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A Note on Automorphisms of Finite p-Groups

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Abstract. Let G be a group and $\mathrm{Aut}^{\Phi}(G)$ denote the group of all automorphisms of G centralizing $G/\Phi(G)$ elementwise. In this paper, we give a necessary and sufficient condition on a finite p-group G for the group $\mathrm{Aut}^{\Phi}(G)$ to be elementary abelian.

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1 Introduction

Let G be a finite group and N a characteristic subgroup of G. We let $\mathrm{Aut}^N(G)$ denote the subgroup of $\mathrm{Aut}(G)$, the automorphism group of G, which centralize G/N. Clearly $\mathrm{Aut}^N(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ and $\alpha \in \mathrm{Aut}^N(G)$ if and only if $x^{-1}x^{\alpha} \in N$ for all $x \in G$. Now let M be a normal subgroup of G. Let us denote by $C_{\mathrm{Aut}^N(G)}(M)$ the group of all automorphisms of $\mathrm{Aut}^N(G)$ centralizing M. It is well-known that if G is a finite p-group, then so is the group $\mathrm{Aut}^{\Phi}(G)$, where Φ denotes the Frattini subgroup of G, the intersection of all the maximal subgroups of G. Clearly $\mathrm{Aut}^{\Phi}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ containing $\mathrm{Inn}(G)$, the group of inner automorphisms of G. In [5], Liebeck gave an upper bound for the nilpotency class of $\mathrm{Aut}^{\Phi}(G)$. Müller in [7] proved

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that if G is a finite non-abelian p-group, then $C_{\mathrm{Aut}^{\Phi}(G)}(Z) = \mathrm{Inn}(G)$ if and only if $\Phi \leq Z$ and Φ is cyclic, where $Z = Z(G)$. This turns out that $\mathrm{Aut}^{\Phi}(G)/\mathrm{Inn}(G)$ is non-trivial if and only if G is neither elementary abelian nor extraspecial. In this paper we give a necessary and sufficient condition on a finite p-group G for the group $\mathrm{Aut}^{\Phi}(G)$ to be elementary abelian. A similar description has been made by Jafari in [3] and Kaboutari Farimani and Nasrabadi in [4] for the occurrence of elementary abelian group in the central automorphism group and the absolute central automorphism of finite purely non-abelian and finite p-groups, respectively.

Throughout this paper all groups are assumed to be finite groups. Our notation is standard and follows that of [2]. In particular, a p-group G is said to be extraspecial if $G' = Z(G) = \Phi(G)$ is of order p, where G' stands for the derived subgroup of G. Also a non-abelian group that has no non-trivial abelian direct factor is said to be purely non-abelian. Recall an abelian p-group A has invariants or is of type $(a_1, a_2, ..., a_k)$ if it is the direct product of cyclic subgroups of orders $p^{a_1}, p^{a_2}, ..., p^{a_k}$, where $a_1 \geq a_2 \geq ... \geq a_k > 0$. For a finite group G, $\Omega_i(G)$, $\exp(G)$ and $o(x)$ respectively denote the subgroup of G generated by those x in G with $x^{p^i} = 1$, the exponent of G and the order of $x \in G$. We use $\mathrm{Hom}(G, A)$ to denote the group of homomorphisms of G into an abelian group A and $\mathbb{Z}_n$ for the cyclic group of order n. Finally if α is an automorphism of G and x is an element of G, we write x^{α} for the image of x under α.

2 Some Basic results

In this section we give some basic results which will be used in the rest of the paper.

Lemma 2.1. *[9, Lemma 2.4] Let G be a finite group with $\Phi(G) \leq Z(G)$. Then there is a bijection from $\mathrm{Hom}(G/G', \Phi(G))$ onto $\mathrm{Aut}^{\Phi}(G)$ associating to every homomorphism $f : G \to \Phi(G)$ the automorphism α_f by $x^{\alpha_f} = xf(x)$ of G. In particular, if G is a p-group and $\exp(\Phi(G)) = p$, then $\mathrm{Aut}^{\Phi}(G) \cong \mathrm{Hom}(G/G', \Phi(G))$.*

Let G be a finite group and $\Phi(G) \leq Z(G)$. For any $\alpha \in \mathrm{Aut}^{\Phi}(G)$, the map $f_{\alpha}(x) = x^{-1}x^{\alpha}$ defines a homomorphism from G into $\Phi(G)$. By

