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Certain Results on $N(k)-$Contact Metric Manifolds and Torse-Forming Vector Fields

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Abstract. In this paper, we study $N(k)-$contact metric manifolds endowed with a torse-forming vector field and give some characterizations for such manifolds. Then, we deal with $N(k)-$contact metric manifolds admitting a Ricci soliton and find that the potential vector field V of the Ricci soliton is a constant multiple of ξ. Also, we obtain a necessary condition for a torse-forming vector field to be recurrent and Killing on M.

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1 Introduction

The first study on $N(k)-$contact metric manifolds was given by Tanno in [19]. In this study, Tanno obtained that if the structure vector field ξ belongs to the $k-$nullity distribution on an Einstein compact Riemannian manifold M of dimension $2n+1 \geq 5$, then $k=1$ and M is Sasakian. Then, Blair et al. extended $N(k)-$contact metric manifolds to the $(k,\mu)-$contact metric manifolds in 1995 [5]. After these

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works, $N(k)-$contact metric manifolds and $(k,\mu)-$contact metric manifolds have been studied extensively by many mathematicians on many context. For more details (see [11], [12] and [16]-[18]).

The notion of Ricci soliton in Riemannian geometry was introduced by Hamilton as a natural generalization of Einstein metric in 1988 [14]. This notion corresponds to the self-similar solution of Hamilton's Ricci flow: $\frac{\partial}{\partial t}g = -2S$, viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphims and scaling. Also, it models the formation of singularities in the Ricci flow. A Riemannian manifold (M,g) is called a Ricci soliton if the following condition is satisfied for arbitrary vector fields X,Y on M

$$(\pounds_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \quad (1)$$

where $\pounds_V g$ denotes the Lie-derivative of the metric tensor g along vector field V, S is the Ricci tensor of M and λ is a constant. If $\pounds_V g = 0$ and $\pounds_V g = \rho g$, then potential vector field V is said to be Killing and conformal Killing, respectively, where ρ is a function. Also, when V is zero or Killing in (1), then Ricci soliton reduces to Einstein manifold. In addition, a Ricci soliton is called gradient if the potential field V is the gradient of a potential function $-f$ (i.e., $V = -\nabla f$) and is called shrinking, steady or expanding depending on $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively.

On the other hand, vector fields have been used for studying differential geometry of manifolds since they determine most geometric properties of the related object. One of these vector fields is torse-forming vector fields. They appear in many areas of diferential geometry and physics. Torse-forming vector fields were firstly defined and studied by Yano [20]. In recent years, they were studied by different authors such as Chen [8], Blaga et al. [1], Mihai et al. [15] and Crasmareanu [9], [10]. According to Yano, a vector field v on a Riemannian manifold (M,g) is called torse-forming if it satisfies the following condition

$$\nabla_X v = fX + \alpha(X)v, \quad (2)$$

where ∇ is the Levi-Civita connection on M, α is a $1-$form and f is a smooth function on M, for any $X \in \Gamma(TM)$. If the $1-$form α vanishes identically in (2), the vector field v is called concircular [7]. If $\alpha = 0$ and $f = 1$ in (2), then v is called a concurrent vector field [6], [21]. Also, the vector field v is called recurrent if it satisfies (2) with $f = 0$.

The present paper is organized as follows:

Section 1 is concerned with introduction.

In section 2, we give some basic notions about almost contact metric manifolds and $N(k)-$contact metric manifolds.

In section 3, we investigate $N(k)-$contact metric manifolds endowed with a torse-forming vector field and analyze these manifolds admitting a Ricci soliton. We obtain some important characterizations for such manifolds.

2 Preliminaries

In this section, we recall some fundamental notations and formulas of almost contact metric manifolds from [2] and [3].

A differentiable manifold M of dimension $(2n+1)$ is said to be an almost contact metric manifold if it admits an almost contact metric structure (φ, ξ, η, g) and the Riemannian metric g satisfies the following relations:

$$\varphi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1, \ \varphi\xi = 0, \ \eta \circ \varphi = 0, \ \eta(X) = g(X, \xi) \quad (3)$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\varphi X, Y) = -g(X, \varphi Y) \quad (4)$$

for any $X, Y \in \Gamma(TM)$, where ξ is a vector field of type $(0, 1)$, (which is so-called the characteristic vector field), $1-$ form η is the $g-$dual of ξ of type $(1, 0)$ and φ is a tensor field of type $(1, 1)$ on M.

