Journal of Mathematical Extension
Vol. 16, No. 8, (2022) (9)1-27
URL: https://doi.org/10.30495/JME.2022.1503
ISSN: 1735-8299
Original Research Paper

Interval Network Malmquist Productivity Index for Examining Productivity Changes of Insurance Companies under Data Uncertainty: A Case Study

FS. Seyed Esmaeili
Science and Research Branch, Islamic Azad University

M. Rostamy-Malkhalifeh*
Science and Research Branch, Islamic Azad University

F. Hosseinzadeh Lotfi
Science and Research Branch, Islamic Azad University

Abstract. The insurance industry is one of the important financial institutions that has a significant place in the economic growth and development of the country. Given the industry's influential role in the financial markets, it is imperative to evaluate the performance and calculate changes in insurance companies' productivity over time. It is necessary to explain that the internal structure of insurance companies can be considered as a two-stage process involving marketing and investment. The purpose of the current study is to propose a novel approach to calculate the changes in insurance companies' productivity by considering their two-stage structure as well as the inherent uncertainties in the data. It should be noted that in order to propose of new interval network Malmquist Productivity Index, the network data

Received: December 2019; Accepted: June 2020
*Corresponding Author

envelopment analysis approach (NDEA), Malmquist productivity index (MPI), and interval programming are applied. The implementation of the proposed research approach is also evaluated using real data of 10 insurance companies in Iran. According to the obtained results, most of the companies have regressed from the first stage and marketing perspective, but in the second stage and from the investment perspective, the majority of companies have represented an acceptable improvement in their productivity.

AMS Subject Classification: 90C08; 90C05.
Keywords and Phrases: Insurance Companies, Malmquist Productivity Index, Two-Stage Structure, Network Data Envelopment Analysis, Interval Data, Non-Discretionary Factor.

1 Introduction

Insurance companies are undoubtedly one of the most important pillars of the financial markets, whose great performance will drive the economy of the country. Therefore, evaluating the performance of insurance companies with the aim of identifying the extent and trend of their productivity changes can be pretty useful for the related managers. Because by taking advantage of the results of evaluating corporate performance and how their productivity changes, they can make acceptable decisions about improving the performance of poor performing companies. Thus, by identifying the inefficiencies of each company and modeling efficient companies, they can initiate a corrective process for all insurance companies.

Calculating productivity changes with the aim of identifying the progress and regress of different decision-making units (DMUs) is one of the attractive applications of Data Envelopment Analysis (DEA). In this way, the decision-maker can measure the extent and how DMU productivity changes over time by combining the DEA approach and Malmquist productivity index (MPI). It is necessary to explain that DEA is a method for estimating an efficient frontier, comparing the performance of the DMUs under investigation with this boundary, and finally estimating the efficiency of DMU under investigation. The CCR model is the first model in the field of DEA introduced by Charnes et al. (1978). Although the most straightforward model of DEA, it is the basis of many modern and advanced models of DEA. It is necessary to explain that the

CCR model assumes constant returns to scale (CRS). This model was subsequently developed by Banker et al. (1984) under variable returns to scale (VRS), which became known as the BCC model.
As mentioned, one of the applications of DEA is the ability to calculate the productivity changes of DMUs over time. In this way, the DMU in the new period compared to the previous period, has progressed, regressed, or stagnated compared to other DMUs.
It is important to note that two important points must be taken into account when changes in insurance companies' productivity are calculated. The first is to consider the structure and internal relationships of the company, and the second is to consider the uncertainties in the data. Failure to include these points in the evaluation process may lead to invalid results. Therefore, the purpose of the present study is to introduce a new index in order to calculate productivity changes in insurance companies considering the internal structure and uncertainties in the data. To achieve this purpose, a two-stage DEA model, Malmquist productivity index, and interval programming will be employed in order to propose the interval network Malmquist productivity index (INMPI).
The rest of this paper is organized as follows: Section 2 introduces the theoretical backgrounds of the current study. Then in Section 3, the proposed approach is presented to calculate the productivity changes of insurance companies. Section 4 shows how to implement the index using real data of 10 insurance companies in Iran. Finally, Section 5 presents conclusions and future research directions.

2 Theoretical Backgrounds

In this section, the internal structure of insurance companies, as well as the basic network data envelopment analysis (NDEA) model, and traditional Malmquist productivity index, are presented.

2.1 Insurance companies (ICs) structure

The internal structure of insurance companies can be considered as a two-stage system shown in Figure 1, including the marketing and investment phases (Kao and Huang, 2008). It should be noted that up to

Table 1: The Application of Two-Stage DEA Approach in Insurance Companies

Authors	NDEA Form		NDEA Approach							Feature			
(Publication years)	MF	EF	MMA	AMA	GBMA	RMA	CWMA	SBMA	PPSBMA	DA	MPI	DU	NDF
Kao and Huang (2008)	✓		✓										
Chen et al. (2009)	✓			✓									
Kao (2009)	✓		✓										
Du et al. (2011)	✓				✓								
An et al. (2016)		✓							✓				
Despotis et al. (2016)	✓					✓							
Gharakhani et al. (2018)	✓						✓			✓			
Li et al. (2018)	✓				✓								
Nourani et al. (2018)		✓						✓		✓			
Almulhim (2019)	✓					✓							
Anandarao et al. (2019)	✓				✓								
Fang (2019)	✓				✓								
Hatami-Marbini and Saati (2019)	✓						✓						
Krupa et al. (2019)		✓						✓					
Tone et al. (2019)		✓						✓		✓			
Our Work		✓							✓	✓	✓	✓	✓

Abbreviations

NDEA: Network Data Envelopment Analysis; MF: Multiplier Form; EF: Envelopment Form; MMA: Multiple Modeling Approach; AMA: Additive Modeling Approach; GBMA: Game Based Modeling Approach; RMA: Reverse Modeling Approach; CWMA: Common Weight Modeling Approach; SBMA: Slack Based Modeling Approach; PPSBMA: Production Possibility Set Based Modeling Approach; DA: Dynamic Analysis; MPI: Malmquist Productivity Index; DU: Data Uncertainty, Non-Discretionary Factor.

now there have been considerable studies in the field of performance evaluation of insurance companies using networked DEA approach, which can be found in Table 1. Moreover, the characteristics of this study have also been introduced in the last row of Table 1.

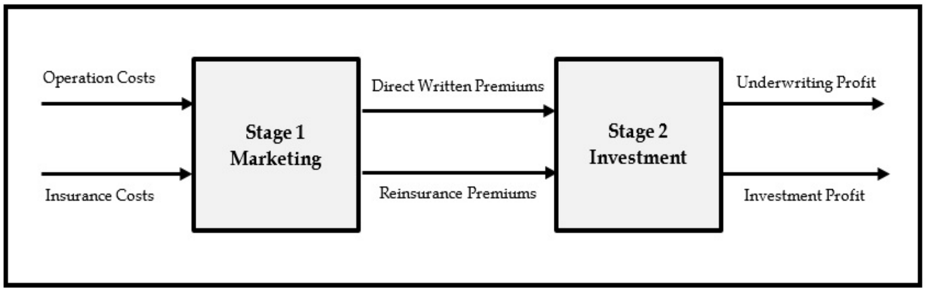

Figure 1: The Structure of Insurance Companies

The explanations for each of the cases presented in figure 1 are given as follows. Operation costs: employee salaries and various types of expenses incurred in daily work. Insurance costs: costs paid to agencies, brokers and lawyers, and other costs associated with the marketing of insurance services. Direct written premium: premium received from insured customers. Reinsurance premium: premium received from transfer companies. Underwriting profit: profit from the insurance business. Investment profit: profit from the portfolio.

