Cheng-Yau operator introduced in [4] for the study of hypersurfaces with constant scalar curvature in Euclidean spaces. This operator is defined by $L_k(f) = \mathrm{tr}(P_k \circ \mathrm{Hess} f)$ for any $f \in C^\infty(M)$, where P_k is the kth Newton transformation. We refer the reader to the surveys [15, 16, 17, 18, 22, 23, 24] on the progress in this topic. Following the ideas of Chen and Piccinni in [3], the notion of L_k-finite type can be generalized to the Gauss map of a hypersurface of a pseudo-Euclidean space. From this point of view, some classes of surfaces in $\mathbb{E}^3$ with L_1-pointwise 1-type Gauss map were recently studied in [11, 12, 13, 25, 27]. Generalizing these study results to the hypersurfaces with L_k-pointwise 1-type Gauss map were presented in [20, 21] by the first author.

Following the ideas from Garay in [6], it is natural to study the hypersurfaces in pseudo-Euclidean $(n+1)$-space whose Gauss map G is of coordinate L_k-finite type, i.e. G satisfies the following condition

$$L_k G = AG, \qquad 0 \leq k \leq n-1, \tag{1}$$

where $A \in \mathbb{R}^{(n+1)\times(n+1)}$. The study on the different types of hypersufaces in pseudo-Euclidean space satisfying the condition (1) when $k = 0$ (i.e. the condition $\Delta G = AG$) over the past two decades has shown that these kinds of hypersurfaces which are not minimal are rare, they are either the open parts of the hyperspheres or the open parts of the generalized circular cylinders (see [1, 5, 10]). There has been recent interest in studying surfaces in $\mathbb{E}^3$ satisfying the condition (1) when $k = 1$ (i.e. the condition $\Box G = L_1 G = AG$, where $A \in \mathbb{R}^{3\times 3}$). The results state [8, 9] that an element of the above family of surfaces which is not flat is either an open part of a sphere or an open part of a circular cylinder.

Here, by extending this result to higher-dimensional pseudo-Euclidean space, we consider an important class of hypersurfaces, namely, translation hypersurfaces in the Lorentz-Minkowski space, $\mathbb{L}^{n+1}$, we give a

classification of these hyppersurfaces satisfying the condition (1) when $k = n-1$. In view of the history of this issue in [19], the authors completely classified hypersurfaces in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ whose position vector Ψ satisfying $L_k\Psi = A\Psi + b$. They have proved that the only hypersurfaces satisfying this condition are one of the following hypersurfaces in $\mathbb{L}^{n+1}$
(a) a hypersurface with zero $(k+1)$th mean curvature.
(b) an open piece of the totally umbilical hypersurface $\mathbb{S}_1^n(r)$ or $\mathbb{H}^n(-r)$.
(c) an open piece of a generalized cylinder $\mathbb{S}_1^m(r) \times \mathbb{R}^{n-m}$, with $k+1 \leq m \leq n-1$ or $\mathbb{L}^m \times \mathbb{S}^{n-m}(r)$, with $k+1 \leq n-m \leq n-1$.

In the particular case where the hypersurfaces considered are translation, when $k = n-1$, we recover from the above classification result, then M is congruent to a cylinder, and hence Gauss-Kronecker curvature $H_n = 0$. We would like to remark that we get the same result. Our main theorem is:

Theorem 1.1. *Let M be a translation hypersurface in $\mathbb{L}^{n+1}$ whose Gauss map G satisfies $L_{n-1}G = AG$, where $A \in \mathbb{R}^{(n+1)\times(n+1)}$. Then M is congruent to a cylinder, and hence Gauss-Kronecker curvature $H_n = 0$.*

Theorem 1.1 is a natural continuation of the classification theorem that has been presented by Yoon in [28] for translation surfaces in Minkowski 3-space satisfying the condition $\Delta G = AG$ where $A \in \mathbb{R}^{3\times3}$. We obtain the following interesting consequence of Theorem 1.1 when $n = 2$.

Corollary 1.2. *Let M be a translation surface in $\mathbb{L}^3$ whose Gauss map G satisfies $L_1G = \Box G = AG$, where $A \in \mathbb{R}^{3\times3}$. Then M is congruent to a cylinder, and hence M is flat.*

2 Preliminaries and basic results

In this section, we recall the basic definitions and results from [14, 19].

Let $\mathbb{L}^{n+1}$ be the $(n+1)$-dimensional Lorentz-Minkowski space, i.e., the space $\mathbb{R}^{n+1}$ equipped with the following metric

$$ds^2 = -dx_1^2 + dx_2^2 + \cdots + dx_{n+1}^2,$$

where $(x_1, \ldots, x_{n+1})$ shows the natural coordinates of $\mathbb{R}^{n+1}$.

