Cousin Complexes and Almost Flat Rings

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Abstract. Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring and T be a commutative strict algebra with unit element 1_T over R such that $\mathfrak{m}T \neq T$. We define almost exact sequences of T-modules and characterize almost flat T-modules. Moreover, we define almost (faithfully) flat homomorphisms between R-algebras T and W, where W has similar properties that T has as an R-algebra. By almost (faithfully) flat homomorphisms and almost flat modules, we investigate Cousin complexes of T and W-modules. Finally, for a finite filtration $\mathcal{F} = (F_i)_{i \geq 0}$ of length less than d of $\mathrm{Spec}(T)$ such that it admits a T-module X, we show that ${}^I\mathrm{E}^2_{p,q} := \mathrm{Tor}^T_p\left(M, \mathrm{H}^{d-q}\left(\mathcal{C}_T\left(\mathcal{F}, X\right)\right)\right) \overset{p}{\Rightarrow} \mathrm{H}_{p+q}(\mathrm{Tot}(\mathcal{T}))$ and ${}^{II}\mathrm{E}^2_{p,q} := \mathrm{H}^{d-p}\left(\mathrm{Tor}^T_q\left(M, \mathcal{C}_T\left(\mathcal{F}, X\right)\right)\right) \overset{p}{\Rightarrow} \mathrm{H}_{p+q}(\mathrm{Tot}(\mathcal{T}))$, where M is an any flat T-module and as a result we show that ${}^I\mathrm{E}^2_{p,q}$ and ${}^{II}\mathrm{E}^2_{p,q}$ are almost zero, when M is almost flat.

1   Introduction

Let R be a ring and $\mathfrak{m}$ be a unique maximal ideal of R. Throughout of this paper $(R, \mathfrak{m})$ is a d-dimensional Noetherian local ring with a system of parameters $\underline{x} := x_1, \ldots, x_d$ and T be a commutative strict algebra with unit element 1_T (not necessarily an integral domain) over R, i.e., $\mathfrak{m}T \neq T$. Then T is called equipped with a value map, if there is a map $v : T \longrightarrow \mathbb{R} \cup \{\infty\}$ such that satisfying the following conditions:

(i) $v(ab) = v(a) + v(b)$, for all $a, b \in T$;

(ii) $v(a + b) \geq \min\{v(a), v(b)\}$, for all $a, b \in T$;

(iii) $v(a) = \infty$ if $a = 0$.

Moreover, v is called *normalized* if $v(c) \geq 0$, for all $c \in T$ and $v(c) > 0$, for all non-unit $c \in T$. Let M be a module over an algebra T which is equipped with a value map v. Then M is called *almost zero* with respect to v, if $m \in M$ and $\varepsilon > 0$ are given, then there exists $b \in T$ such that $b \cdot m = 0$ and $v(b) < \varepsilon$. For almost ring theory we refer to [4].

A T-module M is called *almost Cohen-Macaulay* over R, if $H_{\mathfrak{m}}^i(M)$ is almost zero for all $i \neq d$, but $M/\mathfrak{m}M$ is not almost zero. This notion is considered by some authors, see [1, 3, 5, 6, 7, 9], for more details.

Let M and N be T-modules, then they are called *almost isomorphic*, if there is a T-homomorphism $f : M \longrightarrow N$ (or $g : N \longrightarrow M$) such that both ker f and coker$f$ (or both ker g and coker$g$) are almost zero and we call f (g) is an almost isomorphism. Moreover, M and N are in the same class, if there is a T-module L such that there exist T-homomorphisms $L \longrightarrow M$ and $L \longrightarrow N$, both of which are almost isomorphic in the above sense, which we denote by $M \approx N$.

Authors in [1] showed that $\mathrm{Tor}_n^T(H_{\mathfrak{m}}^d(T), M) \approx H_{\mathfrak{m}}^{d-n}(M)$ for all $n \geq 0$. The T-module M is said to be *almost flat*, if $\mathrm{Tor}_i^T(M, N) \approx 0$ for all $i > 0$ and all T-modules N. Moreover, M is called *almost faithfully flat*, if M is almost flat and for each T-module, N, $M \otimes_T N \approx 0$ implies that $N \approx 0$. If T is an almost Cohen-Macaulay algebra over a local ring $(R, \mathfrak{m})$ and M is an almost faithfully flat T-module, then $H_{\mathfrak{m}}^{d-i}(M) \approx H_{\mathfrak{m}}^{d-i}(T) \otimes_T M$ and M is almost Cohen-Macaulay [1, Theorem 4.3].

In Section 2, we consider almost flat modules over T-algebras defined over R. We investigate almost flat modules by defining almost

exact sequences and we show that this investigation is equivalent to the definition of the almost flat modules where defined in [1]. Moreover, we define (faithfully) almost homomorphism between R-algebras and by this definition, we give some results to transferring almost flatness property. In Section 3, we consider the Cousin complex $\mathcal{C}_{T_{\mathfrak{p}}}(M_W \otimes_T T_{\mathfrak{p}})$, $\mathcal{C}_{T_{\mathfrak{p}}}(M_{\mathfrak{p}} \otimes_T \varphi(T \setminus \mathfrak{p})^{-1}W)$ and we show that they are almost equivalent. Section 4 deals with Cousin complex defined by filtration and we show that ${}^{I}\mathrm{E}^2_{p,q}$ and ${}^{II}\mathrm{E}^2$ related to T, are almost zero. Finally, we ask some question related to almost flat modules and almost flat homomorphism.

