Journal of Mathematical Extension
Vol. 16, No. 6, (2022) (8)1-16
URL: https://doi.org/10.30495/JME.2022.1501
ISSN: 1735-8299
Original Research Paper

Perfect 4-Colorings of the 3-Regular Graphs of Order at Most 8

Z. Vahedi
Karaj Branch, Islamic Azad University

M. Alaeiyan*
Iran University of Science and Technology

M. Maghasedi
Karaj Branch, Islamic Azad University

Abstract. The perfect m-coloring with matrix $A = [a_{ij}]_{i,j\in\{1,\cdots,m\}}$ of a graph $G = (V, E)$ with $\{1, \cdots, m\}$ color is a vertex coloring of G with m-color so that the number of vertices in color j adjacent to a fixed vertex in color i is a_{ij}, independent of the choice of vertex in color i. The matrix $A = [a_{ij}]_{i,j\in\{1,\cdots,m\}}$ is called the parameter matrix.

We study the perfect 4-colorings of the 3-regular graphs of order at most 8, that is, we determine a list of all color parameter matrices corresponding to perfect 4-colorings of 3-regular graphs of orders 4, 6, and 8.

AMS Subject Classification: 03E02, 05C15, 68R05
Keywords and Phrases: Perfect coloring, Parameter matrices, Cubic graph, Equitable partition.

Received: December 2019; Accepted: September 2020
*Corresponding Author

1 Introduction

The concept of a perfect m-coloring plays an important role in graph theory, algebraic combinatorics, and coding theory (Completely regular codes). There is another term for this concept in the literature as "equitable partition" (see [8]).

The existence of completely regular codes in graphs is a historical problem in mathematics. Completely regular codes are a generalization of perfect codes. In 1973, Delsarte conjectured the non-existence of perfect codes in Johnson graphs. Therefore, some effort has been done on enumerating the parameter matrices of some Johnson graphs, including $J(6,3)$, $J(7,3)$, $J(8,3)$, $J(8,4)$, and $J(n,3)$ (n odd) (see [2], [3] and [7]).

Fon-Der-Flaass enumerated the parameter matrices of n-dimensional hypercube Q_n for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the n-dimensional hypercube with a given parameter matrix (see [4], [5] and [6]).

In [1] all perfect 3-colorings of the cubic graphs of order 10 were described.

In this paper we enumerate the parameter matrices of all perfect 4-colorings of the 3-regular graphs of order at most 8.

2 Preliminaries

In this section we use the following definition.

Definition 2.1. *For each graph G and each integer m, a mapping $T : V(G) \rightarrow \{1, \cdots, m\}$ is called a perfect m-coloring with matrix $A = [a_{ij}]_{i,j\in\{1,\cdots,m\}}$, if it is surjective and for all i, j for every vertex of color i , the number of its neighbors of color j is equal to a_{ij}. The matrix A is called the parameter matrix of a perfect coloring.*

The spectrum of a matrix A, denoted by $\sigma(A)$ is the set of all eigenvalues of A. The set of eigenvalues of the adjacency matrix of graph G is called the spectrum of G.

We denote $M_r(4)$ for all parameter matrices of the perfect 4-colorings of r-regular graphs. Note that if $A \in M_r(4)$, then the sum of entries for each row is equal to r.

If $A = [a_{ij}]_{n\times n}$ is a perfect 4-colorings matrix for a 3-regular graph $G = (V, E)$, then $\sum_{j=1}^{4} a_{ij} = 3$ for all $1 \leq i \leq 4$. So there are 20 different models for each row of matrices. Hence there are 20^4 matrices.