On the other hand, in [2], D.E. Blair defined the fundamental $2-$form Φ of M as follows:

$$\Phi(X, Y) = g(X, \varphi Y)$$

for any $X, Y \in \Gamma(TM)$. Furthermore, an almost contact metric manifold M is called a contact metric manifold if it satisfies

$$\Phi(X, Y) = d\eta(X, Y).$$

The Nijenhuis tensor field of φ is defined by

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] + \varphi^2[X,Y] - \varphi[X,\varphi Y] - \varphi[\varphi X, Y]$$

for all $X, Y \in \Gamma(TM)$. If M is an almost contact metric manifold and the Nijenhuis tensor of φ satisfies

$$N_\varphi + 2d\eta \otimes \xi = 0$$

then, M is called a normal contact metric manifold. A normal contact metric manifold M is called Sasakian. It is well known that an almost contact metric manifold M is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

For a Sasakian manifold, we also have

$$\begin{aligned} \nabla_X \xi &= -\varphi X, \\ R(X,Y)\xi &= \eta(Y)X - \eta(X)Y, \end{aligned}$$

where ∇ and R are the Levi-Civita connection and the Riemannian curvature tensor on M, respectively.

The $(k,\mu)-$nullity distribution on contact metric manifolds was introduced by Blair et al. and defined by [5]

$$\begin{aligned} N(k,\mu) : p \to N_p(k,\mu) &= \{Z \in T_pM | R(X,Y)Z \\ &= (kI + \mu h)(g(Y,Z)X - g(X,Z)Y)\}, \end{aligned} \quad (5)$$

where $(k,\mu) \in \mathbb{R}^2$, I is an identity map and h is the tensor field of type $(1,1)$ defined by $h = \frac{1}{2}\pounds_\xi \varphi$. This tensor field satisfy

$$h\xi = 0, \qquad h\varphi + \varphi h = 0, \qquad \nabla_X \xi = -\varphi X - \varphi h X \quad (6)$$

and

$$\begin{aligned} g(hX,Y) &= g(X,hY), & (7) \\ \eta(hX) &= 0. & (8) \end{aligned}$$

A contact metric manifold M is called a $(k,\mu)-$contact metric manifold, if ξ belongs to $(k,\mu)-$nullity distribution $N(k,\mu)$. If μ vanishes identically in (5), then the $(k,\mu)-$nullity distribution $N(k,\mu)$ reduces to $k-$nullity distribution $N(k)$ and is given by [19]

$$\begin{aligned} N(k): p \to N_p(k) &= \{Z \in T_pM | R(X,Y)Z \\ &= k(g(Y,Z)X - g(X,Z)Y)\}. \end{aligned}$$

Also, if $\xi \in N(k)$, then a contact metric manifold M is called an $N(k)-$contact metric manifold [19]. If $k = 1$, then an $N(k)-$contact metric manifold is Sasakian. If $k = 0$, then the manifold is locally isometric to the product $E^{n+1} \times S^4$ for $n > 1$ and flat for $n = 1$ [4].

In a contact metric manifold, the $(1,1)$ tensor field h is said to be recurrent if it satisfies the condition

$$(\nabla_X h)Y = \eta(Y)hX,$$

where η is the $1-$form of contact metric manifold [13].

For an $N(k)-$contact metric manifold, the followings are satisfied [4]:

$$\begin{aligned} h^2 &= (k-1)\varphi^2, && (9) \\ (\nabla_X \varphi)Y &= g(X + hX, Y)\xi - \eta(Y)hX, \\ R(X,Y)\xi &= k(\eta(Y)X - \eta(X)Y), \\ R(\xi,X)Y &= k(g(X,Y)\xi - \eta(Y)X), && (10) \\ S(X,Y) &= 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) \\ &\quad +[2nk - 2(n-1)]\eta(X)\eta(Y), \quad n \geq 1 && (11) \\ S(X,\xi) &= 2nk\eta(X), && (12) \\ Q\xi &= 2nk\xi, \end{aligned}$$

where S is the Ricci tensor and Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$.