2.2 Two-stage data envelopment analysis (TSDEA) model

Consider the two-stage structure is shown in figure 2. There are n network DMUs with a two-stage structure that each DMU_j has I inputs $x_{ij}(i = 1, ..., I)$ in stage 1, D intermediate (linking) variables $z_{dj}(d = 1, ..., D)$ and R outputs $y_{rj}(r = 1, ..., R)$ in stage 2.

Chen and Zhou (2004) proposed a two-stage DEA approach by integrating the envelopment form of the first and second stages. The extended NDEA approach based on study of Chen and Zhou (2004) for DMU under investigation is as Model (1):

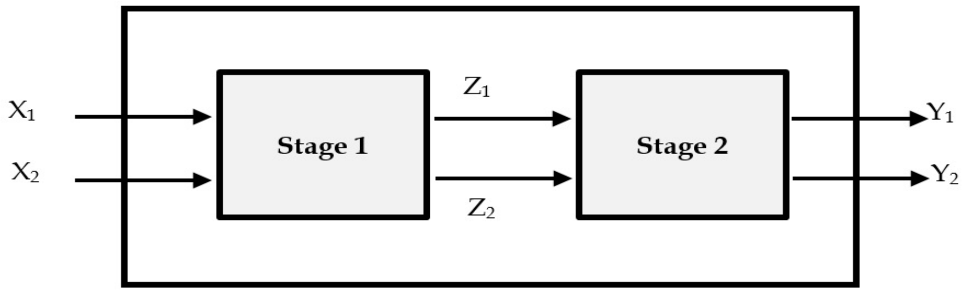

Figure 2: Two-Stage Structure

$$
\begin{aligned}
\min \quad & \frac{\theta}{\varphi} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io} \quad \forall i \\
& \sum_{j=1}^{n} \lambda_j z_{dj} \geq \hat{z}_{do} \quad \forall d \\
& \theta \leq 1 \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \sum_{j=1}^{n} \mu_j z_{dj} \leq \hat{z}_{do} \quad \forall d \\
& \sum_{j=1}^{n} \mu_j y_{rj} \geq \varphi y_{ro} \quad \forall r \\
& \varphi \geq 1 \\
& \sum_{j=1}^{n} \mu_j = 1 \\
& \lambda_j, \mu_j \geq 0 \quad \forall j
\end{aligned}
\tag{1}
$$

It is necessary to explain that $\hat{z}_{do}$ which is presented in Model (1) is a decision variable. Also, the model is introduced under variable returns to scale assumption.

2.3 Malmquist productivity index (MPI)

Făre and Grosskopf (1992) have used the Malmquist productivity index and data envelopment analysis to calculate the extent of productivity changes. They proposed this index by considering two time periods $(t, t+1)$ and calculating technology changes and performance changes in these two time periods. Thus, to obtain efficiency changes, it is sufficient the envelopment form of the input-oriented CCR model for two time periods be solved. Based on this explanation, Models (2) and (3) are used to calculate the efficiency of DMU in time periods t and $t+1$, respectively:

$$\begin{aligned} D_o^t(x_o^t, y_o^t) = &\min \theta \\ s.t. \quad &\sum_{j=1}^{n} \lambda_j x_{ij}^t \le \theta x_{io}^t \quad \forall i \\ &\sum_{j=1}^{n} \lambda_j y_{rj}^t \ge y_{ro}^t \quad \forall r \\ &\lambda_j \ge 0 \quad \forall j \end{aligned} \tag{2}$$

$$\begin{aligned} D_o^{t+1}(x_o^{t+1}, y_o^{t+1}) = &\min \theta \\ s.t. \quad &\sum_{j=1}^{n} \lambda_j x_{ij}^{t+1} \le \theta x_{io}^{t+1} \quad \forall i \\ &\sum_{j=1}^{n} \lambda_j y_{rj}^{t+1} \ge y_{ro}^{t+1} \quad \forall r \\ &\lambda_j \ge 0 \quad \forall j \end{aligned} \tag{3}$$

Also, Models (4) and (5) are solved in order to calculate technology changes. It should be noted that the result of Model (4), represents the distance of DMU_o in time periods t with the efficient frontier in time period $t+1$ and the result of Model (5), indicates the distance of DMU_o in time periods $t+1$ with the efficient frontier in time period t.

$$\begin{aligned} D_o^t(x_o^{t+1}, y_o^{t+1}) = &\min \theta \\ s.t. \quad &\sum_{j=1}^{n} \lambda_j x_{ij}^t \le \theta x_{io}^{t+1} \quad \forall i \\ &\sum_{j=1}^{n} \lambda_j y_{rj}^t \ge y_{ro}^{t+1} \quad \forall r \\ &\lambda_j \ge 0 \quad \forall j \end{aligned} \tag{4}$$

$$\begin{aligned} D_o^{t+1}(x_o^t, y_o^t) = &\min \theta \\ s.t. \quad &\sum_{j=1}^{n} \lambda_j x_{ij}^{t+1} \le \theta x_{io}^t \quad \forall i \\ &\sum_{j=1}^{n} \lambda_j y_{rj}^{t+1} \ge y_{ro}^t \quad \forall r \\ &\lambda_j \ge 0 \quad \forall j \end{aligned} \tag{5}$$

Finally, the Malmquist Productivity Index is calculated using Equation (6):

$$MPI_o = \sqrt{\frac{D_o^t(x_o^{t+1}, y_o^{t+1}) \times D_o^{t+1}(x_o^{t+1}, y_o^{t+1})}{D_o^t(x_o^t, y_o^t) \times D_o^{t+1}(x_o^t, y_o^t)}} \tag{6}$$

It should be explained that based on the value of the Malmquist Productivity Index which can be greater, equal to or less than one, the trend and how the interest rate changes on the unit under consideration are as follows:

- If $MPI_o > 1$, it indicates on progress in productivity DMU under investigation.
- If $MPI_o = 1$, it indicates that no change in the productivity of the DMU_o has occurred.
- If $MPI_o < 1$, it indicates on regress in productivity DMU under investigation.

3 Interval Network Malmquist Productivity Index (INMPI)

In this section, a novel index in order to calculate productivity changes in insurance companies considering the internal structure in the presence of imprecise data and nondiscretionary factors will be proposed. It is necessary to explain that in order to present the proposed research index, the NDEA approach presented by Chen and Zhou (2004) is used as the basic model to the capability of used in the presence of a two-stage structure. Also, the NDEA approach is extended for handling non-discretionary factors and imprecise data. It should be noted that the indexes 1 and 2 denote on discretionary and non-discretionary factors, respectively. As noted earlier, it is important to note that all data used in the process of calculating the productivity index have interval uncertainties $x_{ij} \in \left[\underline{x}_{ij}, \overline{x}_{ij}\right]$, $z_{dj} \in \left[\underline{z}_{dj}, \overline{z}_{dj}\right]$ and $y_{rj} \in \left[\underline{y}_{rj}, \overline{y}_{rj}\right]$ where only the upper and lower bounds are known. In other words, the uncertain models used to calculate the performance of a two-stage DMU over time periods t and $t+1$, are in the form of Models (7) and (8), respectively:

$$
\begin{array}{lll}
\tilde{N}_o^t(\tilde{x}_o^t, \tilde{z}_o^t, \tilde{y}_o^t) = \min \frac{\theta}{\varphi} & & \\
s.t. \quad \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}^t \leq \theta \tilde{x}_{io}^t & \forall i \in I_1 \\
\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}^t \leq \tilde{x}_{io}^t & \forall i \in I_2 \\
\sum_{j=1}^{n} \lambda_j \tilde{z}_{dj}^t \geq \hat{z}_{do} & \forall d \in D_1 \cup D_2 \\
\theta \leq 1 & \\
\sum_{j=1}^{n} \lambda_j = 1 & \\
\sum_{j=1}^{n} \mu_j \tilde{z}_{dj}^t \leq \hat{z}_{do} & \forall d \in D_1 \cup D_2 \\
\sum_{j=1}^{n} \mu_j \tilde{y}_{rj}^t \geq \varphi \tilde{y}_{ro}^t & \forall r \in R_1 \\
\sum_{j=1}^{n} \mu_j \tilde{y}_{rj}^t \geq \tilde{y}_{ro}^t & \forall r \in R_2 \\
\varphi \geq 1 & \\
\sum_{j=1}^{n} \mu_j = 1 & \\
\lambda_j, \mu_j \geq 0 & \forall j
\end{array} \tag{7}
$$