Let $A = [a_{ij}]_{4\times 4}$ be a 4-color parameter matrix for a graph $G = (V, E)$. The first observation says A must possess a weak form of symmetry, described in the following lemma:

Lemma 2.2. *Suppose $A = [a_{ij}]_{n\times n}$ is a parameter matrix for a graph $G = (V, E)$. Then, $a_{ij} = 0$ if and only if $a_{ji} = 0$ for $1 \leq i, j \leq n$.*

Definition 2.3. *Let A and B are two parameter matrices of the perfect 4-colorings of graph G. We define A and B are equivalent if A transformed to B by a permutation on colors and we use the symbol $\sim$ to show it.*

We have the obvious lemmas:

Lemma 2.4. *Let $A = [a_{ij}]_{4\times 4}$ and $A \in M_3(4)$ and $\sigma \in S_4$ (where S_4 is the symmetric group of degree 4), then $[a_{ij}]_{4\times 4} \sim [a_{i\sigma(j)}]_{4\times 4}$.*

Lemma 2.5. *Let $A = [a_{ij}]_{4\times 4} \in M_3(4)$. Then the following cases do not happen.*

1) $a_{14} = 0$, $a_{13} = 0$, $a_{12} = 0$;

2) $a_{24} = 0$, $a_{23} = 0$, $a_{21} = 0$;

3) $a_{34} = 0$, $a_{32} = 0$, $a_{31} = 0$;

4) $a_{43} = 0$, $a_{42} = 0$, $a_{41} = 0$;

Lemma 2.6. *Suppose $A \in M_3(4)$. Then there is not $\sigma \in S_4$ such that* $[a_{i\sigma(j)}] = \begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$

Proof. It is clear with connectivity. □

Remark 2.7. *Suppose $A \in M_3(4)$ is a parameter matrix for a 3-regular graph G. If there is $\sigma \in S_4$ such that* $A = [a_{i\sigma(j)}] = \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix}$ *, then G is bipartite.*

To see this, V is the set of vertices of G. Divide V into two independent sets V_1 and V_2 include vertices with color number 3, 4 and 1, 2 respectively. According to matrix structure, thus vertices V_1 are nonadjacent. Similarly for the vertices V_2. Therefore G is a bipartite graph.

It is easy to see that each perfect coloring on a graph G, create an equitable partition. So by ([8], lemma 1.1), we have the following lemma.

Lemma 2.8. *Suppose $A \in M_3(4)$ is a coloring matrix for graph G. Then the spectrum of A is a subset of the spectrum of G.*

Lemma 2.9. *If $A \in M_3(4)$, then all of the eigenvalues of A are real.*

Proof. By symmetry of adjancy matrices of G is obvious. □

Proposition 2.10. *Let* $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$ *be a color incidence matrix of some connected graph $G = (V, E)$. Let $|v|$ denote the number of vertices of G and v_i denote color i; ($1 \le i \le 4$).*

1) If $b \neq 0$, $c \neq 0$ and $d \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{c}{i} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{ec}{bi} + \frac{ed}{bm}}$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{ib}{ce} + 1 + \frac{id}{cm}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{mb}{de} + \frac{mc}{di} + 1}$$

2) If $b \neq 0$, $c \neq 0$ and $h \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{c}{i} + \frac{bh}{en}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{ec}{bi} + \frac{h}{n}}$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{ib}{ce} + 1 + \frac{ibh}{cen}}, v_4 = \frac{|v|}{\frac{ne}{hb} + \frac{n}{h} + \frac{nec}{hbi} + 1}$$

3) If $b \neq 0$, $c \neq 0$ and $l \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{c}{i} + \frac{cl}{io}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{ec}{bi} + \frac{ecl}{bio}}$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{ib}{ce} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{oi}{lc} + \frac{oib}{lce} + \frac{o}{l} + 1}$$

4) If $b \neq 0$, $d \neq 0$ and $g \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{bg}{ej} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{g}{j} + \frac{ed}{bm}}$$

$$v_3 = \frac{|v|}{\frac{je}{gb} + \frac{j}{g} + 1 + \frac{jed}{gbm}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{mb}{de} + \frac{mbg}{dej} + 1}$$

5) If $b \neq 0$, $d \neq 0$ and $l \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{do}{ml} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{edo}{bml} + \frac{ed}{bm}}$$

$$v_3 = \frac{|v|}{\frac{lm}{od} + \frac{lmb}{ode} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{mb}{de} + \frac{o}{l} + 1}$$