Example 2.1. [12] We consider the three-dimensional manifold

$$M = \{(x,y,z) \in \mathbb{R}^3, (x,y,z) \neq (0,0,0)\},$$

where (x,y,z) are the Cartesian coordinates in $\mathbb{R}^3$. Let e_1, e_2 and e_3 be the linearly independent vector fields in $\mathbb{R}^3$ which satisfies

$$[e_1, e_2] = (1+a)e_3, \quad [e_1, e_3] = -(1-a)e_2 \quad \textit{and} \quad [e_2, e_3] = 2e_1,$$

where a is a real number. Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_i, e_i) &= 1 \\ g(e_i, e_j) &= 0 \quad for \quad i \neq j. \end{aligned}$$

Also, let η, φ be the $1-$ form and the $(1,1)-$tensor field, respectively defined by

$$\eta(Z) = g(Z, e_1), \quad \varphi(e_2) = e_3, \quad \varphi(e_3) = -e_2, \quad \varphi(e_1) = 0$$

for any $Z \in \Gamma(TM)$. Furthermore,

$$he_1 = 0, \quad he_2 = ae_2, \quad and \quad he_3 = -ae_3.$$

On the other hand, using Koszul's formula for the Riemannian metric g, we have:

$$\begin{aligned} &\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = \nabla_{e_1} e_3 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0, \\ &\nabla_{e_3} e_2 = -(1-a)e_1, \quad \nabla_{e_3} e_1 = (1-a)e_2, \\ &\nabla_{e_2} e_1 = -(1+a)e_3, \quad \nabla_{e_2} e_3 = (1+a)e_1. \end{aligned}$$

Therefore, $(M, \varphi, \xi, \eta, g)$ is a $3-$dimensional contact metric manifold. Using the above equations, one has

$$\begin{aligned} &R(e_1, e_2)e_3 = 0, \quad R(e_1, e_3)e_2 = 0, \quad R(e_2, e_3)e_1 = 0, \\ &R(e_1, e_2)e_2 = (1-a^2)e_1, \quad R(e_1, e_2)e_1 = -(1-a^2)e_2, \\ &R(e_1, e_3)e_3 = (1-a^2)e_1, \quad R(e_1, e_3)e_1 = -(1-a^2)e_3, \\ &R(e_2, e_3)e_3 = -(1-a^2)e_2, \quad R(e_2, e_3)e_2 = (1-a^2)e_3. \end{aligned}$$

Hence, in view of the expressions of curvature tensors, the manifold M is a $3-$dimensional $N(1-a^2)-$contact metric manifold.

3 Main Results

In this section, we deal with an $N(k)-$contact metric manifold endowed with a torse-forming vector field and admitting a Ricci soliton. Also, we give some important characterizations which classify such a manifold.

Now, we begin to this section with the following:

Theorem 3.1. *Let M be an $N(k)-$contact metric manifold. If the tensor field h is recurrent, then the manifold M is Sasakian.*

Proof. It follows from the equation (8), we have

$$g(hY, \xi) = 0 \tag{13}$$

for any $Y \in \Gamma(TM)$. Taking the covariant derivative of (13) along arbitrary vector field X, one has

$$g(\nabla_X hY, \xi) + g(hY, \nabla_X \xi) = 0. \tag{14}$$

Using the equation (6) in (14) yields

$$g((\nabla_X h)Y + h(\nabla_X Y), \xi) = g(hY, \varphi X) + g(hY, \varphi hX). \tag{15}$$

Since the tensor field h is recurrent, the equation (15) becomes

$$g(\eta(Y)hX + h(\nabla_X Y), \xi) = g(hY, \varphi X) + g(hY, \varphi hX). \tag{16}$$

Also, making use of (6) and (7) in (16) we get

$$g(hY, \varphi X) + g(hY, \varphi hX) = 0. \tag{17}$$

Replacing Y by hY in (17) and using the equalities (3), (4), (9) one has

$$(k-1)g(Y, \varphi X) + (k-1)g(Y, \varphi hX) = 0. \tag{18}$$

Interchanging the roles of X and Y in (18) gives

$$(k-1)g(X, \varphi Y) + (k-1)g(X, \varphi hY) = 0. \tag{19}$$

Adding (18) and (19) and using (4), (6), (7) we have

$$(k-1)g(hY, \varphi X) = 0. \tag{20}$$

Again, replacing Y by hY in (20) and then using (3), (4), (9) we write

$$(k-1)^2 g(Y, \varphi X) = 0$$

and hence

$$(k-1)^2 d\eta(Y, X) = 0.$$

Removing X and Y in the above equation we have

$$(k-1)^2 d\eta = 0.$$

In a contact metric manifold, since $d\eta \neq 0$, we get $k = 1$. This is the required result. □

The next theorem provides a characterization for a torse-forming vector field to be recurrent.