$$
\begin{array}{lll}
\tilde{N}_o^{t+1}(\tilde{x}_o^{t+1}, \tilde{z}_o^{t+1}, \tilde{y}_o^{t+1}) = \min \frac{\theta}{\varphi} & & \\
s.t. \quad \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}^{t+1} \leq \theta \tilde{x}_{io}^{t+1} & \forall i \in I_1 \\
\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}^{t+1} \leq \tilde{x}_{io}^{t+1} & \forall i \in I_2 \\
\sum_{j=1}^{n} \lambda_j \tilde{z}_{dj}^{t+1} \geq \hat{z}_{do} & \forall d \in D_1 \cup D_2 \\
\theta \leq 1 & \\
\sum_{j=1}^{n} \lambda_j = 1 & \\
\sum_{j=1}^{n} \mu_j \tilde{z}_{dj}^{t+1} \leq \hat{z}_{do} & \forall d \in D_1 \cup D_2 \\
\sum_{j=1}^{n} \mu_j \tilde{y}_{rj}^{t+1} \geq \varphi \tilde{y}_{ro}^{t+1} & \forall r \in R_1 \\
\sum_{j=1}^{n} \mu_j \tilde{y}_{rj}^{t+1} \geq \tilde{y}_{ro}^{t+1} & \forall r \in R_2 \\
\varphi \geq 1 & \\
\sum_{j=1}^{n} \mu_j = 1 & \\
\lambda_j, \mu_j \geq 0 & \forall j
\end{array} \tag{8}
$$

Also, Models (4) and (5), which represent the distance of DMU under consideration with the efficient frontier, are rewritten as Models (9) and (10), respectively:

$$\begin{array}{lll}
\tilde{N}_o^{t+1}(\tilde{x}_o^t, \tilde{z}_o^t, \tilde{y}_o^t) = \min \frac{\theta}{\varphi} & & \\
s.t. \quad \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}^{t+1} \le \theta \tilde{x}_{io}^t & \forall i \in I_1 \\
\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}^{t+1} \le \tilde{x}_{io}^t & \forall i \in I_2 \\
\sum_{j=1}^{n} \lambda_j \tilde{z}_{dj}^{t+1} \ge \hat{z}_{do} & \forall d \in D_1 \cup D_2 \\
\sum_{j=1}^{n} \lambda_j = 1 & \\
\sum_{j=1}^{n} \mu_j \tilde{z}_{dj}^{t+1} \le \hat{z}_{do} & \forall d \in D_1 \cup D_2 \\
\sum_{j=1}^{n} \mu_j \tilde{y}_{rj}^{t+1} \ge \varphi \tilde{y}_{ro}^t & \forall r \in R_1 \\
\sum_{j=1}^{n} \mu_j \tilde{y}_{rj}^{t+1} \ge \tilde{y}_{ro}^t & \forall r \in R_2 \\
\sum_{j=1}^{n} \mu_j = 1 & \\
\lambda_j, \mu_j \ge 0 & \forall j
\end{array} \tag{9}$$

$$\begin{array}{lll}
\tilde{N}_o^{t}(\tilde{x}_o^{t+1}, \tilde{z}_o^{t+1}, \tilde{y}_o^{t+1}) = \min \frac{\theta}{\varphi} & & \\
s.t. \quad \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}^{t} \le \theta \tilde{x}_{io}^{t+1} & \forall i \in I_1 \\
\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}^{t} \le \tilde{x}_{io}^{t+1} & \forall i \in I_2 \\
\sum_{j=1}^{n} \lambda_j \tilde{z}_{dj}^{t} \ge \hat{z}_{do} & \forall d \in D_1 \cup D_2 \\
\sum_{j=1}^{n} \lambda_j = 1 & \\
\sum_{j=1}^{n} \mu_j \tilde{z}_{dj}^{t} \le \hat{z}_{do} & \forall d \in D_1 \cup D_2 \\
\sum_{j=1}^{n} \mu_j \tilde{y}_{rj}^{t} \ge \varphi \tilde{y}_{ro}^{t+1} & \forall r \in R_1 \\
\sum_{j=1}^{n} \mu_j \tilde{y}_{rj}^{t} \ge \tilde{y}_{ro}^{t+1} & \forall r \in R_2 \\
\sum_{j=1}^{n} \mu_j = 1 & \\
\lambda_j, \mu_j \ge 0 & \forall j
\end{array} \tag{10}$$

Due to the uncertainty in the data, the Interval Data Envelopment Analysis (IDEA) approach proposed by Despotis and Smirlis (2002) is now applied in modeling, which is widely used approaches in the field of uncertain DEA. According to the interval programming approach, Models (11) and (12) are used to calculate the lower and upper bounds of the performance of DMU_o in time periods t, respectively.

$$
\begin{array}{lll}
\underline{N}_o^t(x_o^t, z_o^t, y_o^t) = \min \frac{\theta}{\varphi} & & \\
s.t. \quad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \underline{x}_{ij}^t + \lambda_o \overline{x}_{io}^t \leq \theta \overline{x}_{io}^t & \forall i \in I_1 \\
\qquad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \underline{x}_{ij}^t + \lambda_o \overline{x}_{io}^t \leq \overline{x}_{io}^t & \forall i \in I_2 \\
\qquad \sum_{j=1}^{n} \lambda_j \overline{z}_{dj}^t \geq \hat{z}_{do} & \forall d \in D_1 \cup D_2 \\
\qquad \theta \leq 1 & \\
\qquad \sum_{j=1}^{n} \lambda_j = 1 & \\
\qquad \sum_{j=1}^{n} \mu_j \underline{z}_{dj}^t \leq \hat{z}_{do} & \forall d \in D_1 \cup D_2 \\
\qquad \sum_{\substack{j=1\\ j\neq o}}^{n} \mu_j \overline{y}_{rj}^t + \mu_o \underline{y}_{ro}^t \geq \varphi \underline{y}_{ro}^t & \forall r \in R_1 \\
\qquad \sum_{\substack{j=1\\ j\neq o}}^{n} \mu_j \overline{y}_{rj}^t + \mu_o \underline{y}_{ro}^t \geq \underline{y}_{ro}^t & \forall r \in R_2 \\
\qquad \varphi \geq 1 & \\
\qquad \sum_{j=1}^{n} \mu_j = 1 & \\
\qquad \lambda_j, \mu_j \geq 0 & \forall j
\end{array} \tag{11}
$$

$$
\begin{array}{lll}
\overline{N}_o^t(x_o^t, z_o^t, y_o^t) = \min \frac{\theta}{\varphi} & & \\
s.t. \quad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \overline{x}_{ij}^t + \lambda_o \underline{x}_{io}^t \leq \theta \underline{x}_{io}^t & \forall i \in I_1 \\
\qquad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \overline{x}_{ij}^t + \lambda_o \underline{x}_{io}^t \leq \underline{x}_{io}^t & \forall i \in I_2 \\
\qquad \sum_{j=1}^{n} \lambda_j \underline{z}_{dj}^t \geq \hat{z}_{do} & \forall d \in D_1 \cup D_2 \\
\qquad \theta \leq 1 & \\
\qquad \sum_{j=1}^{n} \lambda_j = 1 & \\
\qquad \sum_{j=1}^{n} \mu_j \overline{z}_{dj}^t \leq \hat{z}_{do} & \forall d \in D_1 \cup D_2 \\
\qquad \sum_{\substack{j=1\\ j\neq o}}^{n} \mu_j \underline{y}_{rj}^t + \mu_o \overline{y}_{ro}^t \geq \varphi \overline{y}_{ro}^t & \forall r \in R_1 \\
\qquad \sum_{\substack{j=1\\ j\neq o}}^{n} \mu_j \underline{y}_{rj}^t + \mu_o \overline{y}_{ro}^t \geq \overline{y}_{ro}^t & \forall r \in R_2 \\
\qquad \varphi \geq 1 & \\
\qquad \sum_{j=1}^{n} \mu_j = 1 & \\
\qquad \lambda_j, \mu_j \geq 0 & \forall j
\end{array} \tag{12}
$$