2   Some Results on Almost Flat Modules and Homomorphisms

In this section, we give some results related to almost flat modules and almost flat homomorphisms. Moreover, we define almost flat and almost faithfully flat T-homomorphisms between T-modules. For any T-homomorphism of T-modules $f : M \longrightarrow N$ we denote the kernel and the image of f by $\ker f$ and $\mathrm{im} f$, respectively.

Remark 2.1. Let M and N be T-modules. If $M \approx N$, then there is a T-homomorphism $f : M \longrightarrow N$ or there is a T-homomorphism $g : N \longrightarrow M$ such that $\ker f \approx 0$ and $\mathrm{coker}\ f \approx 0$ or $\ker g \approx 0$ and $\mathrm{coker}\ g \approx 0$. In this paper, by $M \approx N$, we mean there are both T-homomorphisms $f : M \longrightarrow N$ and $g : N \longrightarrow M$ such that $f \circ g \approx i_N$ and $g \circ f \approx i_M$. Moreover, we say $f : M \longrightarrow M$ is almost isomorphic with $h : M \longrightarrow M$, if $\ker f \approx \ker h$, $\mathrm{im}\ f \approx \mathrm{im}\ h$ and we denote by $f \approx h$.

Definition 2.2. We say that a finite or infinite sequence of T- homomorphisms and T-modules

$$\cdots \longrightarrow M_{n-1} \xrightarrow{\varphi_n} M_n \xrightarrow{\varphi_{n+1}} M_{n+1} \longrightarrow \cdots ,$$

is an almost exact sequence if $\mathrm{im}\varphi_n$ and $\ker \varphi_{n+1}$ are almost isomorphic.

Remark 2.3. Let M and N be two T-modules. Then

(i) If a sequence $0 \longrightarrow M \xrightarrow{\varphi} N$ is almost exact, then $\ker \varphi$ is almost zero. Clearly, if φ is injective, then the above sequence becomes exact sequence.

(ii) Consider a sequence $M \xrightarrow{\psi} N \longrightarrow 0$. The kernel of $N \longrightarrow 0$ is N, hence, if the above sequence is almost exact sequence, then $\mathrm{im}\psi$ and the kernel of $N \longrightarrow 0$ i.e., N are almost isomorphic.

(iii) Consider a sequence $0 \longrightarrow M \xrightarrow{\varphi} N \longrightarrow 0$. If this sequence is exact, then by the above cases (i) and (ii), M and N are almost isomorphic.

Definition 2.4. A short almost exact sequence of T-homomorphisms and T-modules is an almost exact sequence of the form

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} C \longrightarrow 0. \tag{1}$$

By Remark 2.3, (1), $\ker \varphi$ is almost zero, $\mathrm{im}\ \psi$ and C are almost isomorphic. We now give the following definition related to almost flat modules that we will show that this definition is equivalent to definition of almost flat modules in [1].

Definition 2.5. We say a T-module M is almost flat, whenever

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0 \tag{2}$$

is an almost exact sequence of T-modules, then

$$0 \longrightarrow M \otimes_T A \xrightarrow{i_M \otimes \varphi} M \otimes_T B \xrightarrow{i_M \otimes \psi} M \otimes_T C \longrightarrow 0 \tag{3}$$

is an almost exact sequence. A T-module M is called almost faithfully flat if (2) is almost exact if and only if (3) is almost exact sequence.

By the following result we show that the above definition and [1, Definition 4.1] are equivalent.

Proposition 2.6. *A T-module M is almost flat in sense of Definition 2.5 if and only if $\mathrm{Tor}_i^T(M, N) \approx 0$, for all $i > 0$ and all T-modules N.*

Proof. Let M be an almost flat T-module and N be an arbitrary T-module. Let $\mathbf{P}_N$ be a projective resolution of N as follows

$$\mathbf{P}_N : \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\nu} N \longrightarrow 0.$$

Almost flatness of M implies that

$$M \otimes_T \mathbf{P}_N : \longrightarrow M \otimes_T P_2 \xrightarrow{i_M \otimes d_2} M \otimes_T P_1 \xrightarrow{i_M \otimes d_1} M \otimes_T P_0 \xrightarrow{i_M \otimes \nu} M \otimes_T N \longrightarrow 0$$

is an almost exact sequence. Thus, $\ker(i_M \otimes d_n)$ is almost isomorphic with $\operatorname{im}(i_M \otimes d_{n+1})$. This means that $\frac{\ker(i_M \otimes d_n)}{\operatorname{im}(i_M \otimes d_{n+1})} \approx 0$. On the other hand, since $\operatorname{Tor}_n^T(M, N) = \operatorname{H}_n(M \otimes \mathbf{P}_N) = \frac{\ker(i_M \otimes d_n)}{\operatorname{im}(i_M \otimes d_{n+1})}$, we get that $\operatorname{Tor}_n^T(M, N) \approx 0$, for all $n > 0$. The converse is trivial. $\square$

Lemma 2.7. *Let I, J be two ideals of T and M be an almost flat T-module. Then $IM \cap JM \approx (I \cap J)M$.*