Theorem 3.2. *Let M be an $N(k)-$contact metric manifold endowed with a torse-forming vector field v. If the vector field v is orthogonal to the characteristic vector field ξ, then v is a recurrent vector field on M.*

Proof. Let the vector field v be a torse-forming on M. Then, from the definition of Lie-derivative and from (2), we have

$$\begin{aligned}(\pounds_v g)(X,Y) &= \pounds_v g(X,Y) - g(\pounds_v X, Y) - g(X, \pounds_v Y) \\ &= g(\nabla_v X, Y) + g(X, \nabla_v Y) - g(\nabla_v X, Y) \\ &\quad + g(\nabla_X v, Y) - g(X, \nabla_v Y) + g(\nabla_Y v, X) \\ &= g(\nabla_X v, Y) + g(\nabla_Y v, X) \\ &= 2f g(X,Y) + \alpha(X) g(v,Y) + \alpha(Y) g(v,X) \qquad (21)\end{aligned}$$

for any $X, Y \in \Gamma(TM)$. Substituting $X = Y = \xi$ in (21) implies

$$(\pounds_v g)(\xi,\xi) = 2f + 2\alpha(\xi)\eta(v). \qquad (22)$$

On the other hand, with the help of the equalities (2)-(4) and (6) one has

$$\begin{aligned}(\pounds_v g)(\xi,\xi) &= -2g(\pounds_v \xi, \xi) \\ &= -2g(\nabla_v \xi, \xi) + 2g(\nabla_\xi v, \xi) \\ &= -2g(-\varphi v - \varphi h v, \xi) + 2g(\nabla_\xi v, \xi) \\ &= 2g(\nabla_\xi v, \xi) \qquad (23)\end{aligned}$$

Also, it is easy to see that $\nabla_\xi(g(v,\xi)) = g(\nabla_\xi v, \xi)$. Therefore, from (22) and (23) we get

$$\begin{aligned}2f + 2\alpha(\xi)\eta(v) &= (\pounds_v g)(\xi,\xi) \\ &= 2g(\nabla_\xi v, \xi) \\ &= 2\nabla_\xi(g(v,\xi)).\end{aligned}$$

If the vector field v is orthogonal to ξ, we have that $f = 0$. This means that v becomes recurrent vector field on M. Thus, the proof is completed. $\square$

As a consequence of the Theorem 3.2, we can state the following corollary:

Corollary 3.3. *Let M be an $N(k)-$contact metric manifold endowed with a concircular vector field v. If the vector field v is orthogonal to the characteristic vector field ξ, then v is Killing on M.*

Theorem 3.4. *Let M be an $N(k)-$contact metric manifold. Then, the characteristic vector field ξ is not torse-forming on M.*

Proof. Let us assume that the structure vector field ξ is torse-forming on M. Then, we have

$$\nabla_X \xi = fX + \alpha(X)\xi \tag{24}$$

for any $X \in \Gamma(TM)$. From (6) and (24), we write

$$fX + \alpha(X)\xi = -\varphi X - \varphi hX. \tag{25}$$

Taking the inner product of (25) with vector field ξ, then we get

$$\alpha(X) = -f\eta(X). \tag{26}$$

Similarly, taking the inner product of (25) with vector field φY and using (3) one has

$$fg(X, \varphi Y) = -g(\varphi X, \varphi Y) - g(\varphi hX, \varphi Y). \tag{27}$$

Also, interchanging the roles of X and Y in (27) gives

$$fg(Y, \varphi X) = -g(\varphi Y, \varphi X) - g(\varphi hY, \varphi X). \tag{28}$$