Also, Models (13) and (14) are used to calculate the efficiency of DMU_o in time periods $t+1$, respectively.

$$
\begin{aligned}
\underline{N}_o^{t+1}(x_o^{t+1}, z_o^{t+1}, y_o^{t+1}) = \min \tfrac{\theta}{\varphi} & \\
s.t. \quad & \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \underline{x}_{ij}^{t+1} + \lambda_o \bar{x}_{io}^{t+1} \leq \theta \bar{x}_{io}^{t+1} && \forall i \in I_1 \\
& \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \underline{x}_{ij}^{t+1} + \lambda_o \bar{x}_{io}^{t+1} \leq \bar{x}_{io}^{t+1} && \forall i \in I_2 \\
& \sum_{j=1}^{n} \lambda_j \bar{z}_{dj}^{t+1} \geq \hat{z}_{do} && \forall d \in D_1 \cup D_2 \\
& \theta \leq 1 \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \sum_{j=1}^{n} \mu_j \underline{z}_{dj}^{t+1} \leq \hat{z}_{do} && \forall d \in D_1 \cup D_2 \\
& \sum_{\substack{j=1\\ j\neq o}}^{n} \mu_j \overline{y}_{rj}^{t+1} + \mu_o \underline{y}_{ro}^{t+1} \geq \varphi \underline{y}_{ro}^{t+1} && \forall r \in R_1 \\
& \sum_{\substack{j=1\\ j\neq o}}^{n} \mu_j \overline{y}_{rj}^{t+1} + \mu_o \underline{y}_{ro}^{t+1} \geq \underline{y}_{ro}^{t+1} && \forall r \in R_2 \\
& \varphi \geq 1 \\
& \sum_{j=1}^{n} \mu_j = 1 \\
& \lambda_j, \mu_j \geq 0 && \forall j
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\overline{N}_o^{t+1}(x_o^{t+1}, z_o^{t+1}, y_o^{t+1}) = \min \tfrac{\theta}{\varphi} & \\
s.t. \quad & \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \bar{x}_{ij}^{t+1} + \lambda_o \underline{x}_{io}^{t+1} \leq \theta \underline{x}_{io}^{t+1} && \forall i \in I_1 \\
& \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \bar{x}_{ij}^{t+1} + \lambda_o \underline{x}_{io}^{t+1} \leq \underline{x}_{io}^{t+1} && \forall i \in I_2 \\
& \sum_{j=1}^{n} \lambda_j \underline{z}_{dj}^{t+1} \geq \hat{z}_{do} && \forall d \in D_1 \cup D_2 \\
& \theta \leq 1 \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \sum_{j=1}^{n} \mu_j \overline{z}_{dj}^{t+1} \leq \hat{z}_{do} && \forall d \in D_1 \cup D_2 \\
& \sum_{\substack{j=1\\ j\neq o}}^{n} \mu_j \underline{y}_{rj}^{t+1} + \mu_o \overline{y}_{ro}^{t+1} \geq \varphi \overline{y}_{ro}^{t+1} && \forall r \in R_1 \\
& \sum_{\substack{j=1\\ j\neq o}}^{n} \mu_j \underline{y}_{rj}^{t+1} + \mu_o \overline{y}_{ro}^{t+1} \geq \overline{y}_{ro}^{t+1} && \forall r \in R_2 \\
& \varphi \geq 1 \\
& \sum_{j=1}^{n} \mu_j = 1 \\
& \lambda_j, \mu_j \geq 0 && \forall j
\end{aligned}
\tag{14}
$$

To calculate the maximum and the minimum distance of DMU_o in time periods t with the efficient frontier in time period $t+1$, Models (15) and (16) are used, respectively.

$$
\begin{aligned}
&\underline{N}_o^t(x_o^{t+1}, z_o^{t+1}, y_o^{t+1}) = \min \tfrac{\theta}{\varphi} \\
&s.t. \quad \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j \underline{x}_{ij}^t + \lambda_o \bar{x}_{io}^t \leq \theta \bar{x}_{io}^{t+1} && \forall i \in I_1 \\
&\qquad \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j \underline{x}_{ij}^t + \lambda_o \bar{x}_{io}^t \leq \bar{x}_{io}^{t+1} && \forall i \in I_2 \\
&\qquad \sum_{j=1}^{n} \lambda_j \bar{z}_{dj}^t \geq \hat{z}_{do} && \forall d \in D_1 \cup D_2 \\
&\qquad \theta \leq 1 \\
&\qquad \sum_{j=1}^{n} \lambda_j = 1 \\
&\qquad \sum_{j=1}^{n} \mu_j \underline{z}_{dj}^t \leq \hat{z}_{do} && \forall d \in D_1 \cup D_2 \\
&\qquad \sum_{\substack{j=1 \\ j\neq o}}^{n} \mu_j \overline{y}_{rj}^t + \mu_o \underline{y}_{ro}^t \geq \varphi \underline{y}_{ro}^{t+1} && \forall r \in R_1 \\
&\qquad \sum_{\substack{j=1 \\ j\neq o}}^{n} \mu_j \overline{y}_{rj}^t + \mu_o \underline{y}_{ro}^t \geq \underline{y}_{ro}^{t+1} && \forall r \in R_2 \\
&\qquad \varphi \geq 1 \\
&\qquad \sum_{j=1}^{n} \mu_j = 1 \\
&\qquad \lambda_j, \mu_j \geq 0 && \forall j
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&\overline{N}_o^t(x_o^{t+1}, z_o^{t+1}, y_o^{t+1}) = \min \tfrac{\theta}{\varphi} \\
&s.t. \quad \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j \bar{x}_{ij}^t + \lambda_o \underline{x}_{io}^t \leq \theta \underline{x}_{io}^{t+1} && \forall i \in I_1 \\
&\qquad \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j \bar{x}_{ij}^t + \lambda_o \underline{x}_{io}^t \leq \underline{x}_{io}^{t+1} && \forall i \in I_2 \\
&\qquad \sum_{j=1}^{n} \lambda_j \underline{z}_{dj}^t \geq \hat{z}_{do} && \forall d \in D_1 \cup D_2 \\
&\qquad \theta \leq 1 \\
&\qquad \sum_{j=1}^{n} \lambda_j = 1 \\
&\qquad \sum_{j=1}^{n} \mu_j \bar{z}_{dj}^t \leq \hat{z}_{do} && \forall d \in D_1 \cup D_2 \\
&\qquad \sum_{\substack{j=1 \\ j\neq o}}^{n} \mu_j \underline{y}_{rj}^t + \mu_o \overline{y}_{ro}^t \geq \varphi \overline{y}_{ro}^{t+1} && \forall r \in R_1 \\
&\qquad \sum_{\substack{j=1 \\ j\neq o}}^{n} \mu_j \underline{y}_{rj}^t + \mu_o \overline{y}_{ro}^t \geq \overline{y}_{ro}^{t+1} && \forall r \in R_2 \\
&\qquad \varphi \geq 1 \\
&\qquad \sum_{j=1}^{n} \mu_j = 1 \\
&\qquad \lambda_j, \mu_j \geq 0 && \forall j
\end{aligned}
\tag{16}
$$