Let $\psi : M \to \mathbb{L}^{n+1}$ be an isometric immersion from an orientable hypersurface M to $\mathbb{L}^{n+1}$ with Gauss map G, $\langle G, G\rangle = \varepsilon = \pm 1$, where $\varepsilon = 1$ or $\varepsilon = -1$ according to M is equipped with a Lorentzian or Riemannian metric, respectively. Denote by S, ∇ and $\bar{\nabla}$ the shape operator with respect to G, the Levi-Civita connection on M, and the usual flat connection on $\mathbb{L}^{n+1}$, respectively.

The characteristic polynomial $\mathcal{Q}_s(t)$ of the shape operator S is given by

$$\mathcal{Q}_s(t) = \det(tI - S) = \sum_{k=0}^{n} a_k t^{n-k}, \quad \text{with } a_0 = 1.$$

The k-th mean curvature of M is defined by

$$\binom{n}{k} H_k = (-\varepsilon)^k a_k, \quad H_0 = 1,$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. In particular, when $k = 1$,

$$nH_1 = -\varepsilon a_1 = \varepsilon \mathrm{tr}(S),$$

we see that H_1 is nothing but the mean curvature H of M. When $k = n$, $H_n = (-\varepsilon)^n a_n = (-\varepsilon)^n \det(S)$ is called the Gauss-Kronecker curvature of M.

The k-th Newton transformation of M is the operator $P_k : \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$P_k = \sum_{j=0}^{k} a_{k-j} S^j,$$

by Cayley-Hamilton theorem, we see that $P_n = 0$.

Related to the Newton transformation P_k, the second-order linear differential operator $L_k : C^\infty(M) \to C^\infty(M)$ is defined by

$$L_k(f) = \mathrm{tr}(P_k \circ \nabla^2 f),$$

where, $\nabla^2 f : \mathfrak{X}(M) \to \mathfrak{X}(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and is given by

$$\langle \nabla^2 f(V), W\rangle = \langle \nabla_V(\nabla f), W\rangle, \qquad V, W \in \mathfrak{X}(M).$$

We can naturally extend the definition of the operator L_k from functions to vector functions $F = (f_1, \dots, f_{n+1}),\ f_i \in C^\infty(M)$, as follows

$$L_k F = (L_k f_1, \dots, L_k f_{n+1}).$$

Invoking the formula (18) of [19], we have

$$L_k G = -\varepsilon C_k \nabla H_{k+1} - \varepsilon C_k (nHH_{k+1} - (n-k-1)H_{k+2})G,$$

where $C_k = (-\varepsilon)^k \binom{n}{k+1}$. Therefore for $k = n-1$, we obtain that

$$L_{n-1} G = (-\varepsilon)^n \nabla H_n + (-\varepsilon)^n nHH_n G.$$

Then it is obvious that every hypersurface with vanishing Gauss-Kronecker curvature satisfies the condition $L_{n-1}G = AG$, with $A = 0$. Also, it is easy to see that the hyperquadric $M_{(a)=\{x\in\mathbb{L}^{n+1}|\langle x,x\rangle=\varepsilon a^2\}}$, $\varepsilon = \mp 1$, satisfies the condition $L_{n-1}G = AG$, with $A = \frac{\varepsilon}{a^{n+1}} I$.

Let $\psi : M \to \mathbb{L}^{n+1}$ be a translation hypersurface in $\mathbb{L}^{n+1}$. Then M is parameterized by

$$\psi(x_1, \dots, x_n) = (x_1, \dots, x_n, f_1(x_1) + \dots + f_n(x_n)),$$

where $f_1, \dots, f_n$ are smooth functions of the variables $x_1, \dots, x_n$, respectively.

The following lemma plays an essential role in the proof of Theorem 1.1.

Lemma 2.1. *Let $\psi : M \to \mathbb{L}^{n+1}$ be a translation hypersurface in $\mathbb{L}^{n+1}$ with the Gauss map G, defined as*

$$\psi(x_1, \dots, x_n) = (x_1, \dots, x_n, f_1(x_1) + \dots + f_n(x_n)).$$

The mean and the Gauss-Kronecker curvatures of M are given by

$$H = \frac{-1}{ng\sqrt{g}}\Big(f_1''\big(1 + \sum_{i=2}^{n} {f_i'}^2\big) + \sum_{i=2}^{n} f_i''\big({f_1'}^2 - 1 - \sum_{\substack{2\le j\le n \\ j\neq i}} {f_j'}^2\big)\Big), \tag{2}$$

$$H_n = \frac{(-\varepsilon)^{n-1}}{g\sqrt{g^n}}(f_1'' \cdots f_n''), \tag{3}$$

where ε denotes the sign of G, $f_i' = \frac{df_i}{dx_i}$ and $g = \varepsilon(-{f_1'}^2 + {f_2'}^2 + \dots + {f_n'}^2)$.

