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A Condition of Reflexivity on some Sequence Spaces

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Abstract. In this paper we will give sufficient conditions for the powers of the multipliction operator M_z to be reflexive on formal Laurent sequence spaces.

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1. Introduction

Let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers satisfying $\beta(0) = 1$. If $1 < p < \infty$, the space $L^p(\beta)$ consists of all Laurent power series

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n,$$

such that the norm

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p,$$

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is finite. These are reflexive Banach spaces with the norm $\| \cdot \|_\beta$ and $L^p(\beta)^* = L^q(\beta^{-1})$, where $\frac{1}{p} + \frac{1}{q} = 1$ ([18]). Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_k$ is a basis for $L^p(\beta)$ such that $\|f_k\| = \beta(k)$. We denote the set of multipliers

$$\{\varphi \in L^p(\beta) : \ \varphi L^p(\beta) \subseteq L^p(\beta)\},$$

by $L^p_\infty(\beta)$ and the linear operator of multiplication by φ on $L^p(\beta)$ by M_φ.

We say that a complex number λ is a bounded point evaluation on $L^p(\beta)$ if the functional $e(\lambda) : L^p(\beta) \longrightarrow \mathbb{C}$ defined by $e(\lambda)(f) = f(\lambda)$ is bounded.

Recall that if E is a separable Banach space and A is a bounded linear operator on E, i.e., $A \in B(E)$, then $Lat(A)$ is by definition the set of all invariant subspaces of A, and $AlgLat(A)$ is the algebra of all operators B in $B(E)$ such that $Lat(A) \subset Lat(B)$. For the algebra $B(E)$, the weak operator topology is the one induced by the family of seminorms

$$p_{x^*,x}(A) = |< Ax, x^* >|,$$

where $x \in E$, $x^* \in E^*$ and $A \in B(E)$. Hence $A_\alpha \longrightarrow A$ in the weak operator topology if and only if $A_\alpha x \longrightarrow Ax$ weakly. Also similarly $A_\alpha \longrightarrow A$ in the strong operator topology if and only if $A_\alpha x \longrightarrow Ax$ in the norm topology. An operator A in $B(E)$ is said to be reflexive if

$$AlgLat(A) = W(A),$$

where $W(A)$ is the smallest subalgebra of $B(E)$ that contains A and the identity I and is closed in the weak operator topology. For some source on weighted sequence spaces, we refer the reader to [1 – 20].

2. Main Results

In this section we will investigate the reflexivity of the powers of the operator M_z acting on $L^p(\beta)$. First, we note that the multiplication operator M_z on $L^p(\beta)$ ($H^p(\beta)$) is unitarily equivalent to an injective bilateral (unilateral) weighted shift and conversely, every injective bilateral (unilateral) weighted shift is unitarily equivalent to M_z acting on $L^p(\beta)$ $(H^p(\beta))$ for a suitable choice of β (the proof is similar to the case p=2 that was proved in [3]).

We use the following notations:

$$\begin{aligned} r_0 &= \overline{\lim}\beta(-n)^{-1/n}, \\ r_1 &= \underline{\lim}\beta(n)^{1/n}, \\ \Omega_0 &= \{z \in \mathbf{C} : |\mathbf{z}| > \mathbf{r_0}\}, \\ \Omega_1 &= \{z \in \mathbf{C} : |\mathbf{z}| < \mathbf{r_1}\}, \\ \Omega &= \Omega_0 \cap \Omega_1. \end{aligned}$$

From now on we consider that M_z is bounded on $L^p(\beta)$.

Theorem 2.1. *Let* $0 < r_0 < r_1 = 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If*

$$\sum_{n<0} \frac{{r_0}^{nq}}{\beta(n)^q} < \infty \qquad ; \qquad \sum_{n\geqslant 0} \frac{1}{\beta(n)^q} < \infty,$$

then M_{z^k} *is reflexive on* $L^p(\beta)$ *for all positive integers* k.