Proof. Consider the exact sequence $0 \longrightarrow I \cap J \longrightarrow T \longrightarrow T/I \oplus T/J \longrightarrow 0$. Since, M is almost flat, we obtain the following almost exact sequence of T-modules:

$$0 \longrightarrow M \otimes_T (I \cap J) \longrightarrow M \otimes_T T = M \longrightarrow M/IM \oplus M/JM \longrightarrow 0.$$

The kernel of $g : M \longrightarrow M/IM \oplus M/JM$ is equal to $IM \cap JM$ and the image of $f : M \otimes_T (I \cap J) \longrightarrow M$ is equal to $(I \cap J)M$. Since $\operatorname{im} f \approx \ker g$, we obtain that $IM \cap JM \approx (I \cap J)M$. $\square$

Definition 2.8. Let T and W be two algebras over R equipped with value maps v_T and v_W, respectively. We say a ring map $\varphi : T \longrightarrow W$ is almost flat, if W is almost flat as a T-module. Moreover, we say φ is almost faithfully flat, if W is an almost faithfully flat T-module.

Proposition 2.9. *Let T and W be two algebras with units 1_T and 1_W over R equipped with value maps v_T and v_W. Let $\varphi : T \longrightarrow W$ be an almost flat map and M be a finitely generated T-module. Then $\operatorname{Ann}_T(M)W \approx \operatorname{Ann}_W(M \otimes_T W)$.*

Proof. Let $m \in M$ and set $I = \operatorname{Ann}_T(m)$. Consider the exact sequence $0 \longrightarrow I \xrightarrow{i} T \xrightarrow{\psi} M \longrightarrow 0$ where $\psi(t) = tm$, for all $t \in T$. By

tensoring W to the above exact sequence, we have the following almost exact sequence

$$0 \longrightarrow I \otimes_T W \overset{i \otimes i_W}{\longrightarrow} T \otimes_T W = W \overset{\psi \otimes i_W}{\longrightarrow} M \otimes_T W \longrightarrow 0,$$

where $\psi \otimes i_W(w) = m \otimes w$, for all $w \in W$. Moreover, $\mathrm{im}(i \otimes i_W) = IW = \mathrm{Ann}_T(m)W$ and

$$
\begin{aligned}
\ker(\psi \otimes i_W) &= \{w \in W : \psi \otimes i_W(w)(m \otimes 1_W) \cdot w\} \\
&= \mathrm{Ann}_W(m \otimes 1_W).
\end{aligned}
$$

This implies that $\mathrm{Ann}_T(m)W \approx \mathrm{Ann}_W(m \otimes_T 1_W)$. Now, suppose that $\{m_1, m_2, \ldots, m_n\}$ be a set of generators of M. Then by the above obtained result we get that $\mathrm{Ann}_T(m_i)W \approx \mathrm{Ann}_W(m_i \otimes_T 1_W)$, for all $1 \leq i \leq n$. By setting $I_i = \mathrm{Ann}_T(m_i)$ and applying Lemma 2.7, we obtain $\bigcap_{i=1}^{n}(I_i W) \approx (\bigcap_{i=1}^{n} I_i) W$. This completes the proof. $\square$

Proposition 2.10. *Let T and W be two algebras with units 1_T and 1_W over R equipped with value maps v_T and v_W. Let $\varphi : T \longrightarrow W$ be an almost (faithfully) flat map. If M is an almost (faithfully) flat W-module, then M is an almost (faithfully) flat T-module.*

Proof. Let $0 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \longrightarrow 0$ be an almost exact sequence of T-modules. Since φ is almost flat, W is an almost flat T-module. This implies that

$$0 \longrightarrow W \otimes_T A \overset{i_W \otimes f}{\longrightarrow} W \otimes_T B \overset{i_W \otimes g}{\longrightarrow} W \otimes_T C \longrightarrow 0$$

is an almost exact sequence of T-modules. Since M is an almost flat W-module, by tensoring M to the above almost exact sequence, we have the following almost exact sequence

$$0 \longrightarrow M \otimes_W (W \otimes_T A) \overset{i_M \otimes (i_W \otimes f)}{\longrightarrow} M \otimes_W (W \otimes_T B)$$
$$\overset{i_M \otimes (i_W \otimes g)}{\longrightarrow} M \otimes_W (W \otimes_T C) \longrightarrow 0.$$

We can rewrite the above almost exact sequence as follows

$$0 \longrightarrow (M \otimes_W W) \otimes_T A \overset{(i_M \otimes i_W) \otimes f}{\longrightarrow} (M \otimes_W W) \otimes_T B$$
$$\overset{(i_M \otimes i_W) \otimes g}{\longrightarrow} (M \otimes_W W) \otimes_T C \longrightarrow 0.$$

Then by $M \otimes_W W \cong M$, we have

$$0 \longrightarrow M \otimes_T A \overset{i_M \otimes f}{\longrightarrow} M \otimes_T B \overset{i_M \otimes g}{\longrightarrow} M \otimes_T C \longrightarrow 0,$$

which is an almost exact sequence. Tracing this argument backwards, we can show that if φ is almost faithfully flat and if M is an almost faithfully W-module, then M is an almost faithfully T-module. $\square$

Lemma 2.11. *Let T, W be two algebras with units 1_T, 1_W, respectively over R equipped with value maps v_T, v_W and $\varphi : T \longrightarrow W$ be a ring homomorphism. If M is an almost (faithfully) flat T-module, then $M \otimes_T W$ is an almost (faithfully) flat W-module.*