Proof. The matrix of the metric $\mathbf{g}$ has the following form

$$\mathbf{g} = (g_{ij}) = \begin{bmatrix} -1 + {f'_1}^2 & f'_1 f'_2 & \dots & f'_1 f'_n \\ f'_2 f'_1 & 1 + {f'_2}^2 & \dots & f'_2 f'_n \\ \vdots & \vdots & \vdots & \vdots \\ f'_n f'_1 & f'_n f'_2 & \dots & 1 + {f'_n}^2 \end{bmatrix}. \tag{4}$$

An easy calculation shows that the Gauss map G of the hypersurfaces of M satisfies

$$G = (G_1, \dots, G_{n+1}) = \frac{1}{\sqrt{g}}(f'_1, -f'_2, \dots, -f'_n, 1), \tag{5}$$

where $\langle G, G \rangle = \varepsilon$ and $g = \varepsilon \det(g_{ij})$. Then, the second fundamental form Π_{ij} of M is given by

$$\Pi_{ij} = \langle G, \frac{\partial^2 \psi}{\partial x_i \partial x_j} \rangle = \frac{1}{\sqrt{g}} \delta_i^j f''_i,$$

therefore, the matrix of Π is diagonal

$$\Pi = (\Pi_{ij}) = \frac{1}{\sqrt{g}} \mathrm{diag}(f''_1, \dots, f''_n).$$

Then, we have $S = \mathbf{g}^{-1}\Pi$, where S denotes the shape operator of M. Hence, by easy computation (2) and (3) follow from the definitions of H and H_n. $\square$

3 Proof of the Theorem 1.1

We now assume that the hypersurface M satisfies the condition $L_{n-1}G = AG$, where $A = (a_{ij}) \in \mathbb{R}^{(n+1)\times(n+1)}$. Then using (5), we get

$$L_{n-1}G = (L_{n-1}G_1, \dots, L_{n-1}G_{n+1}) = \frac{1}{\sqrt{g}}(A_1, \dots, A_{n+1}), \tag{6}$$

where A_i is a polynomial in n variables $f'_1, \dots, f'_n$. By using (2) and (3), we can write

$$H = \frac{1}{g\sqrt{g}}(f''_1 \mathbf{H} + \tilde{\mathbf{H}}), \quad H_n = \frac{1}{g\sqrt{g^n}}(f''_1 \mathbf{H}_n),$$

where $\mathbf{H}$ is a polynomial in $n-1$ variables $f_2', \dots, f_n'$, $\tilde{\mathbf{H}}$ is a polynomial in $2n-1$ variables $f_1', \dots, f_n', f_2'', \dots, f_n''$ and $\mathbf{H}_n$ is a polynomial in $n-1$ variables $f_2'', \dots, f_n''$. Also, by using (4) we have

$$\mathbf{g}^{-1} = (g_{ij})^{-1} = (g^{ij}) = \frac{1}{g}(\mathbf{g}^{ij}), \tag{7}$$

where $\mathbf{g}^{ij}$ is a polynomial in n variables $f_1', \dots, f_n'$. From (3), we obtain

$$\frac{\partial H_n}{\partial x_1} = \frac{1}{g^2\sqrt{g^n}}(f_1'''\mathbf{H}_{n_1} + f_1''^2\tilde{\mathbf{H}}_{n_1}), \qquad \frac{\partial H_n}{\partial x_i} = \frac{1}{g^2\sqrt{g^n}} f_1''\mathbf{H}_{n_i}, \tag{8}$$

where $\mathbf{H}_{n_1}$, $\tilde{\mathbf{H}}_{n_1}$ and $\mathbf{H}_{n_i}$ are polynomials in $2n-1$ variables $f_1', \dots, f_n'$, $f_2'', \dots, f_n''$.

Moreover, by using (7) and (8), the gradient of H_n is given by

$$\begin{aligned}\nabla H_n = \frac{1}{g^3\sqrt{g^n}}(P_1 f_1''' + \tilde{P}_1 f_1''^2, P_2 f_1'', \dots,\\ P_n f_1'', P_{n+1} f_1''' + \tilde{P}_{n+1} f_1''^2 + Q_{n+1} f_1''),\end{aligned} \tag{9}$$

where $P_1 = (\sum_{i=1}^n \mathbf{g}^{i1})\mathbf{H}_{n_1}$, $\tilde{P}_1 = (\sum_{i=1}^n \mathbf{g}^{i1})\tilde{\mathbf{H}}_{n_1}$, $P_{n+1} = f_1' P_1$, $P_j = (\sum_{i=1}^n \mathbf{g}^{ij})\mathbf{H}_{n_j}$, $\tilde{P}_{n+1} = f_1'\tilde{P}_1$ and $Q_{n+1} = \sum_{i=2}^n P_i f_i'$. Substituting (2), (3), (5) and (9) into (2), and applying (6), we find