Proof. Let $X \in AlgLat(M_{z^k})$. Since $Lat(M_z) \subset Lat(M_{z^k})$, thus $Lat(M_z) \subset Lat(X)$. This implies that $X \in AlgLat(M_z)$. It is well known that $X = M_\psi$ for some $\psi \in L^p_\infty(\beta)$. Now set

$$\mathcal{N} = H^\infty(\Omega_1) \bigcap L^p_\infty(\beta).$$

Then $\mathcal{N} \neq \emptyset$, since $1 \in \mathcal{N}$. It is a closed subspace of $L^p(\beta)$, since if $\{h_n\}_n \subset \mathcal{N}$ and $h_n \longrightarrow f$ in $L^p(\beta)$, then for all n, $\|h_n\|_p \leqslant c_2$ for some $c_2 > 0$. Note that $\lambda \in \Omega$ is a bounded point evaluation on $L^p(\beta)$ if and only if $\{\lambda^n/\beta(n)\} \in \ell^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Now, since

$$c_3 = \sum_{n<0} \frac{{r_0}^{nq}}{\beta(n)^q} < \infty$$

and

$$c_4 = \sum_{n\geqslant 0} \frac{1}{\beta(n)^q} < \infty,$$

each point of Ω is a bounded point evaluation on $L^p(\beta)$. By boundedness of point evaluations, for all λ in Ω we have

$$h_n(\lambda) =< h_n, e(\lambda) > \longrightarrow < f, e(\lambda) >= f(\lambda).$$

Also, for all λ in Ω,

$$\begin{aligned} |h_n(\lambda)| &= | < h_n, e(\lambda) > | \\ &\leqslant \|h_n\|_p \|e(\lambda)\| \\ &\leqslant (c_3 + c_4)\|h_n\|_p, \end{aligned}$$

because

$$\sup_{\lambda \in \Omega} \|e(\lambda)\| \leqslant c_3 + c_4.$$

Thus

$$\|h_n\|_{\Omega_1} = \|h_n\|_{\Omega} \leqslant c_3 \|h_n\|_p \leqslant c_2(c_3 + c_4),$$

for all n. This implies that $\{h_n\}_n$ is a normal family in $H^\infty(\Omega_1)$ and by passing to a subsequence if necessary, we may suppose that $h_n \longrightarrow f$ uniformly on compact subsets of Ω_1. Thus $f \in H^\infty(\Omega_1)$. Note that

$$\|M_{h_n}\| \leqslant c_1 \|h_n\|_{\Omega_1} \leqslant c_1 c_2 (c_3 + c_4),$$

for all n, and ball $B(\mathcal{H})$ is compact in the weak operator topology. Hence $M_{h_n} \longrightarrow A$ in the weak operator topology for some $A \in B(\mathcal{H})$. Since $h_n(\lambda) \longrightarrow f(\lambda)$, we see that $A = M_f$ and so $\mathcal{N}$ is indeed a closed subspace of $L^p(\beta)$. Now clearly $\mathcal{N} \in Lat(M_z)$, thus $X\mathcal{N} \subset \mathcal{N}$. Since $1 \in \mathcal{N}$, we get

$$X1 = \psi \in \mathcal{N} = H^\infty(\Omega_1) \bigcap L^p_\infty(\beta).$$

This implies that $M_{P_n(\psi)} \rightarrow M_\psi$ in the weak operator topology, where

$$P_n(\psi) = \sum_{k=0}^{n} (1 - \frac{k}{n+1}) \hat{\psi}(k) z^k, \ n \geqslant 0,$$