6) If $b \neq 0$, $g \neq 0$ and $h \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{bg}{ej} + \frac{bh}{en}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{g}{j} + \frac{h}{n}}$$

$$v_3 = \frac{|v|}{\frac{je}{gb} + \frac{j}{g} + 1 + \frac{jh}{gn}}, v_4 = \frac{|v|}{\frac{ne}{hb} + \frac{n}{h} + \frac{ng}{hj} + 1}$$

7) If $b \neq 0$, $g \neq 0$ and $l \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{bg}{ej} + \frac{bgl}{ejo}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{g}{j} + \frac{gl}{jo}}$$

$$v_3 = \frac{|v|}{\frac{je}{gb} + \frac{j}{g} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{oje}{lgb} + \frac{oj}{lg} + \frac{o}{l} + 1}$$

8) If $b \neq 0$, $h \neq 0$ and $l \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{b}{e} + \frac{bho}{enl} + \frac{bh}{en}}, v_2 = \frac{|v|}{\frac{e}{b} + 1 + \frac{ho}{nl} + \frac{h}{n}}$$

$$v_3 = \frac{|v|}{\frac{lne}{ohb} + \frac{ln}{oh} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{ne}{hb} + \frac{n}{h} + \frac{o}{l} + 1}$$

9) If $c \neq 0$, $d \neq 0$ and $g \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{cj}{ig} + \frac{c}{i} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{gi}{cj} + 1 + \frac{g}{j} + \frac{gid}{jcm}}$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{j}{g} + 1 + \frac{id}{cm}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{mcj}{dig} + \frac{mc}{di} + 1}$$

10) If $c \neq 0$, $d \neq 0$ and $h \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{dn}{mh} + \frac{c}{i} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{hm}{dn} + 1 + \frac{hmc}{ndi} + \frac{h}{n}}$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{idn}{cmh} + 1 + \frac{id}{cm}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{n}{h} + \frac{mc}{di} + 1}$$

11) If $c \neq 0$, $g \neq 0$ and $h \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{cj}{ig} + \frac{c}{i} + \frac{cjh}{ign}}, v_2 = \frac{|v|}{\frac{gi}{jc} + 1 + \frac{g}{j} + \frac{h}{n}}$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{j}{g} + 1 + \frac{jh}{gn}}, v_4 = \frac{|v|}{\frac{ngi}{hjc} + \frac{n}{h} + \frac{ng}{hj} + 1}$$

12) If $c \neq 0$, $g \neq 0$ and $l \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{cj}{ig} + \frac{c}{i} + \frac{cl}{io}}, v_2 = \frac{|v|}{\frac{gi}{jc} + 1 + \frac{g}{j} + \frac{gl}{jo}}$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{j}{g} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{oi}{lc} + \frac{oj}{lg} + \frac{o}{l} + 1}$$

13) If $c \neq 0$, $h \neq 0$ and $l \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{cln}{ioh} + \frac{c}{i} + \frac{cl}{io}}, v_2 = \frac{|v|}{\frac{hoi}{nlc} + 1 + \frac{ho}{nl} + \frac{h}{n}}$$

$$v_3 = \frac{|v|}{\frac{i}{c} + \frac{ln}{oh} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{oi}{lc} + \frac{n}{h} + \frac{o}{l} + 1}$$

14) If $d \neq 0$, $g \neq 0$ and $h \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{dn}{mh} + \frac{dng}{mhj} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{hm}{nd} + 1 + \frac{g}{j} + \frac{h}{n}}$$

$$v_3 = \frac{|v|}{\frac{jhm}{gnd} + \frac{j}{g} + 1 + \frac{jh}{gn}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{n}{h} + \frac{ng}{hj} + 1}$$

15) If $d \neq 0$, $g \neq 0$ and $l \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{doj}{mlg} + \frac{do}{ml} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{glm}{jod} + 1 + \frac{g}{j} + \frac{gl}{jo}}$$

$$v_3 = \frac{|v|}{\frac{lm}{od} + \frac{j}{g} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{oj}{lg} + \frac{o}{l} + 1}$$