Finally, Models (17) and (18) are used, respectively, to measure the maximum and minimum distance of DMU_o in time periods $t+1$ with the efficient frontier in the time period t.

$$
\begin{array}{lll}
\underline{N}_o^{t+1}(x_o^t, z_o^t, y_o^t) = \min \frac{\theta}{\varphi} & & \\
s.t. \quad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \underline{x}_{ij}^{t+1} + \lambda_o \bar{x}_{io}^{t+1} \leq \theta \bar{x}_{io}^{t} & \forall i \in I_1 & \\
\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \underline{x}_{ij}^{t+1} + \lambda_o \bar{x}_{io}^{t+1} \leq \bar{x}_{io}^{t} & \forall i \in I_2 & \\
\sum_{j=1}^{n} \lambda_j \overline{z}_{dj}^{t+1} \geq \hat{z}_{do} & \forall d \in D_1 \cup D_2 & \\
\theta \leq 1 & & \\
\sum_{j=1}^{n} \lambda_j = 1 & & \\
\sum_{j=1}^{n} \mu_j \underline{z}_{dj}^{t+1} \leq \hat{z}_{do} & \forall d \in D_1 \cup D_2 & \\
\sum_{\substack{j=1\\ j\neq o}}^{n} \mu_j \overline{y}_{rj}^{t+1} + \mu_o \underline{y}_{ro}^{t+1} \geq \varphi \underline{y}_{ro}^{t} & \forall r \in R_1 & \\
\sum_{\substack{j=1\\ j\neq o}}^{n} \mu_j \overline{y}_{rj}^{t+1} + \mu_o \underline{y}_{ro}^{t+1} \geq \underline{y}_{ro}^{t} & \forall r \in R_2 & \\
\varphi \geq 1 & & \\
\sum_{j=1}^{n} \mu_j = 1 & & \\
\lambda_j, \mu_j \geq 0 & \forall j & (17)
\end{array}
$$

$$
\begin{array}{lll}
\overline{N}_o^{t+1}(x_o^t, z_o^t, y_o^t) = \min \frac{\theta}{\varphi} & & \\
s.t. \quad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \bar{x}_{ij}^{t+1} + \lambda_o \underline{x}_{io}^{t+1} \leq \theta \underline{x}_{io}^{t} & \forall i \in I_1 & \\
\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \bar{x}_{ij}^{t+1} + \lambda_o \underline{x}_{io}^{t+1} \leq \underline{x}_{io}^{t} & \forall i \in I_2 & \\
\sum_{j=1}^{n} \lambda_j \underline{z}_{dj}^{t+1} \geq \hat{z}_{do} & \forall d \in D_1 \cup D_2 & \\
\theta \leq 1 & & \\
\sum_{j=1}^{n} \lambda_j = 1 & & \\
\sum_{j=1}^{n} \mu_j \overline{z}_{dj}^{t+1} \leq \hat{z}_{do} & \forall d \in D_1 \cup D_2 & \\
\sum_{\substack{j=1\\ j\neq o}}^{n} \mu_j \underline{y}_{rj}^{t+1} + \mu_o \overline{y}_{ro}^{t+1} \geq \varphi \overline{y}_{ro}^{t} & \forall r \in R_1 & \\
\sum_{\substack{j=1\\ j\neq o}}^{n} \mu_j \underline{y}_{rj}^{t+1} + \mu_o \overline{y}_{ro}^{t+1} \geq \overline{y}_{ro}^{t} & \forall r \in R_2 & \\
\varphi \geq 1 & & \\
\sum_{j=1}^{n} \mu_j = 1 & & \\
\lambda_j, \mu_j \geq 0 & \forall j & (18)
\end{array}
$$

Finally, the upper and lower bounds of the interval network Malmquist productivity index are calculated using Models (19) and (20):

$$INMPI_o^N(L) = \sqrt{\frac{\underline{N}_o^t(x_o^{t+1}, y_o^{t+1}) \times \underline{N}_o^{t+1}(x_o^{t+1}, y_o^{t+1})}{\overline{N}_o^t(x_o^t, y_o^t) \times \overline{N}_o^{t+1}(x_o^t, y_o^t)}} \tag{19}$$

$$INMPI_o^N(U) = \sqrt{\frac{\overline{N}_o^t(x_o^{t+1}, y_o^{t+1}) \times \overline{N}_o^{t+1}(x_o^{t+1}, y_o^{t+1})}{\underline{N}_o^t(x_o^t, y_o^t) \times \underline{N}_o^{t+1}(x_o^t, y_o^t)}} \tag{20}$$

Theorem 3.1. *If* $(\theta^*, \varphi^*, \lambda^*, \mu^*), (\underline{\theta}, \underline{\varphi}, \underline{\lambda}, \underline{\mu})$ *and* $(\overline{\theta}, \overline{\varphi}, \overline{\lambda}, \overline{\mu})$ *are the optimal solution of Models (7), (11) and (12), respectively, then* $\frac{\underline{\theta}}{\underline{\varphi}} \leq \frac{\theta^*}{\varphi^*} \leq \frac{\overline{\theta}}{\overline{\varphi}}$.

Proof. To prove this proposition, it is sufficient to show that each optimal solution of Model (7) is a feasible solution of Model (11), and each optimal solution of Model (12) is a feasible solution of Model (7). Assume that the optimal solution of Model (7) is $(\theta^*, \varphi^*, \lambda^*, \mu^*)$ and $i \in I_1$. We show that this optimal solution can also established in the constraints of Model (11).
Because of $\lambda_j^* \geq 0$ and $\underline{x}_{ij}^t \geq \tilde{x}_{ij}^t, j = 1, ..., n$, we have:

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^* \underline{x}_{ij}^t + \lambda_o^* \overline{x}_{io}^t \leq \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^* \tilde{x}_{ij}^t + \lambda_o^* \overline{x}_{io}^t + \lambda_o^* \tilde{x}_{io}^t - \lambda_o^* \tilde{x}_{io}^t$$

Now, with respect to λ^* is the optimal solution of Model (7), then it is the feasible solution of this model and with respect to $\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}^t \leq \theta \tilde{x}_{io}^t$, therefore we have:

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^* \underline{x}_{ij}^t + \lambda_o^* \overline{x}_{io}^t \leq \sum_{j=1}^{n} \lambda_j^* \tilde{x}_{ij}^t + \lambda_o^* \overline{x}_{io}^t - \lambda_o^* \tilde{x}_{io}^t \leq \theta^* \tilde{x}_{io}^t + \lambda_o^* \overline{x}_{io}^t - \lambda_o^* \tilde{x}_{io}^t$$
$$= \theta^* \overline{x}_{io}^t - \theta^* \overline{x}_{io}^t + \theta^* \tilde{x}_{io}^t + \lambda_o^* \overline{x}_{io}^t - \lambda_o^* \tilde{x}_{io}^t$$

Since,

$$\theta^* (\tilde{x}_{io} - \overline{x}_{io}^t) - \lambda_o^* (\tilde{x}_{io} - \overline{x}_{io}^t) = (\tilde{x}_{io} - \overline{x}_{io}^t)(\theta^* - \lambda_o^*)$$

and $\overline{x}_{io}^t \geq \tilde{x}_{io}$, we have:

$$\tilde{x}_{io} - \overline{x}_{io}^t \leq 0 \tag{21}$$

and with respect to the first constraint of Model (7):

$$\sum_{j=1}^{n} \lambda_j^* \tilde{x}_{ij}^t \leq \theta^* \tilde{x}_{io}^t \Rightarrow \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^* \tilde{x}_{ij}^t + \lambda_o^* \tilde{x}_{io}^t \leq \theta^* \tilde{x}_{io}^t \overset{\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j^* \tilde{x}_{ij}^t \geq 0}{\Longrightarrow} \lambda_o^* \tilde{x}_{io}^t \leq \theta^* \tilde{x}_{io}^t$$