Lemma 2.1, the mappings $\alpha \mapsto f_\alpha$ and $f \mapsto \alpha_f$ are inverses of each other, so that a bijection between $\mathrm{Aut}^\Phi(G)$ and $\mathrm{Hom}(G, \Phi(G))$ exists. Hence $\mathrm{Aut}^\Phi(G) = \{\alpha_f | f \in \mathrm{Hom}(G, \Phi(G))\}$ and $|\mathrm{Aut}^\Phi(G)| = |\mathrm{Hom}(G, \Phi(G))|$. We note that if $f, g \in \mathrm{Hom}(G, \Phi(G))$, then $\alpha_f\alpha_g = \alpha_g\alpha_f$ if and only if $f \circ g = g \circ f$.

We recall that if $X \leq \Phi(G) \leq Z(G)$ and A/G' is a direct factor of G/G', then any element f of $\mathrm{Hom}(A/G', X)$ induces an element $\bar{f}$ of $\mathrm{Hom}(G/G', X)$ which is trivial on the complement of A/G' in G/G'. To simplify the notation we shall identify f with the corresponding homomorphism from G into X which is induced by $\bar{f}$. Throughout the paper we shall make use of this convention without any further explanation.

As an application of Lemma 2.1, we get the following corollary.

Corollary 2.2. *[1, Lemma 3.2] Let G be a finite p-group. Then $\mathrm{Aut}^\Phi(G) = 1$ if and only if G is elementary abelian.*

Proof. Let $\mathrm{Aut}^\Phi(G) = 1$. Since $\mathrm{Inn}(G) \leq \mathrm{Aut}^\Phi(G)$, it follows that G is abelian and by Lemma 2.1, $\Phi(G) = 1$, which shows that G is elementary abelian. The converse is evident. □

In proving the following lemma, we have used the argument given by Jafari ([3, Lemma 2.2]).

Lemma 2.3. *Let G be a finite group with $\Phi(G) \leq Z(G)$, $\Phi(G) = X_1 \times X_2$ and $G/G' = A_1/G' \times A_2/G'$. If for $1 \leq i, j \leq 2$,*

$$B_{ij} = \{\alpha_f | f \in \mathrm{Hom}(A_i/G', X_j)\},$$

then

(i) the subgroups B_{ij} of $\mathrm{Aut}^\Phi(G)$ having mutually trivial intersections,

(ii) $|B_{ij}| = |\mathrm{Hom}(A_i/G', X_j)|$ and $|\mathrm{Aut}^\Phi(G)| = |B_{11}||B_{21}||B_{12}||B_{22}|$,

(iii) $\mathrm{Aut}^\Phi(G) = B_{11}B_{21}B_{12}B_{22}$,

(iv) $\mathrm{Aut}^\Phi(G)$ is abelian if and only if $[B_{ij}, B_{kl}] = 1$ for $i, j, k, l \in \{1, 2\}$,

(v) if $\mathrm{Aut}^\Phi(G)$ is abelian, then $\mathrm{Aut}^\Phi(G) = B_{11} \times B_{21} \times B_{12} \times B_{22}$.

Proof. (i) It is straightforward.
(ii) This is immediate by using Lemma 2.1.
(iii) It is readily seen that the correspondence $\psi : B_{11}\times B_{21}\times B_{12}\times B_{22} \to B_{11}B_{21}B_{12}B_{22}$ defined by $(\alpha_f, \alpha_g, \alpha_h, \alpha_k) \mapsto \alpha_f\alpha_g\alpha_h\alpha_k$ is one-to-one. Hence $|\mathrm{Aut}^{\Phi}(G)| = |B_{11} \times B_{21} \times B_{12} \times B_{22}| \leq |B_{11}||B_{21}||B_{12}||B_{22}|$, which shows that $\mathrm{Aut}^{\Phi}(G) = B_{11}B_{21}B_{12}B_{22}$, as required.
(iv)-(v) These are easily proved. $\square$

3 Elementary Abelian p-Groups of The $\mathrm{Aut}^{\Phi}(G)$

In this section, we determine the finite p-group G for the group $\mathrm{Aut}^{\Phi}(G)$ to be elementary abelian. By Corollary 2.2, we may assume that G is not an elementary abelian.

First we consider the case when p is odd and G be a finite p-group.