Proof. Let $0 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \longrightarrow 0$ be an almost exact sequence of W-modules. Then by tensoring $M \otimes_T W$ to the above almost exact sequence, by a conclusion isomorphism $(M \otimes_T W) \otimes A = M \otimes_T A$ and by almost flatness of M as a T-module, we have the following almost exact sequence

$$0 \longrightarrow M \otimes_T A \overset{i_M \otimes f}{\longrightarrow} M \otimes_T B \overset{i_M \otimes g}{\longrightarrow} M \otimes_T C \longrightarrow 0.$$

Thus,

$$0 \longrightarrow (M \otimes_T W) \otimes_W A \overset{i_M \otimes f}{\longrightarrow} (M \otimes_T W) \otimes_W B$$
$$\overset{i_M \otimes g}{\longrightarrow} (M \otimes_T W) \otimes_W C \longrightarrow 0$$

is almost sequence and so $M \otimes_T W$ is an almost flat W-module. Tracing this argument backwards, we can show that if M is an almost faithfully T-module, then $M \otimes_T W$ is an almost faithfully W-module. $\square$

Proposition 2.12. *Let T, W be two algebras with units 1_T, 1_W, respectively over R equipped with value maps v_T, v_W and $\varphi : T \longrightarrow W$ be a ring homomorphism. If φ is an almost faithfully map, then M is an almost flat T-module if and only if $M_W = M \otimes_T W$ is an almost flat W-module.*

Proof. If M is an almost flat T-module, then by Lemma 2.11, M_W is an almost flat W-module.

Conversely, suppose that M_W is an almost flat W-module and $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an almost exact sequence of T-modules. From almost flatness of φ and W as a T-module, we have

$$0 \longrightarrow W \otimes_T A \longrightarrow W \otimes_T B \longrightarrow W \otimes_T C \longrightarrow 0$$

is an almost exact sequence. Almost flatness of M_W as a W-module implies that

$$0 \longrightarrow M_W \otimes_W W \otimes_T A \longrightarrow M_W \otimes_W W \otimes_T B \longrightarrow M_W \otimes_W W \otimes_T C \longrightarrow 0$$

is an almost exact sequence. Therefore, we can write the above sequence as follows

$$0 \longrightarrow W \otimes_T M \otimes_T A \longrightarrow W \otimes_T M \otimes_T B \longrightarrow W \otimes_T M \otimes_T C \longrightarrow 0,$$

which is an almost sequence. By almost faithfully flatness of W as a T-module, we obtaine that

$$0 \longrightarrow M \otimes_T A \longrightarrow M \otimes_T B \longrightarrow M \otimes_T C \longrightarrow 0$$

is an almost sequence of T-modules. This implies that M is almost flat. $\square$

3   Cousin Complexes and Almost Flat Ring Extensions

Let $\varphi : T \longrightarrow W$ be an almost faithfully flat homomorphism and M be a T-module. In this section, we investigate the relation between the Cousin complexes $\mathcal{C}_{T_{\mathfrak{p}}}(M_W \otimes_T T_{\mathfrak{p}})$ and $\mathcal{C}_{T_{\mathfrak{p}}}(M_{\mathfrak{p}} \otimes_T \varphi(T \setminus \mathfrak{p})^{-1}W)$. Let $\varphi : T \longrightarrow W$ be an almost flat homomorphism, I be an ideal of T and J be an ideal of W. Then we denote the ideals $\varphi(I)R$ and $\varphi^{-1}(J)$ by I^{φ} and J_{φ}, respectively. Let M be a T-module. Then we denote $M \otimes_T W$ by M_W.

Lemma 3.1. *Let T, W be two algebras over R equipped with value maps v_T, v_W and $\varphi : T \longrightarrow W$ be an almost flat R-algebras homomorphism and $\mathfrak{p} \in \mathrm{Spec}\,(T)$ such that $\mathfrak{p}^{\varphi} \neq W$. Then the induced R-algebras homomorphism $\tilde{\varphi} : T_{\mathfrak{p}} \longrightarrow \varphi(T \setminus \mathfrak{p})^{-1}W$ defined by $\tilde{\varphi}\left(\frac{t}{s}\right) = \frac{\varphi(t)}{\varphi(s)}$, for all $t \in T$ and $s \in T \setminus \mathfrak{p}$, is an almost faithfully R-algebras homomorphism.*

Proof. Clearly, $\varphi(T \setminus \mathfrak{p})^{-1}$ is a multiplicatively closed subset of W and $\tilde{\varphi}$ is an R-algebras homomorphism. Consider the following almost exact sequence of T-modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0. \tag{4}$$

Set $\varphi(T \setminus \mathfrak{p})^{-1}W = \tilde{W}$. By tensoring $\tilde{W}$ to the above almost exact sequence, we show that the following sequence of T-modules is an almost exact sequence:

$$0 \longrightarrow \tilde{W} \otimes_T A \xrightarrow{i_{\tilde{W}} \otimes f} \tilde{W} \otimes_T B \xrightarrow{i_{\tilde{W}} \otimes g} \tilde{W} \otimes_T C \longrightarrow 0. \tag{5}$$

Let $X \in \ker(i_{\tilde{W}} \otimes f)$. Then there are $w \in W$, $s \in \varphi(T \setminus \mathfrak{p})^{-1}$ and $a \in A$ such that $X = \frac{w}{\varphi(s)} \otimes a$. Since φ is almost flat, W is an almost flat T-module. Hence, for every $\varepsilon > 0$, there exists $b \in T$ such that $b \cdot w = \varphi(b)w = 0$ and $v_T(b) < \varepsilon$. This implies that $b \cdot X = b \cdot \left(\frac{w}{\varphi(s)} \otimes a \right) = \frac{\varphi(b)w}{\varphi(s)} \otimes a = 0$. Thus, $\ker(i_{\tilde{W}} \otimes f)$ is an almost zero T-module.