16) If $d \neq 0$, $h \neq 0$ and $l \neq 0$ then

$$v_1 = \frac{|v|}{1 + \frac{dn}{mh} + \frac{do}{ml} + \frac{d}{m}}, v_2 = \frac{|v|}{\frac{hm}{nd} + 1 + \frac{ho}{nl} + \frac{h}{n}}$$

$$v_3 = \frac{|v|}{\frac{lm}{od} + \frac{ln}{oh} + 1 + \frac{l}{o}}, v_4 = \frac{|v|}{\frac{m}{d} + \frac{n}{h} + \frac{o}{l} + 1}$$

By using above proposition and lemmas for n= 4, 6, 8 we only have the following matrices, which we have shown with M_1, ..., M_{42}.

$$M_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_3 = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, M_5 = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, M_6 = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

$$M_7 = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, M_8 = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_9 = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix},$$

$$M_{10} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, M_{11} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}, M_{12} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

$$M_{13} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_{14} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}, M_{15} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

$$M_{16} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_{17} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, M_{18} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

$$M_{19} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_{20} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}, M_{21} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix},$$

$$M_{22} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}, M_{23} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}, M_{24} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix},$$

$$M_{25} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}, M_{26} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, M_{27} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

$$M_{28} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}, M_{29} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \end{bmatrix}, M_{30} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

$$M_{31} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}, M_{32} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}, M_{33} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix},$$

$$M_{34} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_{35} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix}, M_{36} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix},$$

$$M_{37} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}, M_{38} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_{39} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix},$$

$$M_{40} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_{41} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}, M_{42} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

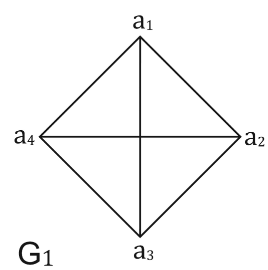

Figure 1: Connected 3-regular graph of order 4

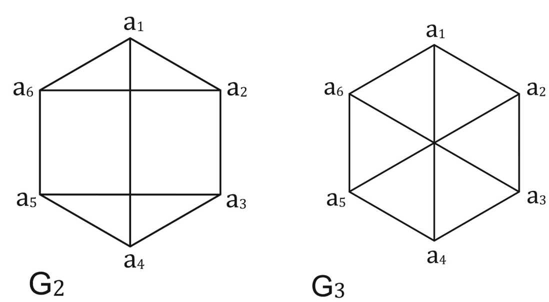

Figure 2: Connected 3-regular graphs of order 6

3 Main Results

None isomorphic 3-regular connected graphs of order 4, 6 and 8 are shown below in figures 1, 2 and 3.

Theorem 3.1. *The parameter matrix of 3-regular graph of order* 4 *is just* M_1.

Proof. Because each vertex is colored with one color. □

Theorem 3.2. *If* M *is a perfect 4-colorings matrix of the 3-regular graph of order 6, then only the matrices* M_{23}, M_{30} *for* G_2 *and* M_{20} *for* G_3 *can be parameter matrices.*