Theorem 3.1. *Let G be a finite p-group, p odd, then $\mathrm{Aut}^{\Phi}(G)$ is elementary abelian if and only if $\Phi(G) \leq Z(G)$ and $\exp(G/G') = p$ or $\exp(\Phi(G)) = p$.*

Proof. Suppose that $\mathrm{Aut}^{\Phi}(G)$ is elementary abelian p-group. Then $\Phi(G) \leq Z(G)$ and $\exp(G') = p$ by [6, Lemma 0.4]. Now since the map $\alpha : g \mapsto g^{p+1}$, defines an automorphism of G which lies in $\mathrm{Aut}^{\Phi}(G)$, we see that $\exp(G) \leq p^2$ and so $\exp(G/G') = p$ or $\exp(\Phi(G)) = p$.

Conversely, suppose that $\Phi(G) \leq Z(G)$. If $\exp(G/G') = p$, then $\Phi(G) = G'$ and $\mathrm{Aut}^{\Phi}(G) = \mathrm{Aut}^{G'}(G) \cong \mathrm{Hom}(G/G', G')$, by [8, Lemma 3.1], which implies that $\mathrm{Aut}^{\Phi}(G)$ is elementary abelian. If $\exp(\Phi(G)) = p$, then by Lemma 2.1, the result follows at once. $\square$

The following definition is taken from [3].

Definition 3.2. Let G be a finite abelian p-group and $G = A \times B$ with $B \cong \mathbb{Z}_p \times ... \times \mathbb{Z}_p$. Then B is said to be an elementary part of G. Also if A is cyclic of order p^n with $n > 1$, then it is called a cyclic part of G.

Lemma 3.3. *Let G be a finite p-group with $\Phi(G) \leq Z(G)$. Suppose that A/G' and X are direct factors of G/G' and $\Phi(G)$, respectively and $T = \{\alpha_f | f \in \mathrm{Hom}(A/G', X)\}$. If $\exp(A/G') = p$ or $\exp(X) = p$ and G' contains an elementary part of $\Phi(G)$, then T is elementary abelian.*

Proof. It is easy to see that the map $\varphi : \mathrm{Hom}(A/G', X) \to T$ defined by $f \mapsto \alpha_f$ is one-to-one. Suppose that $G/G' = L/G' \times A/G'$, $y \in G$ and $f, g \in \mathrm{Hom}(A/G', X)$. If $\exp(A/G') = p$, then $f \circ g(y) = f(g(y)G') = 1$, since

$$g(y)G' \in \Phi(G)/G' = \Phi(G/G') \leq \Phi(L/G').$$

Also if $\exp(X) = p$, then $f \circ g(y) = f(g(y)G') = f(G') = 1$, since an elementary part of $\Phi(G)$ is contained in G'. Hence φ is an isomorphism from $\mathrm{Hom}(A/G', X)$ onto T, so the result follows. $\square$

Notation. Let G be a finite p-group with $\Phi(G) \leq Z(G)$, $G/G' = \langle aG' \rangle \times B/G'$ and $x \in \Phi(G)$ with $o(x) \mid o(aG')$. We define the map $f \in \mathrm{Hom}(G/G', \Phi(G))$ by $f : a^i G' \mapsto x^i$ $(i \geq 0)$, which is trivial on the complement of $\langle aG' \rangle$ and denote it by $f_{a,x}$ in the rest of the paper.

We next consider the case $p = 2$ and give a necessary and sufficient condition on a 2-group G for the group $\mathrm{Aut}^{\Phi}(G)$ to be elementary abelian.

Theorem 3.4. *Let G be a finite 2-group. Then $\mathrm{Aut}^{\Phi}(G)$ is elementary abelian if and only if $\Phi(G) \leq Z(G)$ and one of the following conditions holds:*

(i) $\exp(G/G') = 2$ *or* $\exp(\Phi(G)) = 2$*;*

(ii) $\exp(\Phi(G)) = 4$ *and* $(n, n_1, n_2, ..., n_s)$*, where* $n > 1$*, and* $n_i = 1$ *for* $1 \leq i \leq s$*, and* $(m, m_1, m_2, ..., m_r)$*, where* $m > 1$ *and* $m_i = 1$ *for* $1 \leq i \leq r$*, be the invariants of of* G/G' *and* $\Phi(G)$*, respectively. Furthermore* G' *contains an elementary part of* $\Phi(G)$*, and if an elementary part of* $\Phi(G)$ *is equal to* G'*, then* $\exp(G/G') = 8$*; otherwise,* $\exp(G/G') = 4$*.*