Adding (27) and (28) and using (3), (4), (6)-(8) we obtain

$$g(\varphi X, \varphi Y) = -g(hX, Y). \tag{29}$$

Replacing X by hX in (29) and then using (3), (4), (8) and (9) yields

$$g(hX, Y) = (k-1)g(\varphi X, \varphi Y). \tag{30}$$

From (29) and (30), we have

$$kg(hX, Y) = 0. \tag{31}$$

Again, replacing X by hX in (31) and making use of (4), (9) one has

$$k(k-1)g(\varphi X, \varphi Y) = 0$$

equivalent to

$$k(k-1)d\eta(\varphi X, Y) = 0.$$

Since $d\eta \neq 0$, either $k = 0$ or $k = 1$. If $k = 1$, then $h = 0$. From (29), one write

$$d\eta(\varphi X, Y) = g(\varphi X, \varphi Y) = -g(hX, Y) = 0.$$

This is a contradiction. Therefore, we have $k = 0$.

On the other hand, if we use (4) and (29) in (27) we get

$$fg(X, \varphi Y) = fd\eta(X, Y) = 0$$

which implies that $f = 0$. So, from (26), we have $\alpha = 0$. Then, from (6) and (24) we find that

$$0 = (\pounds_\xi g)(X, Y) = 2g(hX, \varphi Y)$$

Since $h \neq 0$, This is a contradiction. Thus, the vector field ξ is not torse-forming on M and which completes the proof of the theorem. □

The next theorem presents a characterization for an $N(k)-$contact metric manifold.

Theorem 3.5. *Let M be an $N(k)-$contact metric manifold endowed with a torse-forming vector field v. If the vector field v is orthogonal to the characteristic vector field ξ, then M is locally isometric to the product $E^{n+1} \times S^4$ for $n > 1$ and flat for $n = 1$ provided $\alpha(\xi) \neq 0$.*

Proof. In a Riemannian manifold, we have

$$g(R(X, Y)Z, W) + g(R(X, Y)W, Z) = 0 \tag{32}$$

for any $X, Y, Z, W \in \Gamma(TM)$. If we take the Lie-derivative of (32) with respect to the vector field v, we write

$$(\pounds_v g)(R(X,Y)Z,W) + (\pounds_v g)(R(X,Y)W,Z) = 0. \qquad (33)$$

If we put $X = Z = W = \xi$ in (33) and use the equations (3), (10) we find

$$(\pounds_v g)(k(\eta(Y)\xi - Y), \xi) = 0,$$

that is,

$$(\pounds_v g)(k\eta(Y)\xi, \xi) = (\pounds_v g)(kY, \xi). \qquad (34)$$

For the left side of (34), from definition of Lie-derivative, one has

$$(\pounds_v g)(k\eta(Y)\xi, \xi) = 2k\eta(X)g(\nabla_\xi v, \xi). \qquad (35)$$

Since the vector v is torse-forming on M, for the right side of (34), we get

$$\begin{aligned}(\pounds_v g)(kY,\xi) &= g(\nabla_{kY} v, \xi) + g(kY, \nabla_\xi v) \\ &= 2fk\eta(Y) + k\alpha(Y)\eta(v) + k\alpha(\xi)g(Y,v) \qquad (36)\end{aligned}$$

By virtue of (34), (35) and (36), we have

$$2k\eta(X)g(\nabla_\xi v, \xi) = 2fk\eta(Y) + k\alpha(Y)\eta(v) + k\alpha(\xi)g(Y,v). \qquad (37)$$

If the vector field v is orthogonal to ξ, then equation (37) becomes

$$2k\eta(X)g(\nabla_\xi v, \xi) = 2fk\eta(Y) + k\alpha(\xi)g(Y,v).$$

Also, as a result of the Theorem 3.2, the above equation reduces

$$0 = k\alpha(\xi)g(Y,v).$$

Since $g(Y,v) \neq 0$ and $\alpha(\xi) \neq 0$, we have $k = 0$. Thus, we get the requested result. □

The next theorem gives an important characterization for a Ricci soliton to be shrinking.

Theorem 3.6. *Let M be an $N(k)-$contact metric manifold admitting a Ricci soliton whose the non-zero potential vector field V is pointwise collinear with the structure vector field ξ. Then, the followings are satisfied:*

i) The vector field V is constant multiple of ξ.

ii) The Ricci soliton (M, g, V, λ) is shrinking.