Now, define $F : \operatorname{im}(i_{\tilde{W}} \otimes g) \longrightarrow \tilde{W} \otimes_T C$ by $F\left(\frac{w}{\varphi(s)} \otimes x \right) = \frac{w}{\varphi(s)} \otimes g(x)$, for all $w \in W$, $s \in \varphi(T \setminus \mathfrak{p})^{-1}$ and $x \in B$. Clearly, F is a T-homomorphism and the coker F is almost zero. Moreover, according to almost flatness of W as a T-module and by the above reason, there exists $b \in T$ such that $b \cdot w = \varphi(b)w = 0$ and $v_T(b) < \varepsilon$, for every $\varepsilon > 0$. Let $\frac{w}{\varphi(s)} \otimes x \in \ker F$. Then

$$b \cdot F\left(\frac{w}{\varphi(s)} \otimes x \right) = F\left(\frac{\varphi(b)w}{\varphi(s)} \otimes x \right) = 0.$$

This means that $\ker F$ is almost zero. Thus, $\tilde{W}$ is an almost flat T-module and consequently $\tilde{\varphi}$ is almost flat. Now, we show that almost exactness of (5) implies almost exactness of (4). Let $x \in \ker f$. Therefore, for any $\frac{w}{\varphi(s)}$, where $w \in W$ and $s \in \varphi(T \setminus \mathfrak{p})^{-1}$, we have $\frac{w}{\varphi(s)} \otimes x \in \tilde{W} \otimes_T A$. This implies that $\frac{w}{\varphi(s)} \otimes x \in \ker\left(i_{\tilde{W}} \otimes f \right)$. Almost exactness of (5) implies that for every $\varepsilon > 0$, there exists $b \in T$ such that $b \cdot \left(\frac{w}{\varphi(s)} \otimes x \right) = \frac{\varphi(b)w}{\varphi(s)} \otimes x = \frac{w}{\varphi(s)} \otimes bx = 0$ and $v_T(b) < \varepsilon$. Thus, $\ker f$ is an almost zero T-module. Similarly, one can show that $\operatorname{im} g$ is an almost

zero T-module. Let $\mu : \operatorname{im} f \longrightarrow \ker g$ be a T-homomorphism. Then $i_{\tilde{W}} \otimes \mu : \tilde{W} \otimes_T \operatorname{im} f = \operatorname{im} \left(i_{\tilde{W}} \otimes_T f \right) \longrightarrow i_{\tilde{W}} \otimes_T \ker g \subseteq \ker \left(i_{\tilde{W}} \otimes_T g \right)$. Thus, both $\ker i_{\tilde{W}} \otimes \mu$ and $\operatorname{coker} \left(i_{\tilde{W}} \otimes \mu \right)$ are almost zero. Then these show that the $\ker \mu$ and $\operatorname{im} \mu$ are almost zero. This completes the proof. $\square$

Let T be a Noetherian algebra over R with a value map v and X be a T-module. The Cousin complex $\mathcal{C}_T(X)$ for X is of the form

$$0 \xrightarrow{d^{-2}} X \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{i-2}} X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} \cdots ,$$

where, for all $i \geq 0$, we have

$$X^i = \bigoplus_{\mathfrak{p} \in \operatorname{Supp}(X), \operatorname{ht}_X \mathfrak{p} = i} \left(\operatorname{coker} d^{i-2} \right)_{\mathfrak{p}} .$$

Now, let Y be a T-module with the Cousin complex $\mathcal{C}_T(Y)$ and $f : X \longrightarrow Y$ be an almost T-isomorphism. The we have the following diagram

$$
\begin{array}{c}
0 \xrightarrow{d_X^{-2}} X \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} \cdots \\
\Big\downarrow f \quad\quad \Big\downarrow f^0 \quad\quad \Big\downarrow f^1 \quad\quad\quad \Big\downarrow f^{i-1} \quad\quad \Big\downarrow f^i \\
0 \xrightarrow[d_Y^{-2}]{} Y \xrightarrow[d_Y^{-1}]{} Y^0 \xrightarrow[d_Y^0]{} Y^1 \xrightarrow[d_Y^1]{} \cdots \xrightarrow[d_Y^{i-1}]{} Y^i \xrightarrow[d_Y^i]{} Y^{i+1} \xrightarrow[d_Y^{i+1}]{} \cdots
\end{array}
\tag{6}
$$

By induction and according to Remark 2.1, the above diagram commutes and f^i, for all $i \geq -1$, are almost isomorphism. Now, we prove our the main result in this section.