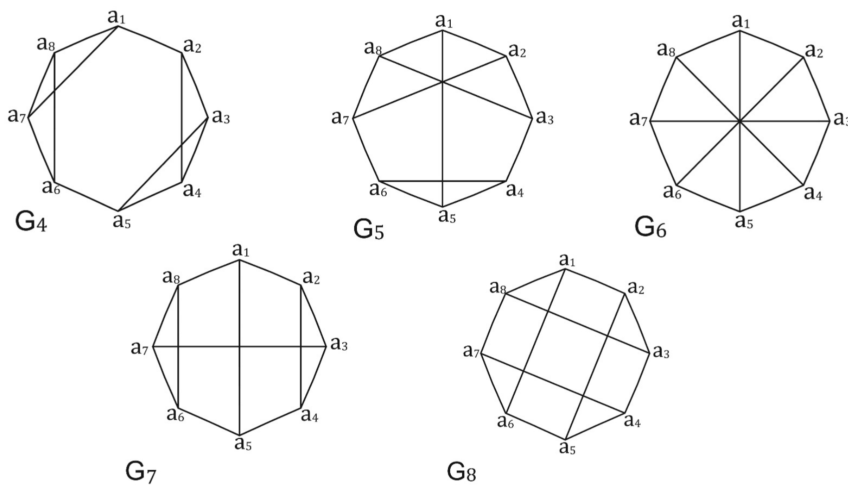

Figure 3: Connected 3-regular graphs of order 8

Proof. With consideration of 3-regular graphs eigenvalues and using Lemma 2.5, it can be seen the connected 3-regular graphs with 6 vertices can have perfect 4-colorings with matrices M_{20}, M_{23} and M_{30}.

So we introduce 3-regular graphs of order 6 that have perfect 4-colorings. Now we introduce the mappings of all graphs that have perfect 4-colorings with the parameter matrices.

The graph G_2 has perfect 4-colorings with matrix M_{23}. Consider the mapping T as follows:

$T(a_1) = 1$, $T(a_3) = T(a_5) = 2$, $T(a_2) = T(a_6) = 3$, $T(a_4) = 4$.

There is no perfect 4-colorings with the matrix M_{30} for the graph G_2. Contrary to our claim, suppose that T is a perfect 4-colorings with the matrix M_{30} for the graph G_2. Then according to the matrix M_{30}, by symmetry if T, is a coloring, then we have 2 cases for the color of number 1 as follows:

$T(a_1) = 1$ or $T(a_2) = 1$.

(1) $T(a_1) = 1$, according to the matrix M_{30}, $T(a_2) = 3$, $T(a_6) = 3$ as a result $T(a_4) = 4$, it follows that $T(a_3) = 4$ or $T(a_5) = 4$, which are a contradiction with the third row and fourth column of the matrix M_{30}.

(2) $T(a_2) = 1$, according to the matrix M_{30}, $T(a_1) = 3$ and $T(a_6) = 3$

as a result $T(a_3) = 4$, it follows that $T(a_4) = 4$ or $T(a_5) = 4$, which are a contradiction with the third row and fourth column of the matrix M_{30}.

Therefore the graph G_2 has no perfect 4-colorings with matrix M_{30}.

The graph G_3 has perfect 4-colorings with matrix M_{20}. Consider the mapping T as follows:

$T(a_1) = T(a_3) = T(a_5) = 1$. $T(a_2) = 2$, $T(a_6) = 4$, $T(a_4) = 3$. □

Theorem 3.3. *If M is a perfect 4-colorings matrix of the 3-regular graph of order 8, then only matrices M_2, M_9, M_{18} for G_4 and M_{35} for G_5 and M_8 for G_6 and M_8, M_{34} for G_7 and M_1, M_2, M_{14}, M_{36} for G_8 can be parameter matrices.*

Proof. With consideration of 3-regular graphs eigenvalues and using Lemma 2.5, it can be seen the connected 3-regular graphs with 8 vertices can have perfect 4-colorings with matrices M_1, M_2, M_8, M_9, M_{14}, M_{18}, M_{34}, M_{35} and M_{36}.

The graph G_4 has perfect 4-colorings with the matrices M_2 and M_9. Consider two mappings T_1 and T_2 as follows:

$T_1(a_1) = T_1(a_6) = 1$, $T_1(a_2) = T_1(a_5) = 4$, $T_1(a_3) = T_1(a_4) = 2$, $T_1(a_7) = T_1(a_8) = 3$.

$T_2(a_1) = T_2(a_4) = 1$, $T_2(a_2) = T_2(a_3) = 4$, $T_2(a_5) = T_2(a_8) = 2$, $T_2(a_6) = T_2(a_7) = 3$.

There is no perfect 4-colorings with the matrix M_{18}.

Contrary to our claim, suppose that T is a perfect 4-colorings with the matrix M_{18} for graph G_4.