Proof. Suppose that the vector field v be a pointwise collinear with the structure vector field ξ. That is, $V = b\xi$, where b is a smooth function on M. Then, from (1) we have

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0$$

for any $X, Y \in \Gamma(TM)$. From (3), (4), (6) and (7), one has

$$X(b)\eta(Y) + Y(b)\eta(X) + 2bg(hX, \varphi Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{38}$$

Putting $X = Y = \xi$ in (38) and using (3), (6), (12), we get

$$\xi(b) = -(\lambda + 2nk). \tag{39}$$

Replacing Y by ξ in (38) and making use of (3), (6), (12), (39) gives

$$X(b) = -(\lambda + 2nk)\eta(X)$$

and hence

$$db(X) = -(\lambda + 2nk)\eta(X). \tag{40}$$

Removing X in equation (40) implies

$$db = -(\lambda + 2nk)\eta. \tag{41}$$

Applying d to the both sides of the equation (41), we get

$$0 = -(\lambda + 2nk)d\eta.$$

Since $d\eta \neq 0$, we have

$$\lambda = -2nk. \tag{42}$$

Using (42) in (41) yields

$$db(X) = X(b) = 0 \tag{43}$$

which means that the function b is a constant.

On the other hand, from (38) and (43) we have

$$-bg(hX, \varphi Y) - \lambda g(X, Y) = S(X, Y). \tag{44}$$

Replacing X by hX in (44) and then using (8), (9), (11) we derive

$$\begin{aligned} -b(k-1)g(X, \varphi Y) - \lambda g(hX, Y) &= 2(n-1)\{g(hX, Y) \\ &\quad -(k-1)g(\varphi X, \varphi Y)\}. \end{aligned} \tag{45}$$

Interchanging the roles of X and Y in (45) one has

$$\begin{aligned} -b(k-1)g(Y, \varphi X) - \lambda g(hY, X) &= 2(n-1)\{g(hY, X) \\ &\quad -(k-1)g(\varphi Y, \varphi X)\}. \end{aligned} \tag{46}$$

Subtracting (46) from (45) and using (4), (7) we find

$$b(k-1)g(X, \varphi Y) = 0,$$

namely,

$$(k-1)d\eta(X, Y) = 0.$$

which shows that $k = 1$. Using the fact that $k = 1$ and from (42), we have that the Ricci soliton is shrinking. This is the desired result. □

Theorem 3.7. *Let M be an $N(k)-$contact metric manifold admitting a Ricci soliton whose the potential vector field V is orthogonal to ξ. Then, the Ricci soliton (M, g, V, λ) is steady if and only if M is locally isometric to the product $E^{n+1} \times S^4$ for $n > 1$ and flat for $n = 1$.*

Proof. It follows from the definition of Lie-derivative and from $\nabla_\xi \xi = 0$, we have

$$(\pounds_V g)(\xi, \xi) = 2g(\nabla_\xi V, \xi) = 2\nabla_\xi (g(V, \xi)) = 0. \tag{47}$$

Since M is a Ricci soliton and from equation (1) we write

$$(\pounds_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0 \tag{48}$$

for any $X, Y \in \Gamma(TM)$. Also, putting $X = Y = \xi$ in (48) and using (12), (47), we get

$$\lambda = -2nk. \tag{49}$$

This result ends the proof of the theorem. □

Using the equality (49), we can state the following.

Corollary 3.8. *Let M be an $N(k)-$contact metric manifold admitting a Ricci soliton whose the potential vector field V is orthogonal to ξ. If M is Sasakian, then the Ricci soliton (M, g, V, λ) is shrinking.*

Example 3.9. From example 2.1, we know that the manifold M is a $3-$dimensional $N(1-a^2)-$contact metric manifold. Using the expressions of the curvature tensors, we find the values of the Ricci tensors as follows:

$$S(e_1, e_1) = 2(1-a^2), \quad S(e_2, e_2) = 0, \quad S(e_3, e_3) = 0, \quad S(e_i, e_j) = 0$$

for all $i, j = 1, 2, 3$ $(i \neq j)$. In this case, M admits a Ricci soliton $(g, V = e_2, \lambda)$ which satisfies the equation (1) for $\lambda = 0$. Similarly, M admits a Ricci soliton $(g, V = e_3, \lambda)$ which satisfies the equation (1) for $\lambda = 0$. From $\lambda = 0$, the manifold M becomes $N(0)-$contact metric manifold. This verifies our Theorem 3.7.