Theorem 3.2. *Let T, W be two algebras over R equipped with value maps v_T, v_W and $\varphi : T \longrightarrow W$ be an almost flat R-algebras homomorphism and $\mathfrak{p} \in \operatorname{Spec}(T)$ such that $\mathfrak{p}^\varphi \neq W$. Then $\mathcal{C}_{T_{\mathfrak{p}}}(M_W \otimes_T T_{\mathfrak{p}}) \approx \mathcal{C}_{T_{\mathfrak{p}}}(M_{\mathfrak{p}} \otimes_T \varphi(T \setminus \mathfrak{p})^{-1} W)$.*

Proof. By Lemma 3.1, $\tilde{\varphi} : T_{\mathfrak{p}} \longrightarrow \varphi(T \setminus \mathfrak{p})^{-1} W$ defined by $\tilde{\varphi} \left(\frac{t}{s} \right) = \frac{\varphi(t)}{\varphi(s)}$, for all $t \in T$ and $s \in T \setminus \mathfrak{p}$, is an almost faithfully flat R-algebras

homomorphism. Similar to the proof of Lemma 3.1, set $\varphi(T \setminus \mathfrak{p})^{-1}W = \tilde{W}$. We now define $\psi : M_W \otimes_T T_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \otimes_{T_{\mathfrak{p}}} \tilde{W}$ by $\psi(m \otimes w \otimes \frac{t}{s}) = \frac{m}{1} \otimes \frac{\varphi(t)w}{\varphi(s)}$, for all $m \in M$, $w \in W$, $t \in T$ and $s \in T \setminus \mathfrak{p}$. We consider $M_W \otimes_T T_{\mathfrak{p}}$ and $M_{\mathfrak{p}} \otimes_{T_{\mathfrak{p}}} \tilde{W}$ as $T_{\mathfrak{p}}$-modules by the following actions

$$
\begin{aligned}
\frac{t'}{s'} \cdot \left(m \otimes w \otimes \frac{t}{s} \right) &= m \otimes w \otimes \frac{t't}{s's} \\
&= m \otimes w\varphi(t') \otimes \frac{t}{s's} \\
&= m \otimes w \cdot t' \otimes \frac{t}{s's}
\end{aligned}
$$

and

$$
(x \otimes \tilde{w}) \cdot \frac{t'}{s'} = x \otimes \tilde{w}\frac{\varphi(t')}{\varphi(s')},
$$

for all $\frac{t'}{s'}, \frac{t}{s} \in T_{\mathfrak{p}}$, $m \in M$, $w \in W$, $x \in M_{\mathfrak{p}}$ and $\tilde{w} \in \tilde{W}$. Then ψ becomes a $T_{\mathfrak{p}}$-homomorphism. By almost flatness of φ, for all $w \in W$ and $\varepsilon > 0$, there exists $b \in T$ such that $b \cdot w = \varphi(b)w = 0$ and $v_T(b) < \varepsilon$. Then for any $m \otimes w \otimes \frac{t}{s} \in \ker \psi$,

$$
\frac{b}{1} \cdot \left(m \otimes w \otimes \frac{t}{s} \right) = m \otimes w \cdot b \otimes \frac{t}{s} = 0.
$$

Thus, $\ker \psi \approx 0$. Similarly, one can show that $\operatorname{coker} \psi \approx 0$. Hence, ψ is an almost isomorphism. Now, by applying a similar argument in (6), we have $\mathcal{C}_{T_{\mathfrak{p}}}(M_W \otimes_T T_{\mathfrak{p}}) \approx \mathcal{C}_{T_{\mathfrak{p}}}(M_{\mathfrak{p}} \otimes_T \tilde{W})$.   □

4   Cousin Complexes Defined by Filtration and Almost Flat Modules

Let T be a Noetherian algebra over R with a value map v. A filtration of $\operatorname{Spec}(T)$ is a descending sequence $\mathcal{F} = (F_i)_{i \geq 0}$ of subsets of $\operatorname{Spec}(T)$ such that

$$
\operatorname{Spec}(T) \supseteq F_0 \supseteq F_1 \supseteq \cdots \supseteq F_i \supseteq F_{i+1} \supseteq \cdots,
$$

with the property that $F_i - F_{i+1}$ is low with respect to F_i, for all $i \geq 0$, i.e., every memeber $F_i - F_{i+1}$ is a minimal member of F_i with respect to

inclusion. If, for a T-module X, $\mathrm{Supp}_T(X) \subseteq F_0$, we say that $\mathcal{F}$ admits the T-module X. Suppose that $\mathcal{F}$ is a filtration of $\mathrm{Spec}(T)$ which admits a T-module X. The Cousin complex $\mathcal{C}_T(\mathcal{F}, X)$ for X with respect to $\mathcal{F}$ is of the form

$$0 \xrightarrow{d^{-2}} X \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{i-2}} X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} \cdots,$$

where, for all $i \geq 0$, we have

(i) $X^i = \bigoplus_{\mathfrak{p} \in F_i \backslash F_{i+1}} \left(\mathrm{coker} d^{i-2} \right)_{\mathfrak{p}}$,

(ii) $d^{i-1}(x) = \left\{ \left(x + \mathrm{Im} d^{i-2} \right) / 1 \right\}_{\mathfrak{p} \in F_i \backslash F_{i+1}}$, for every $x \in X^{i-1}$.