According to the matrix M_{18}, by symmetry we have two cases for the color of number 1 as follows:

(1) If $T(a_1) = 1$, then $T(a_8) = T(a_2) = 3$. It follows that $T(a_7) = 4$, which is a contradiction with the third row of the matrix M_{18}.

(2) If $T(a_2) = 1$, then $T(a_1) = T(a_3) = 3$. It follows that $T(a_8) = 4$, which is a contradiction with the third row of the matrix M_{18}. Therefore the graph G_4 has no perfect 4-colorings with the matrix M_{18}.

The graph G_5 has perfect 4-colorings with the matrix M_{35}.

Consider the mapping T as follows:

$T(a_4) = T(a_5) = T(a_6) = 1$, $T(a_2) = 2$.

$T(a_3) = T(a_1) = T(a_7) = 4$, $T(a_8) = 3$.

The graph G_6 has perfect 4-colorings with the matrix M_8. Consider the mapping T as follows:

$T(a_2) = T(a_7) = 1$, $T(a_4) = T(a_5) = 2$.

$T(a_3) = T(a_6) = 3$, $T(a_1) = T(a_8) = 4$.

The graph G_7 has no perfect 4-colorings with the matrices M_8 and M_{34}.

Contrary to our claim, suppose that T is a perfect 4-colorings with matrix M_8 for the graph G_7. Then according to the matrix M_8, by symmetry we have two cases for the color of number 2 as follows:

(1) If $T(a_1) = 2$, then $T(a_2) = 2$, $T(a_8) = 4$ and $T(a_5) = T(a_3) = 3$. It follows that $T(a_7) = 1$, $T(a_4) = 1$ and $T(a_6) = 1$, which is a contradiction with the first row of the matrix M_8.

(2) If $T(a_2) = 2$, then $T(a_1) = 2$, $T(a_8) = 3$ and $T(a_3) = 4$ according to the $T(a_8)$, 2 vertices should be connected with color 1 so, it's not possible.

Therefore the graph G_7 has no perfect 4-colorings with the matrix M_8. Similarly, we can show that the graph G_7 has no perfect 4-colorings with the matrix M_{34}.

The connected 3-regular graphs G_8 with 8 vertices can have perfect 4-colorings with the matrices M_1, M_2, M_{14}, M_{36}. Now we introduce the mappings of all graphs that have perfect 4-colorings with the parameter matrices.

The graph G_8 has perfect 4-colorings with the matrices M_1, M_2, M_{14} and M_{36}. Consider four mappings T_1, T_2, T_3 and T_4 as follows:

$T_1(a_1) = T_1(a_6) = 1$, $T_1(a_3) = T_1(a_8) = 2$.

$T_1(a_2) = T_1(a_5) = 3$, $T_1(a_4) = T_1(a_7) = 4$.

$T_2(a_1) = T_2(a_6) = 1$, $T_2(a_2) = T_2(a_5) = 2$.

$T_2(a_7) = T_2(a_8) = 3$, $T_2(a_3) = T_2(a_4) = 4$.

$T_3(a_1) = T_3(a_3) = 3$, $T_3(a_2) = T_3(a_4) = 1$.

$T_3(a_6) = T_3(a_8) = 2$, $T_3(a_5) = T_3(a_7) = 4$.

$T_4(a_1) = T_4(a_3) = T_4(a_7) = 1$, $T_4(a_5) = 2$.

$T_4(a_4) = T_4(a_6) = T_4(a_8) = 3$, $T_4(a_2) = 4$. □

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Zeinab Vahedi
PhD Student of Mathematics
Department of Mathematics
Karaj Branch, Islamic Azad University
Karaj, Iran.
E-mail: zeinab.vahedi20@kiau.ac.ir

Mehdi Alaeiyan
Professor of Mathematics
School of Mathematics
Iran University of Science and Technology
Narmak 16846, Tehran, Iran.
E-mail: alaeiyan@iust.ac.ir

Mohammad Maghasedi
Associate Professor of Mathematics
Department of Mathematics
Karaj Branch, Islamic Azad University
Karaj, Iran.
E-mail: maghasedi@kiau.ac.ir