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References

[1] A. M. Blaga, M. Crasmareanu, Torse-forming $\eta-$Ricci Solitons in Almost Paracontact $\eta-$Einstein Geometry, *Filomat*, 31(2) (2017), 499-504.

[2] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, (1976).

[3] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics 203, Birkhouser Boston, Inc., MA, (2002).

[4] D. E. Blair, J. S. Kim, M. M. Tripathi, On the Concircular Curvature Tensor of a Contact Metric Manifold, *J. Korean Math. Soc.*, 42(5) (2005), 883-992.

[5] D. E. Blair, T. Koufogiorgos, B. J. Papantoniou, Contact Metric Manifold Satisfying a Nullity Condition, *Israel J. Math.*, 91 (1995), 189-214.

[6] B.-Y. Chen, Classification of Torqued Vector Fields and Its Applications to Ricci Solitons, *Kragujevac J. Math.*, 41(2) (2017), 239-250.

[7] B.-Y. Chen, Some Results on Concircular Vector Fields and Their Applications to Ricci Solitons, *Bull. Korean Math. Soc.*, 52(5) (2015), 1535-1547.

[8] B.-Y. Chen, L. Verstraelen, A Link Between Torse-forming Vector Fields and Rotational Hypersurfaces, *Int. J. Geo. Methods in Modern Physics*, 14(12) (2017), 1-10.

[9] M. Crasmareanu, Scalar Curvature for Middle Planes in Odd-Dimensional Torse-forming Amost Ricci Solitons, *Kragujevac J. Math.*, 43(2) (2019), 275-279.

[10] M. Crasmareanu, Parallel Tensors and Ricci Solitons in $N(k)-$quasi Einstein Manifolds, *Indian J. Pure Appl. Math.*, 43 (2012), 359-369.

[11] U. C. De, A. K. Gazi, On $\Phi-$Recurrent $N(k)-$Contact Metric Manifolds, *Math. J. Okayama Univ.*, 50 (2008), 101-112.

[12] U. C. De, A. Yildiz, S. Ghosh, On a Class of $N(k)-$Contact Metric Manifolds, *Math. Reports*, 16 (2014), 207-217.

[13] U. C. De, J. Y. Suh, P. Majhi, Ricci Solitons on $\eta-$Einstein Contact Manifolds, *Filomat*, 32(13) (2018), 4679-4687.

[14] R. S. Hamilton, The Ricci Flow on Surfaces, Mathematics and General Relativity (Santa Cruz, CA, 1986), *Contemp. Math.*, A.M.S, 71 (1988), 237-262.

[15] A. Mihai, I. Mihai, Torse forming Vector Fields and Exterior Concurrent Vector Fields on Riemannian Manifolds and Applications, *J. Geom. Phys.*, 73 (2013), 200-208.

[16] P. Majhi, G. Ghosh, Concircular Vector Fields in $(k, \mu)-$Contact Metric Manifolds, *Int. Elect. J. Geom.*, 11(1) (2018), 52-56.

[17] P. Majhi, U. C. De, Classifications of $N(k)-$Contact Metric Manifolds Satisfying Certain Curvature Conditions, *Acta Math. Univ. Comenianae*, 84(1) (2015), 167-178.

[18] C. Özgür, S. Sular, On $N(k)-$Contact Metric Manifolds Satisfying Certain Conditions, *SUT Journal of Mathematics*, 44(1) (2008), 89-99.

[19] S. Tanno, Ricci Curvatures of Contact Riemannian Manifolds, *Tohoku Math. J.*, 40 (1988), 441-448.

[20] K. Yano, On Torse-forming Direction in a Riemannian Space, *Proc. Imp. Acad. Tokyo*, 20 (1944), 340-345.

[21] H. İ. Yoldaş, Ş. E. Meriç, E. Yaşar, On Generic Submanifold of Sasakian Manifold with Concurrent Vector Field, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 68(2) (2019), 1983-1994.

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