(see [3]). For simplicity put $r_n = P_n(\psi)$ and let $\mathcal{M}_k$ be the closed linear span of the set $\{f_{nk} : n \geqslant 0\}$. We have

$$M_{z^k} f_{nk} = f_{(n+1)k} \in \mathcal{M}_k,$$

for all $n \geqslant 0$. Thus $\mathcal{M}_k \in Lat(M_{z^k})$ and so $\mathcal{M}_k \in Lat(M_\psi)$. Let

$$\psi(z) = \sum_{n=0}^{\infty} \hat{\psi}(n) z^n.$$

Since $1 \in \mathcal{M}_k$, thus $M_\psi 1 = \psi \in \mathcal{M}_k$. Hence $\hat{\psi}(i) = 0$ for all $i \neq nk$, $n \geqslant 0$. Now, by the particular construction of r_n, each r_n should be a polynomial in z^k, i.e., $r_n(z) = q_n(z^k)$ for some polynomial q_n. Thus

$$M_{r_n} = r_n(M_z) = q_n(M_{z^k}) \to X,$$

in the weak operator topology. Hence $X \in W(M_{z^k})$. Thus M_{z^k} is reflexive and so the proof is complete. $\square$

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References

[1] K. Jahedi and B. Yousefi, On the weighted Hardy spaces, *Journal of Mathematical Extension*, 1 (1) (2006), 45-55.

[2] S. Jahedi and B. Yousefi, On some properties of weighted composition operators, *Korean Annals of Mathematics*, 24 (2) (2007), 99-106.

[3] K. Seddighi and B. Yousefi, On the reflexivity of operators on function spaces, *Proc. Amer. Math. Soc.*, 116 (1992), 45-52.

[4] A. Shields and L. Wallen, The commutants of certain Hilbert space operators, *Ind. Univ. Math. J.*, 20 (1971), 777-788.

[5] A. L. Shields, Weighted shift operators and analytic functions theory, *Math. Surveys, A.M.S. Providence*, 13 (1974), 49-128.

[6] B. Yousefi, On the space $\ell^p(\beta)$, Rendiconti Del Circolo Matematico Di Palermo, 49 (2000), 115-120.

[7] B. Yousefi, Unicellularity of the multiplication operator on Banach spaces of formal power series, *Studia Mathematica*, 147 (3) (2001), 201-209.

[8] B. Yousefi, Bounded analytic structure of the Banach space of formal power series, *Rendiconti Del Circolo Matematico Di Palermo*, Serie II, Tomo LI, (2002), 403-410.

[9] B. Yousefi and S. Jahedi, Composition operators on Banach spaces of formal power series, *Bollettino Della Unione Matematica Italiano*, (8) 6-B (2003), 481-487.

[10] B. Yousefi, Strictly cyclic algebra of operators acting on Banach spaces $H^p(\beta)$, *Czechoslovak Mathematical Journal*, 54 (129) (2004), 261-266.

[11] B. Yousefi, Multiplication operators on Hilbert spaces of analytic functions, *Archiv der Mathematik*, 83 (6) (2004), 536-539.

[12] B. Yousefi, On the eighteenth question of Allen Shields, *International Journal of Mathematics*, 16 (1) (2005), 1-6.

[13] B. Yousefi and S. Jahedi, Reflexivity of the multiplication operator on the weighted Hardy spaces, *Southeast Asian Bulletin of Mathematics,* 31 (2007), 163-168.

[14] B. Yousefi and J. Doroodgar, Reflexivity on Banach spaces of analytic functions, *Journal of Mathematical Extension,* 3 (1) (2009), 87-93.

[15] B. Yousefi, Reflexivity on some function spaces, *Journal of Mathematical Extension*, 3 (2) (2009), 27-42.

[16] B. Yousefi and A. Khaksari, Multiplication operators on analytic functional spaces, *Taiwanese Journal of Mathematics*, 13 (4) (2009), 1159-1165.

[17] B. Yousefi and A. I. Kashkooly, On the eleventh question of Allen Shields, *Fixed Point Theory and Applications*, (2011), doi:10.1186/1687-1812-2011-16.

[18] B. Yousefi, A certain operator on spaces of formal series, *International Journal of Applied Mathematics*, 24 (4) (2011), 591-595.

[19] B. Yousefi and Sh. Khoshdel, Reflexivity of powers of the multiplication operator on special function spaces, *Acta Mathematica Scientia*, 32 (6) (2012), 2279-2284.

[20] B. Yousefi and K. Jahedi, Numerical range of operators acting on Banach spaces, *Czechoslovak Mathematical Journal*, 62 (137) (2012), 495-503.

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