Therefore, both sides of the relation are divided into $\overline{x}_{io} \neq 0$:

$$\lambda_o^* \leq \theta^* \Rightarrow \theta^* - \lambda_o^* \geq 0 \tag{22}$$

Now, according to Equations (21) and (22), it can be concluded that:

$$-\theta^*\overline{x}_{io}^t + \theta^*\tilde{x}_{io}^t + \lambda_o^*\overline{x}_{io}^t - \lambda_o^*\tilde{x}_{io}^t = \theta^*(\tilde{x}_{io}^t - \overline{x}_{io}^t) - \lambda_o^*(\tilde{x}_{io}^t - \overline{x}_{io}^t)$$
$$= (\tilde{x}_{io}^t - \overline{x}_{io}^t)(\theta^* - \lambda_o^*) \leq 0$$

Therefore,

$$\theta^*\overline{x}_{io}^t - \theta^*\overline{x}_{io}^t + \theta^*\tilde{x}_{io}^t + \lambda_o^*\overline{x}_{io}^t - \lambda_o^*\tilde{x}_{io}^t \leq \theta^*\overline{x}_{io}^t$$

As a result

$$\sum_{\substack{j=1\\j\neq o}}^{n} \lambda_j^*\underline{x}_{ij}^t + \lambda_o^*\overline{x}_{io}^t \leq \theta^*\overline{x}_{io}^t$$

Thus, the optimal solution of Model (7) can be established from the first constraint of Model (11). As a result, the optimal solution of Model (7) is a feasible solution of Model (11), and therefore, the optimum of the objective function of Model (11) is less than or equal to the value of objective function of Model (7), i.e. $\frac{\underline{\theta}}{\underline{\varphi}} \leq \frac{\theta^*}{\varphi^*}$. And in a similar manner, the relation of $\frac{\theta^*}{\varphi^*} \leq \frac{\overline{\theta}}{\overline{\varphi}}$ can be proved and consequently the proof can be completed. Also, if $i \in I_2$, it is enough to put θ^* equal to one in the above proof process and repeat the same proof process. Therefore, we have $\sum_{\substack{j=1\\j\neq o}}^{n} \lambda_j^*\underline{x}_{ij}^t + \lambda_o^*\overline{x}_{io}^t \leq \overline{x}_{io}^t$.

Moreover, in the same manner, theorem it also can be proofed for the output constraints. At the end, the theorem is proofed for the constraints related to intermediate measures. Based on the definition, we have $\sum_{j=1}^{n} \lambda_j^*\overline{z}_{dj} \geq \sum_{j=1}^{n} \lambda_j^*\tilde{z}_{dj}$.

Since λ^* is the optimal solution of Model (7), therefore, it also is the feasible solution of this model, and with respect to $\sum_{j=1}^{n} \lambda_j^*\tilde{z}_{dj} \geq \hat{z}_{do}^*$, we have $\sum_{j=1}^{n} \lambda_j^*\overline{z}_{dj} \geq \sum_{j=1}^{n} \lambda_j^*\tilde{z}_{dj} \geq \hat{z}_{do}^*$. Then, according to the constraint of Model (11), the relation of $\sum_{j=1}^{n} \lambda_j^*\overline{z}_{dj} \geq \sum_{j=1}^{n} \lambda_j^*\tilde{z}_{dj} \geq \hat{z}_{do}^* \geq \sum_{j=1}^{n} \mu_j^*\tilde{z}_{dj}$ is

conducted.
Finally, we have $\sum_{j=1}^{n} \lambda_j^* \overline{z}_{dj} \geq \sum_{j=1}^{n} \lambda_j^* \tilde{z}_{dj} \geq \hat{z}_{do}^* \geq \sum_{j=1}^{n} \mu_j^* \tilde{z}_{dj} \geq \sum_{j=1}^{n} \mu_j^* \underline{z}_{dj}$.
Thus, the optimal solution of Model (7) can be established from the sixth constraint of Model (11). As a result, the optimal solution of Model (7) is a feasible solution of Model (11), and therefore, the optimal value of the objective function of Model (11) is less than or equal to the optimal value of objective function of Model (7) and consequently the proof can be completed. □

Now, according to the results of INMPI for stage 1, stage 2, and overall, that obtained from Equations (19) and (20), the productivity changes of DMUs from first stage, second stage and overall viewpoints can be classified as follows, respectively:

Stage 1 (S1):

- If $INMPI_o^{N(S1)}(L) > 1 \Rightarrow DMU_o^{S1} \in S1^{++}$
- If $\frac{INMPI_o^{N(S1)}(L)+INMPI_o^{N(S1)}(U)}{2} > 1 \Rightarrow DMU_o^{S1} \in S1^{+}$
- If $\frac{INMPI_o^{N(S1)}(L)+INMPI_o^{N(S1)}(U)}{2} = 1 \Rightarrow DMU_o^{S1} \in S1$
- If $\frac{INMPI_o^{N(S1)}(L)+INMPI_o^{N(S1)}(U)}{2} < 1 \Rightarrow DMU_o^{S1} \in S1^{-}$
- If $INMPI_o^{N(S1)}(U) < 1 \Rightarrow DMU_o^{S1} \in S1^{--}$

Stage 2 (S2):

- If $INMPI_o^{N(S2)}(L) > 1 \Rightarrow DMU_o^{S2} \in S2^{++}$
- If $\frac{INMPI_o^{N(S2)}(L)+INMPI_o^{N(S2)}(U)}{2} > 1 \Rightarrow DMU_o^{S2} \in S2^{+}$
- If $\frac{INMPI_o^{N(S2)}(L)+INMPI_o^{N(S2)}(U)}{2} = 1 \Rightarrow DMU_o^{S2} \in S2$
- If $\frac{INMPI_o^{N(S2)}(L)+INMPI_o^{N(S2)}(U)}{2} < 1 \Rightarrow DMU_o^{S2} \in S2^{-}$
- If $INMPI_o^{N(S2)}(U) < 1 \Rightarrow DMU_o^{S2} \in S2^{--}$

Overall (O):

- If $INMPI_o^{N(O)}(L) > 1 \Rightarrow DMU_o^O \in O^{++}$
- If $\frac{INMPI_o^{N(O)}(L)+INMPI_o^{N(O)}(U)}{2} > 1 \Rightarrow DMU_o^O \in O^{+}$
- If $\frac{INMPI_o^{N(O)}(L)+INMPI_o^{N(O)}(U)}{2} = 1 \Rightarrow DMU_o^O \in O$
- If $\frac{INMPI_o^{N(O)}(L)+INMPI_o^{N(O)}(U)}{2} < 1 \Rightarrow DMU_o^O \in O^{-}$
- If $INMPI_o^{N(O)}(U) < 1 \Rightarrow DMU_o^O \in O^{--}$

It should be explained that based on the lower and upper values of the proposed index interval, the trend and how the interest rate changes on the DMU under consideration can be divisible as follows:
Stage 1:

- If $DMU_o \in S1^{++}$, it indicates progress in productivity of the stage 1.
- If $DMU_o \in S1^{+}$, it indicates marginally progress in productivity of the stage 1.
- If $DMU_o \in S1$, it indicates that no change in the productivity of the stage 1 has occurred.
- If $DMU_o \in S1^{-}$, it indicates marginally regress in productivity of the stage 1.
- If $DMU_o \in S1^{--}$, it indicates regress in productivity of the stage 1.

Stage 2:

- If $DMU_o \in S2^{++}$, it indicates progress in productivity of the stage 2.

- If $DMU_o \in S2^{+}$, it indicates marginally progress in productivity of the stage 2.
- If $DMU_o \in S2$, it indicates that no change in the productivity of the stage 2 has occurred.
- If $DMU_o \in S2^{-}$, it indicates marginally regress in productivity of the stage 2.
- If $DMU_o \in S2^{--}$, it indicates regress in productivity of the stage 2.