Proof. Since $\mathrm{Aut}^{\Phi}(G)$ is elementary abelian, $\Phi(G) \leq Z(G)$ and by [6, Lemma 0.4], $\exp(G') = 2$. Let $\exp(G/G') = 2^n$ $(n > 1)$, $\exp(\Phi(G)) = 2^m$ $(m > 1)$ and suppose that aG' is an element of order 2^n in G/G'. Setting $G/G' = \langle aG' \rangle \times K/G'$ and assume that $x \in \Phi(G)$ such that $o(x) = 4$. We let $xG' = a^t kG'$, where $t \geq 0$ and $k \in K$. If $2 \nmid t$, then $\langle aG' \rangle = \langle a^t G' \rangle$, whence $G/G' = \langle xG' \rangle (K/G')$. Since $\langle xG' \rangle \leq \Phi(G/G')$, it follows that $G/G' = K/G'$, a contradiction. By taking the homomorphisms f_{a,a^2} and

$g = f_{a,x}$ from G/G' into $\Phi(G)$, we deduce that $o(a) \leq 8$ and $t/2$ is odd, because $a^{{\alpha_g}^2} = ax^{t+2}$. Next, we claim that K/G' is elementary abelian. To see this, if $\exp(K/G') = 2^l, l > 1$ and uG' is an element of order 2^l in K/G', then we write $G/G' = \langle aG' \rangle \times \langle uG' \rangle \times K_1/G'$. By considering the homomorphisms f_{a,u^2} and f_{u,a^2} from G/G' into $\Phi(G)$, we find that $o(u) = 4$ and $n = 2$. Therefore we may assume that $l = 2$ and setting $xG' = a^t u^r k_1 G'$, where $k_1 \in K$. By taking the homomorphism $h = f_{u,x}$ from G/G' into $\Phi(G)$, we deduce that $2 \mid r$ and $r/2$ is odd, because $u^{{\alpha_h}^2} = ux^{r+2}$. Now

$$h \circ g(a) = h \circ g(aG') = h(x) = h(xG') = h(a^t u^r k_1 G') = x^r \neq 1 = g \circ h(a),$$

a contradiction. This shows that G/G' is of type $(n, n_1, n_2, ..., n_s)$, where $n > 1$ and $n_i = 1$ for $1 \leq i \leq s$. Next, we note that $\Phi(G)$ has invariants $(m, m_1, m_2, ..., m_r)$, where $m > 1$ and $m_i = 1$ for $1 \leq i \leq r$. To see this, let $x_1, x_2 \in \Phi(G)$ such that $o(x_1) = o(x_2) = 4$. It is sufficient to prove that $\langle x_1 \rangle \bigcap \langle x_2 \rangle \neq 1$. Otherwise, since $x_1 G' \neq G'$, it follows that $x_1 G' \in \Phi(G/G') \leq \langle a^2 G' \rangle$ and so $x_1 G' = a^{2i} G'$, where $i \in \{1, 3\}$, similarly $x_2 G' = a^{2j} G'$, with $j \in \{1, 3\}$. Therefore by setting the homomorphisms $f = f_{a,x_1}$ and $h = f_{a,x_2}$ from G/G' into $\Phi(G)$, we observe that for $i \in \{1, 3\}$,

$$h \circ f(a) = h \circ f(aG') = h(x_1) = h(x_1 G') = h(a^{2i} G') = x_2^{2i},$$

and for $j \in \{1, 3\}$,

$$f \circ h(a) = f \circ h(aG') = f(x_2) = f(x_2 G') = f(a^{2j} G') = x_1^{2j}.$$

It follows that $x_1^{2j} = x_2^{2i}$, a contradiction. Hence $\langle x_1 \rangle \bigcap \langle x_2 \rangle \neq 1$, as required.