Let $\mathcal{F} = (F_i)_{i \geq 0}$ be a filtration of $\mathrm{Spec}(T)$. Then we say that $\mathcal{F}$ is finite of length less than d, whenever $F_i = \emptyset$, for all $i > d$. In this case, for all $i \geq 0$, we have

(i) $\mathrm{Supp}_T(X^i) \subseteq F_i$,

(ii) $\mathrm{Supp}_T(\mathrm{H}^{i-1}(\mathcal{C}_T(\mathcal{F}, X))) \subseteq F_{i+1}$,

and

$$\mathcal{C}_T(\mathcal{F}, X): \ 0 \longrightarrow X \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$
$$\longrightarrow X^i \longrightarrow \cdots \longrightarrow X^{d-1} \longrightarrow X^d \longrightarrow 0.$$

Let $\mathcal{F} = (F_i)_{i \geq 0}$ be a filtration of $\mathrm{Spec}(R)$. The set $\mathcal{C}_{\mathcal{F}}(T)$ is the category of T-modules which are admitted by $\mathcal{F}$ and we denote the category of complexes of T-modules by $\mathrm{Comp}(T)$. We recall the following result from [2]:

Corollary 4.1. *Suppose that $\mathcal{F} = (F_i)_{i \geq 0}$ is a filtration of $\mathrm{Spec}(T)$ and*

$$(A^{\bullet}, b^{\bullet}): \cdots \longrightarrow A^{i-1} \xrightarrow{b^{i-1}} A^i \xrightarrow{b^i} A^{i+1} \longrightarrow \cdots$$

is a complex in $\mathcal{C}_{\mathcal{F}}(T)$. Then
$$(C_T(\mathcal{F}, A^{\bullet}), C_T(\mathcal{F}, b^{\bullet})):$$
$$\cdots \longrightarrow C_T\left(\mathcal{F}, A^{i-1}\right) \xrightarrow{C_T(\mathcal{F}, b^{i-1})} C_T\left(\mathcal{F}, A^i\right) \xrightarrow{C_T(\mathcal{F}, b^i)} C_T\left(\mathcal{F}, A^{i+1}\right) \longrightarrow \cdots$$
is a complex in $\mathrm{Comp}(T)$.

Theorem 4.2. *Let $\mathcal{F} = (F_i)_{i \geq 0}$ be a finite filtration of length less than d of $\mathrm{Spec}(T)$ such that admits a T-module X and M be an any flat T-module. Then*

(i) ${}^I\mathrm{E}^2_{p,q} := \mathrm{Tor}^T_p\left(M, \mathrm{H}^{d-q}\left(\mathcal{C}_T\left(\mathcal{F}, X\right)\right)\right) \overset{p}{\Rightarrow} \mathrm{H}_{p+q}(\mathrm{Tot}(\mathcal{T})).$

(ii) ${}^{II}\mathrm{E}^2_{p,q} := \mathrm{H}^{d-p}\left(\mathrm{Tor}^T_q\left(M, \mathcal{C}_T\left(\mathcal{F}, X\right)\right)\right) \overset{p}{\Rightarrow} \mathrm{H}_{p+q}(\mathrm{Tot}(\mathcal{T})).$

Proof. Assume that

$$\mathbb{F}_\bullet : \cdots \longrightarrow \mathbb{F}_{i+1} \longrightarrow \mathbb{F}_i \longrightarrow \mathbb{F}_{i-1} \longrightarrow \cdots \longrightarrow \mathbb{F}_1 \longrightarrow \mathbb{F}_0 \longrightarrow 0$$

is a free resolution of M. For any $p \geq 0$, suppose that $\mathfrak{q} \in \mathrm{Supp}_T\left(\mathbb{F}_p \otimes_T X\right)$. This means that there exists $x = x' \otimes x'' \in \mathbb{F}_p \otimes_T X$ such that $(0 :_T x) \subseteq \mathfrak{q}$, where $x \in \mathbb{F}_p$ and $x'' \in X$. Thus, for any $t \in T \setminus \mathfrak{q}$, $t \cdot x = tx' \otimes x'' = x' \otimes tx'' \neq 0$. This means that $tx'' \neq 0$. Thus, $0 \neq \frac{x''}{1_T} \in X_\mathfrak{q}$. This shows that $X_\mathfrak{q} \neq 0$ and so $\mathfrak{q} \in \mathrm{Supp}_T(X)$. For any $p \geq 0$, $\mathbb{F}_p$ is a free T-module. Then $\mathbb{F}_p \otimes X$ is a free T-module (see [8, Example 4.101]). Now, let $\mathfrak{q} \in \mathrm{Supp}_T(X)$. Then there exists $x \in X$ such that for all $t \in T \setminus \mathfrak{q}$, $tx \neq 0$. Now, choose $y \in \mathbb{F}_p$ such that $y \otimes tx \neq 0$ (note that since $\mathbb{F}_p$ and $\mathbb{F}_p \otimes_T X$ are free T-modules, we can choose such element). This implies that $t(y \otimes x) \neq 0$. Thus $(\mathbb{F}_p \otimes_T X)_\mathfrak{q} \neq 0$. This means that $\mathfrak{q} \in \mathrm{Supp}_T\left(\mathbb{F}_p \otimes_T X\right)$. So for any $p \geq 0$, we have

$$\mathrm{Supp}_T\left(\mathbb{F}_p \otimes_T X\right) = \mathrm{Supp}_T(X).$$

Hence, $\mathcal{F}$ admits T-module $\mathbb{F}_p \otimes_T X$. This implies that $\mathbb{F}_\bullet \otimes_T X$ is in $\mathcal{C}_\mathcal{F}(T)$. Then by [2, Corollary 2.3], $\mathcal{C}_T\left(\mathcal{F}, \mathbb{F}_\bullet \otimes_T X\right)$ is in $\mathrm{Comp}(T)$. Then $\mathcal{T} = \left\{\mathcal{C}_T(\mathcal{F}, \mathbb{F}_p \otimes_T X)^{d-q}\right\}$ is a first quadrant bi-complex and we denote the total of $\mathcal{T}$ by $\mathrm{Tot}(T)$. Now, we prove (i) and (ii) as follow:

(i) Let ${}^I\mathrm{E}^2 = \mathrm{H}'_p\mathrm{H}''_{p,q}$ be the first iterated homology of $\mathcal{T}$ with respect to the first filtration. Then

$$\begin{aligned}
\mathrm{H}''_{p,q}(\mathcal{T}) &= \mathrm{H}^{d-q}\left(\mathcal{C}_T\left(\mathcal{F}, \mathbb{F}_p \otimes_T X\right)\right) \\
&= \mathrm{H}^{d-q}\left(\mathbb{F}_p \otimes_T \mathcal{C}_T\left(\mathcal{F}, X\right)\right) \\
&= \mathbb{F}_p \otimes_T \mathrm{H}^{d-q}\left(\mathcal{C}_T\left(\mathcal{F}, X\right)\right).
\end{aligned}$$

The above equality implies that

$$
\begin{aligned}
{}^{I}\mathrm{E}^2_{p,q} &= \mathrm{H}'_p \mathrm{H}''_{p,q}(\mathcal{T}) \\
&= \mathrm{H}_p\left(\mathbb{F}_p \otimes_T \mathrm{H}^{d-q}\left(\mathcal{C}_T\left(\mathcal{F},X\right)\right)\right) \\
&= \mathrm{Tor}^T_p\left(M, \mathrm{H}^{d-q}\left(\mathcal{C}_T\left(\mathcal{F},X\right)\right)\right).
\end{aligned}
$$

By applying the above obtained equality, we have the following the first quadrant spectral sequence

$$
{}^{I}\mathrm{E}^2_{p,q} := \mathrm{Tor}^T_p\left(M, \mathrm{H}^{d-q}\left(\mathcal{C}_T\left(\mathcal{F},X\right)\right)\right) \overset{p}{\Rightarrow} \mathrm{H}_{p+q}(\mathrm{Tot}(\mathcal{T})).
$$

(ii) Let ${}^{II}\mathrm{E}^2 = \mathrm{H}''_p \mathrm{H}''_{q,p}$ be the second iterated homology of $\mathcal{T}$ with respect to the second filtration. Then

$$
\begin{aligned}
\mathrm{H}'_{q,p} &= \mathrm{H}_q\left(\mathcal{C}_T\left(\mathcal{F}, \mathbb{F}_\bullet \otimes_T X\right)^{d-p}\right) \\
&= \mathrm{H}_q\left(\mathbb{F}_\bullet \otimes_T \mathcal{C}_T\left(\mathcal{F},X\right)^{d-p}\right) \\
&= \mathrm{Tor}^T_q\left(M, \mathcal{C}_T\left(\mathcal{F},X\right)^{d-p}\right).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
{}^{II}\mathrm{E}^2_{p,q} &= \mathrm{H}''_p \mathrm{H}''_{q,p}(\mathcal{T}) \\
&= \mathrm{H}_p\left(\mathrm{Tor}^T_q\left(M, \mathcal{C}_T\left(\mathcal{F},X\right)^{d-p}\right)\right) \\
&= \mathrm{H}^{d-p}\left(\mathrm{Tor}^T_q\left(M, \mathcal{C}_T\left(\mathcal{F},X\right)\right)\right).
\end{aligned}
$$

The above equality implies the first quadrant spectral sequence

$$
{}^{II}\mathrm{E}^2_{p,q} := \mathrm{H}^{d-p}\left(\mathrm{Tor}^T_q\left(M, \mathcal{C}_T\left(\mathcal{F},X\right)\right)\right) \overset{p}{\Rightarrow} \mathrm{H}_{p+q}(\mathrm{Tot}(\mathcal{T})).
$$

$\square$

Corollary 4.3. *Let $\mathcal{F} = (F_i)_{i \geq 0}$ be a finite filtration of length less than d of $\mathrm{Spec}(T)$ such that admits a T-module X. Let M be an almost flat T-module, then for $p, q > 0$,*

(i) ${}^{I}\mathrm{E}^2_{p,q} \approx 0$.

(ii) ${}^{II}\mathrm{E}^2_{p,q} \approx 0$.

5   Problems

In this section we give some problems related to almost flat modules.
Let T and R as before.

1. Let M be a T-module. Lazard's Theorem says that M is flat if
   and only if it is the colimit of a directed system of free finite T-
   modules. What about for the almost flat modules? Clearly, if M
   is the colimit of a directed system of free finite T-modules, then
   by Lazard's Theorem it is almost flat.

2. Let $\varphi : T \longrightarrow W$ be a local homomorphism of Noetherian local
   R-algebras, $I \neq T$ be an ideal in T and M be a finitely generated
   W-module. If $\mathrm{Tor}_1^T(M, T/I) \approx 0$ and M/IM is almost flat T/I-
   module, is M almost flat T-module?

3. Let $\varphi : T \longrightarrow W$ be an almost faithfully flat homomorphism of
   R-algebras and W is reduced (a normal as a ring, a regular as a
   ring, a Nagata ring). Is T reduced (a normal as a ring, a regular
   as a ring, a Nagata ring)?

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