Overall:

- If $DMU_o \in O^{++}$, it indicates progress in productivity of the overall.
- If $DMU_o \in O^{+}$, it indicates marginally progress in productivity of the overall.
- If $DMU_o \in O$, it indicates that no change in the productivity of the overall has occurred.
- If $DMU_o \in O^{-}$, it indicates marginally regress in productivity of the overall.
- If $DMU_o \in O^{--}$, it indicates regress in productivity of the overall.

4 Real-Life Case Study

In this section, the proposed research approach is implemented using real data. For this purpose, data of 10 insurance companies in Iran were extracted for two consecutive years 2014 and 2015.
It should be noted that inputs 1 and 2 are respectively operation costs and insurance costs, intermediate measures 1 and 2 are respectively direct written premium and reinsurance premium and finally outputs 1 and 2 are underwriting profit and investment profit, respectively. Given these explanations, the upper and lower bounds of the data for all 10

insurance companies for the years 2014 and 2015 are set out in Tables 2 and 3, respectively:

Table 2: The Real Data for Insurance Companies - 2014

	Insurance Companies	x_1	x_2	z_1	z_2	y_1	y_2
Lower Bound	IC 01	3562766	854122	8155585	777572	2183815	32371
	IC 02	4143911	778104	10134219	1333132	1771955	1002725
	IC 03	3462208	2105323	14976590	887621	4371693	353410
	IC 04	2632175	1611667	8065327	580714	1708637	441415
	IC 05	3618236	777112	14719272	837500	4083786	659211
	IC 06	2473610	509647	6413810	537261	937123	349171
	IC 07	2544177	1849264	8608785	441331	3661588	482720
	IC 08	976611	661180	4748616	1076082	752263	283708
	IC 09	267175	286849	2156783	526368	609720	61952
	IC 10	284817	65216	421432	281750	454386	39288
Upper Bound	IC 01	3634742	871376	8320345	793280	2227933	33025
	IC 02	4227627	793824	10338951	1360064	1807753	1022983
	IC 03	3532152	2147855	15279148	905553	4460011	360550
	IC 04	2685351	1644225	8228263	592446	1743155	450333
	IC 05	3691332	792812	15016632	854420	4166286	672529
	IC 06	2523582	519943	6543382	548115	956055	356225
	IC 07	2595575	1886622	8782699	450247	3735560	492472
	IC 08	996341	674538	4844548	1097822	767461	289440
	IC 09	272573	292643	2200355	537002	622038	63204
	IC 10	290571	66534	429946	287442	463566	40082

Table 3: The Real Data for Insurance Companies - 2015

	Insurance Companies	x_1	x_2	z_1	z_2	y_1	y_2
Lower Bound	IC 01	3760541	906334	8806692	817234	2289532	34351
	IC 02	4552639	815458	11543228	1445266	1876531	1046415
	IC 03	3827327	2125488	16421621	917314	5212335	375861
	IC 04	3113833	1681562	8876324	629134	1894517	491926
	IC 05	4227158	838483	15051617	904428	4916213	729230
	IC 06	2866644	531617	7094322	596523	986621	411720
	IC 07	2936217	2342321	9173127	507767	4085688	564155
	IC 08	1036515	709428	5184232	1115323	818318	412432
	IC 09	323715	333531	2191521	590906	719332	75226
	IC 10	353417	71734	482088	322232	521224	49766
Upper Bound	IC 01	3836511	924644	8984604	833744	2335786	35045
	IC 02	4644611	831932	11776424	1474464	1914441	1067555
	IC 03	3904647	2168428	16753371	935846	5317635	383455
	IC 04	3176739	1715532	9055644	641844	1932791	501864
	IC 05	4312556	855423	15355691	922700	5015531	743962
	IC 06	2924556	542357	7237642	608573	1006553	420038
	IC 07	2995535	2389641	9358443	518025	4168228	575553
	IC 08	1057455	723760	5288964	1137855	834850	420764
	IC 09	330255	340269	2235795	602844	733864	76746
	IC 10	360557	73184	491828	328742	531754	50772

Now, after collecting the data, the results of solving and implementing the Models (11) to (18) previously presented in Section 3 are shown in Tables 4 to 7, respectively:

Table 4: The Results of Models (11) and (12), $DN_o^t(x_o^t, z_o^t, y_o^t)$

Insurance	Stage 1		Stage 2		Overall	
Companies	Lower	Upper	Lower	Upper	Lower	Upper
IC 01	0.57	0.61	0.58	0.61	0.33	0.37
IC 02	0.61	0.98	1.00	1.00	0.61	0.98
IC 03	0.54	0.55	1.00	1.00	0.54	0.55
IC 04	0.75	0.82	0.60	0.62	0.45	0.51
IC 05	0.62	0.62	1.00	1.00	0.62	0.62
IC 06	1.00	1.00	0.40	0.42	0.40	0.42
IC 07	0.72	0.75	1.00	1.00	0.72	0.75
IC 08	1.00	1.00	0.36	0.38	0.36	0.38
IC 09	1.00	1.00	0.53	0.55	0.53	0.55
IC 10	1.00	1.00	0.12	0.13	0.12	0.13

Table 5: The Results of Models (13) and (14), $DN_o^{t+1}(x_o^{t+1}, z_o^{t+1}, y_o^{t+1})$

Insurance	Stage 1		Stage 2		Overall	
Companies	Lower	Upper	Lower	Upper	Lower	Upper
IC 01	0.68	0.68	0.53	0.57	0.36	0.39
IC 02	0.66	0.98	1.00	1.00	0.66	0.98
IC 03	0.53	0.55	1.00	1.00	0.53	0.55
IC 04	0.73	0.79	0.61	0.63	0.44	0.50
IC 05	0.64	0.64	1.00	1.00	0.64	0.64
IC 06	1.00	1.00	0.42	0.44	0.42	0.44
IC 07	0.68	0.70	1.00	1.00	0.68	0.70
IC 08	0.94	1.00	0.89	0.89	0.83	0.89
IC 09	1.00	1.00	0.58	0.60	0.58	0.60
IC 10	1.00	1.00	0.13	0.99	0.13	0.99

Table 6: The Results of Models (15) and (16), $DN_o^{t}(x_o^{t+1}, z_o^{t+1}, y_o^{t+1})$

Insurance	Stage 1		Stage 2		Overall	
Companies	Lower	Upper	Lower	Upper	Lower	Upper
IC 01	0.54	0.58	0.61	0.64	0.33	0.37
IC 02	0.58	0.93	1.05	1.77	0.61	1.65
IC 03	0.49	0.5	1.4	1.45	0.68	0.73
IC 04	0.65	0.7	0.67	0.69	0.43	0.49
IC 05	0.57	0.57	1.37	1.43	0.78	0.81
IC 06	1.27	1.38	0.45	0.47	0.57	0.65
IC 07	0.63	0.66	1.13	1.13	0.71	0.74
IC 08	0.46	1.02	0.77	1.49	0.36	1.51
IC 09	0.87	0.87	0.61	0.64	0.53	0.55
IC 10	6.36	6.84	0.14	0.15	0.89	0.99

Table 7: The Results of Models (17) and (18), $DN_o^{t+1}(x_o^{t}, z_o^{t}, y_o^{t})$

Insurance	Stage 1		Stage 2		Overall	
Companies	Lower	Upper	Lower	Upper	Lower	Upper
IC 01	0.72	0.73	0.51	0.55	0.37	0.4
IC 02	0.69	1.04	0.97	1.5	0.66	1.57
IC 03	0.58	0.59	1.05	1.09	0.6	0.65
IC 04	0.84	0.91	0.55	0.57	0.46	0.52
IC 05	0.73	0.74	1.04	1.09	0.76	0.8
IC 06	1.44	1.57	0.37	0.39	0.54	0.61
IC 07	0.78	0.81	0.9	0.9	0.7	0.73
IC 08	1.98	2.04	0.36	0.38	0.72	0.78
IC 09	1.21	1.21	0.49	0.51	0.59	0.62
IC 10	1.24	8.54	0.11	0.83	0.13	7.09

Finally, the values of the proposed efficiency index for each marketing and investment process as well as the entire insurance company are shown in Table 8.

