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The Two-Term Abel's Integral Equation

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Abstract. In this article we investigate the two-term Abel's integral equations. We will do this in two different ways and show that such equation is reducible to an integro−differential equation of Volterra type.

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1. Introduction

Abel's integral equation is a special kind of linear Volterra integral equations of the first kind, and is usually solved via the Laplace transform method, which finally reduces it to a differentiation of fractional order [2].
In this paper we investigate the two-term Abel's equation given by

$$\int_0^x \left\{ \frac{A}{(x-t)^\alpha} + \frac{B}{(x-t)^\beta} \right\} u(t)dt = f(x), \qquad x > 0, \qquad 0 < \beta < \alpha < 1 \quad (1)$$

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and will solve it in two different ways, and derive some results about the connection between fractional differentiation and solution of linear Volterra integro−differential equations of the second kind.
The structure of the paper is as follows:
In section 2 we solve (1) via the Laplace transform method and express its solution as an infinite sum of the Riemann−Liouville fractional derivatives of the function f [3, 4, 5].
In section 3 we reduce (1) to a Volterra integro−differential equation of the second kind. In section 4 we summarize some conclusions.

2. Solution by the Laplace Transform Method

We consider the two terms Abel's integral equations in the general form:

$$\int_0^x \left\{ \frac{A}{(x-t)^{\alpha}} + \frac{B}{(x-t)^{\beta}} \right\} u(t)dt = f(x), \qquad x > 0 , \qquad 0 < \beta < \alpha < 1, \quad (2)$$

and will solve it via the Laplace transform method. In this generalized case as the original Abel's equation:

$$\int_0^x \frac{u(t)}{(x-t)^{\gamma}} dt = f(x) , \qquad x > 0, \qquad 0 < \gamma < 1, \quad (3)$$

by using the Laplace transforms and putting $F(z) = \mathcal{L}\{f(x)\}$ and $U(z) = \mathcal{L}\{u(x)\}$ we obtain:

$$\left\{ \frac{A\Gamma(1-\alpha)}{z^{1-\alpha}} + \frac{B\Gamma(1-\beta)}{z^{1-\beta}} \right\} U(z) = F(z), \quad (4)$$

or equivalently:

$$U(z) = \frac{z^{1-\alpha}}{A\Gamma(1-\alpha)} . \frac{1}{1 + \frac{B\Gamma(1-\beta)}{A\Gamma(1-\alpha)} z^{\beta-\alpha}} F(z), \quad (5)$$

and in the domain $|z|^{\beta-\alpha} < \left| \frac{A\Gamma(1-\alpha)}{B\Gamma(1-\beta)} \right|$ we can use the geometric series to obtain:

$$U(z) = \frac{z^{1-\alpha}}{A\Gamma(1-\alpha)} \left(\sum_{n=0}^{\infty} (-1)^n \left(\frac{B\Gamma(1-\beta) z^{\beta-\alpha}}{A\Gamma(1-\alpha)} \right)^n \right) F(z), \quad (6)$$

which by using the Riemann−Liouville's integral formula[3]:

$$ {}_0D_x^{-p} f(t) = \frac{1}{\Gamma(p)} \int_0^x (x-t)^{p-1} f(t)dt, \tag{7} $$

and the convolution theorem for the Laplace transform [2] on (6) gives

$$ \begin{aligned} u(x) &= \textstyle\sum_{n=0}^{\infty} (-1)^n \frac{(B\Gamma(1-\beta))^n}{(A\Gamma(1-\alpha))^{n+1}} \cdot \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} f(t)dt \\ &= \textstyle\sum_{n=0}^{\infty} c_n \, {}_0D_x^{-\eta} f(x), \end{aligned} \tag{8} $$

where $\eta = (n+1)\alpha - n\beta - 1$ and $c_n = (-1)^n \frac{(B\Gamma(1-\beta))^n}{(A\Gamma(1-\alpha))^{n+1}}$.