To continue the proof, suppose that $\langle b \rangle$ is a cyclic part of $\Phi(G)$. We choose an element x of order 4 in $\langle b \rangle$. Since $xG' \neq G'$, it follows that $xG' \in \Phi(G/G') \leq \langle aG' \rangle$. Writing $xG' = a^t G'$, where $t > 0$. According the previously-mentioned points and by setting the homomorphism $f_{a,x}$, we find that $2 \mid t$ and $t/2$ is odd. Therefore $o(xG') = o(a^t G') = 2^{n-1}$ and so $\Phi(G/G') = \Phi(G)/G' = \langle xG' \rangle$. We show that G' contains an elementary part of $\Phi(G)$. Let u be any element of order 2 in $\Phi(G)$. Since $uG' \in \langle xG' \rangle$, it follows that $\Omega_1(\Phi(G)) \leq \langle x \rangle G'$ and hence $\Phi(G) =$

$\langle b\rangle\Omega_1(\Phi(G)) = \langle b\rangle G'$. We are now choose a subgroup A of G' such that $\Phi(G) = \langle b\rangle \times A$, which shows that an elementary part of $\Phi(G)$ is contained in G'. Since $|\langle b\rangle \bigcap G'| \leq 2$, we distinguish two cases.

First we assume that $|\langle b\rangle \bigcap G'| = 1$. In this case $\Phi(G) = \langle b\rangle \times G'$ and an elementary part of $\Phi(G)$ is equal to G'. So we deduce that that $n = 3$ and $m = 2$.

In the other case, $\langle b\rangle \bigcap G' = \langle x^2\rangle$ and $o(xG') = 2$, from which we get $n = m = 2$. Now let u be any element of order 2 in $\Phi(G)$. Hence $uG' \in \langle xG'\rangle$. If $uG' = xG'$, then $o(x) = 2$, a contradiction. Therefore $u \in G'$ and G' contains an elementary part of $\Phi(G)$.

Conversely, assume that $\Phi(G) \leq Z(G)$. If (i) holds, then by similar argument that was applied for Lemma 3.1, we observe that $\mathrm{Aut}^{\Phi}(G)$ is elementary abelian.

Now suppose that (ii) holds. We write $G/G' = \langle aG'\rangle \times K/G'$ and $\Phi(G) = \langle b\rangle \times H$, where $\langle aG'\rangle$ and $\langle b\rangle$ are cyclic parts of G/G' and $\Phi(G)$, respectively. First we assume that $o(aG') = 8$ and $o(b) = 4$. Since G' and an elementary part of $\Phi(G)$ coincide, $o(bG') = 4$ and so $bG' = a^{2l}G'$, where $l \in \{1, 3\}$. Let $X_1 = \langle b\rangle$, $X_2 = H$, $A_1/G' = \langle aG'\rangle$, $A_2 = K$ and for $1 \leq i, j \leq 2$, $B_{ij} = \{\alpha_f | f \in \mathrm{Hom}(A_i/G', X_j)\}$.
We are able to show that $[B_{11}, B_{22}] = 1$; the other cases are treated similarly. To do this, assume that $\alpha_f \in B_{11}$, $\alpha_g \in B_{22}$, where $f = f_{a,b^t}, 0 \leq t \leq 3$ and $g \in \mathrm{Hom}(A_2/G', X_2)$. Then

$$g \circ f(a) = g(b^t) = g(b^tG') = g(a^{2lt}G') = 1 = f(G') = f \circ g(a),$$

where $l \in \{1, 3\}$ and for $k \in K$,

$$g \circ f(k) = g \circ f(kG') = g(G') = 1 = f(G') = f \circ g(kG') = f \circ g(k),$$

since an elementary part of $\Phi(G)$ is equal to G'. This shows that $\alpha_f\alpha_g = \alpha_g\alpha_f$ and so $[B_{11}, B_{22}] = 1$. Therefore $\mathrm{Aut}^{\Phi}(G)$ is abelian, by using Lemma 2.3(iv). We claim that each non-trivial element of B_{11} is of order 2; by Lemma 3.3, the other groups are elementary abelian. To see this, let $\alpha_f \in B_{11}$ where $f = f_{a,b^t}, 0 \leq t \leq 3$. Then for $l \in \{1, 3\}$,

$$a^{\alpha_f^2} = (ab^t)^{\alpha_f} = (ab^t)b^tb^{2lt} = ab^{2t(1+l)} = a,$$

as required, which together with Lemma 2.3(v), $\mathrm{Aut}^{\Phi}(G) = B_{11} \times B_{21} \times B_{12} \times B_{22}$. Now the result follows at once.

For the other case, since $\exp(G/G') = 4$, it follows that $o(bG') = 2$ and hence $bG' = a^2G'$. We omit the proof which is quite similar to the previous case. $\square$

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