Table 8: The Results of INMPI

Insurance Companies	Marketing			Investment			Overall		
	Lower	Upper	Class	Lower	Upper	Class	Lower	Upper	Class
IC 01	0.91	0.98	S1–	0.99	1.11	S2+	0.9	1.09	O-
IC 02	0.61	1.48	S1+	0.84	1.35	S2+	0.51	2	O+
IC 03	0.89	0.94	S1–	1.13	1.18	S2++	1.01	1.11	O++
IC 04	0.79	0.94	S1–	1.07	1.16	S2++	0.84	1.08	O-
IC 05	0.89	0.9	S1–	1.12	1.17	S2++	1	1.05	O++
IC 06	0.9	0.98	S1–	1.08	1.18	S2++	0.98	1.15	O+
IC 07	0.83	0.9	S1–	1.12	1.12	S2++	0.94	1.02	O-
IC 08	0.46	0.72	S1–	2.16	3.19	S2++	1	2.29	O++
IC 09	0.85	0.85	S1–	1.12	1.22	S2++	0.95	1.03	O-
IC 10	0.86	2.35	S1+	0.41	3.3	S2+	0.35	7.75	O+

As shown in Table 8, from the marketing perspective, the majority of insurance companies including IC 01, IC 03, IC 04, IC 05, IC 06, IC 07, IC 08, and IC 09 had regressed and they fall into category $S1$. The main reason of this regress is due to the lack of proper cost management in ICs and this issue should be modified.

Unlike the stage 1, in the stage 2, most of insurance companies including IC 03, IC 04, IC 05, IC 06, IC 07, IC 08, IC 09 from the investment perspective had an acceptable improvement in their productivity and they fall into category $S2$. Also, the rest of ICs including IC 01, IC 02, IC 10 fall into category $S2$. Therefore, it can be concluded that the management of ICs in the investment function is acceptable.

In terms of overall performance, only three insurance companies including IC 03, IC 05 and IC 08 have also achieved a good performance in 2015 compared to 2014 and classified as O . As a result, these insurance companies can be considered as benchmark for other Insurance companies mangers. In other words, the management style of these growing companies can be applied to other companies as a benchmark.

5 Conclusions and Future Research Directions

In this study, we presented a new productivity index called Malmquist Interval Network Productivity Index with the aim of measuring insurance productivity changes over time. It should be explained that the reason for presenting this index was the need to consider the structure and internal relationships of insurance companies, as well as the uncertainty in the data in the process of calculating productivity changes. Because ignoring the internal structure of companies, along with data uncertainty, may lead to invalid results. Finally, the efficiency of the proposed approach was evaluated using real data obtained from 10 insurance companies in Iran, which results indicate the robustness of the new index. For the future studies, the other uncertain programming approaches such as stochastic programming, fuzzy mathematical programming and robust optimization, can also be used to deal with different type of data uncertainty. Moreover, non-discretionary factors can be considered in performance assessment procedure of insurance companies.

References

[1] Almulhim, T. *Analysis of Takaful vs. Conventional insurance firms' efficiency: Two-stage DEA of Saudi Arabia's insurance market.* Cogent Business and Management, 6(1)(2019), 1633807.

[2] An, Q., Yan, H., Wu, J., and Liang, L. *Internal resource waste and centralization degree in two-stage systems: An efficiency analysis.* Omega, 61 (2016), 89-99.

[3] Anandarao, S., Durai, S. R. S., and Goyari, P. *Efficiency decomposition in two-stage data envelopment analysis: an application to life insurance companies in India.* Journal of Quantitative Economics, 17(2)(2019), 271-285.

[4] Banker, R. D., Charnes, A., and Cooper, W. W. *Some models for estimating technical and scale inefficiencies in data envelopment analysis.* Management science, 30(9)(1984), 1078-1092.

[5] Charnes, A., Cooper, W. W., and Rhodes, E. *Measuring the efficiency of decision making units.* European Journal of Operational Research, 2(6)(1978), 429-444.

[6] Chen, Y., and Zhu, J. *Measuring information technology's indirect impact on firm performance.* Information Technology and Management, 5(1-2)(2004), 9-22.

[7] Chen, Y., Cook, W. D., Li, N., and Zhu, J. *Additive efficiency decomposition in two-stage DEA.* European Journal of Operational Research, 196(3)(2009), 1170-1176.

[8] Despotis, D. K., and Smirlis, Y. G. *Data Envelopment Analysis with Imprecise Data.* European Journal of Operational Research, 140(1)(2002), 24-36.

[9] Despotis, D. K., Koronakos, G., and Sotiros, D. *Composition versus decomposition in two-stage network DEA: a reverse approach.* Journal of Productivity Analysis, 45(1)(2016), 71-87.

[10] Du, J., Liang, L., Chen, Y., Cook, W. D., and Zhu, J. *A bargaining game model for measuring performance of two-stage network structures.* European Journal of Operational Research, 210(2)(2011), 390-397.

[11] Fang, L. *Stage efficiency evaluation in a two-stage network data envelopment analysis model with weight priority.* Omega,(2019), 102081.

[12] Färe, R., and Grosskopf, S. *Malmquist productivity indexes and Fisher ideal indexes.* The Economic Journal, 102(410)(1992), 158-160.

[13] Gharakhani, D., Eshlaghy, A. T., Hafshejani, K. F., Mavi, R. K., and Lotfi, F. H. *Common weights in dynamic network DEA with goal programming approach for performance assessment of insurance companies in Iran.* Management Research Review, (2018).

[14] Hatami-Marbini, A., and Saati, S. *Measuring performance with common weights: network DEA.* Neural Computing and Applications, (2019), 1-19.

[15] Kao, C. *Efficiency decomposition in network data envelopment analysis: A relational model.* European Journal of Operational Research, 192(3)(2009), 949-962.

[16] Kao, C., and Hwang, S. N. *Efficiency decomposition in two-stage data envelopment analysis: An application to non-life insurance companies in Taiwan.* European journal of operational research, 185(1)(2008), 418-429.

[17] Krupa, T., Farbarzevics, K., and Salame, B. *Investment efficiency of life insurance companies in Germany: application of a two-stage SBM.* Contemporary Economy, 10(1 (32))(2019), 79-91.

[18] Li, H., Chen, C., Cook, W. D., Zhang, J., and Zhu, J. *Two-stage network DEA: Who is the leader?.* Omega, 74 (2018), 15-19.

[19] Nourani, M., Devadason, E. S., and Chandran, V. G. R. *Measuring technical efficiency of insurance companies using dynamic network DEA: an intermediation approach.* Technological and Economic Development of Economy, 24(5)(2018), 1909-1940.

[20] Tone, K., Kweh, Q. L., Lu, W. M., and Ting, I. W. K. *Modeling investments in the dynamic network performance of insurance companies.* Omega, 88(2019), 237-247.

Fatemeh Sadat Seyed Esmaeili
Ph.D of Mathematics
Department of Mathematics
Science and Research Branch, Islamic Azad University
Tehran, Iran
E-mail: rozita.esmaeeli@gmail.com

Mohsen Rostamy-Malkhalifeh
Associate Professor of Mathematics
Department of Mathematics
Science and Research Branch, Islamic Azad University
Tehran, Iran
E-mail: mohsen_rostamy@yahoo.com

Farhad Hosseinzadeh Lotfi
Full Professor of Mathematics
Department of Mathematics
Science and Research Branch, Islamic Azad University
Tehran, Iran
E-mail: farhad@hosseinzadeh.ir