3. Solution by Transforming to Volterra Integral Equations of the Second Kind

In this section we solve (1) by using integral operators[2]. So we define the two integral operators

$$ [Lu](x) = \int_0^x \frac{A}{(x-t)^{\alpha}} u(t)dt, \tag{9} $$

and

$$ [Mu](x) = \int_0^x \frac{B}{(x-t)^{\beta}} u(t)dt. \tag{10} $$

Then by using (9) and (10) in (1) we have:

$$ [(L+M)u](x) = f(x), \tag{11} $$

and so we obtain

$$ \begin{aligned} [Lu](x) &= f(x) - [Mu](x) \\ &= f(x) - \textstyle\int_0^x \frac{B}{(x-t)^{\beta}} u(t)dt, \end{aligned} \tag{12} $$

and so:

$$ u(x) = [L^{-1}f](x) - L^{-1}\left[\int_0^x \frac{B}{(x-t)^{\beta}} u(t)dt \right], \tag{13} $$

where by using

$$[L^{-1}g](x) = \frac{1}{A}\frac{\sin(\alpha\pi)}{\pi}\frac{d}{dx}\int_0^x \frac{g(t)}{(x-t)^{1-\alpha}}dt, \tag{14}$$

can be expressed as[2]:

$$\begin{aligned} u(x) &= [L^{-1}f](x) - L^{-1}\left([Mu]\right)(x) \\ &= \tfrac{1}{A}\tfrac{\sin(\alpha\pi)}{\pi}\tfrac{d}{dx}\int_0^x \tfrac{f(t)}{(x-t)^{1-\alpha}}dt - \tfrac{B}{A}\tfrac{\sin(\alpha\pi)}{\pi}\tfrac{d}{dx}\int_0^x \tfrac{1}{(x-z)^{1-\alpha}}\int_0^z \tfrac{u(t)}{(z-t)^{\beta}}dtdz, \end{aligned} \tag{15}$$

and changing the order of integration and doing some manipulations we obtain:

$$\begin{aligned} u(x) &= \tfrac{1}{A}\tfrac{\sin(\alpha\pi)}{\pi}\tfrac{d}{dx}\int_0^x \tfrac{f(t)}{(x-t)^{1-\alpha}}dt \\ &\quad -\tfrac{B}{A}\tfrac{\sin(\alpha\pi)}{\pi}\tfrac{\Gamma(\alpha)\Gamma(1-\beta)}{\Gamma(1-\beta+\alpha)}\tfrac{d}{dx}\int_0^x (x-t)^{\alpha-\beta}u(t)dt \end{aligned} \tag{16}$$

which is a Volterra integro−differential equation of the second kind, whose unique solution must be given by (8).

4. Conclusion

In this section we summarize the results of sections 2 and 3.
Comparing (8) and (16) we obtain[1]:

$$u(x) = \sum_{n=0}^{\infty} c_n \, {}_0D_x^{-(n+1)\alpha+n\beta+1}f(x) = \left([I-\lambda Q]^{-1}F\right)(x) \tag{17}$$

where:

$$F(x) = \frac{1}{A}\frac{\sin(\alpha\pi)}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt \tag{18}$$

$$\lambda = -\frac{B}{A}\frac{\sin(\alpha\pi)}{\pi}\frac{\Gamma(\alpha)\Gamma(1-\beta)}{\Gamma(1-\beta+\alpha)} \tag{19}$$

$$[Qu](x) = \frac{d}{dx}\int_0^x (x-t)^{\alpha-\beta}u(t)dt \tag{20}$$

but the volterra equation

$$u(x) = F(x) + \lambda[Qu](x), \tag{21}$$

is of the second kind and can be solved by many methods such as iteration method [7], Adomian's method [6], ..., and given approximate solutions for(1).

References

[1] K. Atkinson and W. Han, *Theoretical Numerical Analysis: A Functional Analysis Framework*, Springer, 2009.

[2] H. Hochstadt, *Integral Equation*, Wiley-interscience publication, New York, 1973.

[3] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[4] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.

[5] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[6] A. M. Wazwaz and S. M. ElSayed, A new modification of the adomian decomposition method for linear and nonlinear operators, *Applied Mathematics and Computation*, 122 (2001), 393-405.

[7] L. Xu, Variational iteration method for solving integral equations, *Computers and Mathematics with Applications*, 54 (2007), 1071-1078.

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