$$\langle L_{n-1}G, e_1\rangle = \frac{(-\varepsilon)^n}{g^2\sqrt{g^{n+1}}}(P_1 f_1''' + (\tilde{P}_1 + n\mathbf{H}\mathbf{H}_n f_1') f_1''^2 + n\tilde{\mathbf{H}}\mathbf{H}_n f_1') f_1'' = A_1, \tag{10}$$

$$\langle L_{n-1}G, e_2\rangle = \frac{(-\varepsilon)^n}{g^2\sqrt{g^{n+1}}}(-n\mathbf{H}\mathbf{H}_n f_2' f_1''^2 + (P_2 - n\tilde{\mathbf{H}}\mathbf{H}_n f_2') f_1'') = A_2, \tag{11}$$

$$\begin{aligned}\langle L_{n-1}G, e_{n+1}\rangle = \frac{(-\varepsilon)^n}{g^2\sqrt{g^{n+1}}}(P_{n+1} f_1''' + (\tilde{P}_{n+1} + n\mathbf{H}\mathbf{H}_n) f_1''^2 +\\ (Q_{n+1} + n\tilde{\mathbf{H}}\mathbf{H}_n) f_1'') = A_{n+1}.\end{aligned} \tag{12}$$

(10) and (12) imply that

$$P f_1''^2 + Q f_1'' = \sqrt{g^{n+1}} R, \tag{13}$$

where we put

$$\begin{aligned} P &= P_1(\tilde{P}_{n+1} + n\mathbf{H}\mathbf{H}_n) - P_{n+1}(\tilde{P}_1 + n\mathbf{H}\mathbf{H}_n f_1'), \\ Q &= P_1(Q_{n+1} + n\tilde{\mathbf{H}}\mathbf{H}_n) - n\tilde{\mathbf{H}}\mathbf{H}_n f_1' P_{n+1}, \\ R &= (-\varepsilon)^n g^2 (P_1 A_{n+1} - P_{n+1} A_1). \end{aligned}$$

Also, it follows from (11)

$$\tilde{P} f_1''^2 + \tilde{Q} f_1'' = \sqrt{g^{n+1}}\tilde{R}, \tag{14}$$

where we set

$$\begin{aligned} \tilde{P} &= -n\mathbf{H}\mathbf{H}_n f_2', \\ \tilde{Q} &= P_2 - n\tilde{\mathbf{H}}\mathbf{H}_n f_2', \\ \tilde{R} &= (-\varepsilon)^n g^2 A_2. \end{aligned}$$

In case

$$P\tilde{Q} - Q\tilde{P} = 0,$$

we see that the function f_1' satisfies a nontrivial polynomial whose coefficients depend only on the functions $f_2', \ldots, f_n'$ and theirs derivatives. Thus, f_1' must be constant.

In case

$$P\tilde{Q} - Q\tilde{P} \neq 0,$$

from (13) and (14) we have

$$g^{n+1}(P\tilde{R} - \tilde{P}R)^4 = (P\tilde{Q} - Q\tilde{P})^2 (R\tilde{Q} - Q\tilde{R})^2,$$

again we conclude that f_1' must be constant.

By the constancy of f_1', we get $H_n = 0$. Also, it follows that f_1' is linear, that is $f_1 = ax_1 + b$. It immediately follows that

$$\begin{aligned} &\psi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f_1(x_1) + \cdots + f_n(x_n)) \\ &\quad = (x_1, \ldots, x_n, ax_1 + b + f_2(x_2) + \cdots + f_n(x_n)) \\ &\quad = x_1(1, 0, \ldots, 0, a) + (0, x_2, \ldots, x_n, b + f_2(x_2) + \cdots + f_n(x_n)), \end{aligned}$$

which implies that M is a cylinder. This completes the proof of Theorem 1.1.

4 Conclusion

In this work, we gave a partial answer to the following problem which arises in a natural way:

Which are the hypersurfaces in $\mathbb{L}^{n+1}$ satisfying the condition $L_kG = AG$, for some fixed $k = 0, 1, \ldots, n-1$, some constant matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$.

As a first attempt to solve this question we considerd an important class of hypersurfaces, namely, translation hypersurfaces and proved that the only translation hypersurface in $\mathbb{L}^{n+1}$ whose Gauss map G satisfies $L_{n-1}G = AG$, where $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is a cylinder.

We would hope to solve this problem for other important hypersurfaces in our future studies.

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