Journal of Mathematical Extension
Vol. 6, No. 4, (2012), 79-88

A Note on Power Values of Derivation in Prime and Semiprime Rings

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Abstract. Let R be a ring with derivation d, such that $(d(xy))^n = (d(x))^n(d(y))^n$ for all $x, y \in R$ and $n \geqslant 1$ a fixed integer. In this paper, we show that if R is prime, then $d = 0$ or R is commutative. If R is semiprime, then d maps R into its center. Moreover in semiprime case let $A = O(R)$ be the orthogonal completion of R and $B = B(C)$ be the Boolian ring of C, where C is the extended centroid of R. Then there exists an idempotent $e \in B$ such that eA is a commutative ring and d induces a zero derivation on $(1 - e)A$.

AMS Subject Classification: 16R50; 16N60; 16D60
Keywords and Phrases: Derivation, prime ring, semiprime ring, Martindale quotient ring

1. Introduction

Let R be an associative ring with center $Z(R)$. Recall that an additive map $d : R \to R$ is called derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. Many results in literature indicate that global structure of a prime (semiprime) ring R is often lightly connected to the behaviour of additive mappings defined on R. A well-known result of Herstein [13] stated that if R is a prime ring and d is an inner derivation of R such that $d(x)^n = 0$ for all $x \in R$ and $n \geqslant 1$ fixed integer, then $d = 0$.

Received: January 2012; Accepted: October 2012
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The number of authors extended this theorem in several ways. In [12] Giambruno and Herstein extended this result to arbitrary derivations in semiprime rings. In [5] Carini and Giambruno proved that if R is a prime ring with derivation d such that $d(x)^{n(x)} = 0$ for all $x \in L$, a Lie ideal of R, then $d(L) = 0$ when R has no non-zero nil right ideal and char $R \neq 2$. The same conclusion holds when $n(x) = n$ is fixed and R is a 2-torsion free semiprime ring. Using the ideas in [5] and the methods in [10] Lanski [16] removed both the bound on the indices of nilpotence and the characteristic assumptions on R. In [4] Bresar gave a generalization of the result due to Herstein and Giambruno [12] in another direction. Explicitly, he proved in semiprime ring R with derivation d and $a \in R$, if $ad(x)^n = 0$ for all $x \in R$, where $n \geqslant 1$ is a fixed integer, then $ad(R) = 0$ when R is an $(n-1)!$-torsion free ring. In recent years, a number of articles discussed derivations in the context of prime and semiprime rings (see [6, 11, 20, 8, 1, 9]).
But here we will extend Herstein result's [13] when the condition is more widespread.
Indeed, we consider the situation when $(d(xy))^n = (d(x))^n(d(y))^n$ for all $x, y \in R$ and $n \geqslant 1$ is a fixed integer.

The main results in this paper are as follows:

Theorem 1.1. *Let R be a prime ring and d a derivation of R. Suppose $(d(xy))^n = (d(x))^n(d(y))^n$ for all $x, y \in R$ and $n \geqslant 1$ is a fixed integer. Then $d = 0$ or R is commutative.*
When R is a semiprime ring, we prove:

Theorem 1.2. *Let R be a semiprime ring and d a non-zero derivation of R. Suppose $(d(xy))^n = (d(x))^n(d(y))^n$ for all $x, y \in R$ and $n \geqslant 1$ is a fixed integer. Then d maps R into its center.*

Theorem 1.3. *Let R be a semiprime ring with derivation d. Consider $(d(xy))^n = (d(x))^n(d(y))^n$ for all $x, y \in R$ and $n \geqslant 1$ is a fixed integer. Further, let $A = O(R)$ be the orthogonal completion of R and $B = B(C)$ where C the extended centroid of R. Then there exists idempotent $e \in B$ such that eA is a commutative ring and d induce a zero derivation on $(1-e)A$.*

Throughout the paper we use the standard notation from [3].
In particular, we denote by Q the two sided Martindale quotient of prime (semiprime) ring R and C the center of Q. We call C the extended centroid of R.

2. Main Results

First, we consider the case when R is a prime ring. The following results are useful tools needed in the proof of Theorem 1.1.

Lemma 2.1. (see [7, Theorem 2]). *Let R be a prime ring and I a nonzero ideal of R. Then I, R and Q satisfy the same generalized polynomial identities with coefficient in Q.*

Lemma 2.2. (see [18, Theorem 2). *Let R be a prime ring and I a nonzero ideal of R. Then I, R and Q satisfy the same differential identities.*

Theorem 2.3. (Kharchenko [15]). *Let R be a prime ring, d a nonzero derivation of R and I a nonzero ideal of R. If I satisfies the differential identity*

$$f(r_1, r_2, \ldots, r_n, d(r_1), d(r_2), \ldots, d(r_n)) = 0,$$

for any $r_1, r_2, \ldots, r_n \in I$, then one of the following holds:
(i) satisfies the generalized polynomial identity

$$f(r_1, r_2, \ldots, r_n, x_1, x_2, \ldots, x_n) = 0.$$

(ii) d is Q-inner, that is, for some $q \in Q$, $d(x) = [q, x]$ and I satisfies the generalized polynomial identity

$$f(r_1, r_2, \ldots, r_n, [q, r_1], [q, r_2], \ldots, [q, r_n]) = 0.$$

We establish the following technical result required in the proof of Theorem 1.1.

Lemma 2.4. *Let R be a prime ring with extended centroid C. Suppose $([a, x]y + x[a, y])^n - [a, x]^n[a, y]^n = 0$, for all $x, y \in R$ and some $a \in R$. Then R is commutative or $a \in C$.*

Proof. If R is commutative there is nothing to prove. Suppose R is not commutative. Set

$$f(x,y) = ([a,x]y + x[a,y])^n - [a,x]^n[a,y]^n.$$

Since R is not commutative, then by Lemma 2.1, $f(x,y)$ is a nontrivial generalized polynomial identity for R and so for Q.
In case C is infinite, we have $f(x,y) = 0$ for all $x, y \in Q \bigotimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q \bigotimes_C \overline{C}$ are prime and centrally closed [14], we may replace R by Q or $Q \bigotimes_C \overline{C}$ according to C finite or infinite. Thus we may assume that R is a centrally closed over C which is either finite or algebraically closed and $f(x,y) = 0$ for all $x, y \in R$. By Martindale's Theorem [19], R is then a primitive ring having nonzero socle H with C as associated division ring. Hence by Jacobson's Theorem [14] R is isomorphic to a dense ring of linear transformations of some vector space V over C, and H consists of the linear transformations in R of finite rank. Let $\dim_C V = k$. Then the density of R on V implies that $R \cong M_k(C)$. If $\dim_C V = 1$, then R is commutative, which is a contradiction.
Suppose that $\dim_C V \geqslant 2$. We show that for any $v \in V$, v and av are linearly dependent over C. Suppose v and av are linearly independent for some $v \in V$. By density of R, there exist $x, y \in R$ such that

$$xv = 0, \quad xav = v,$$

$$yv = 0, \quad yav = v.$$

Since $[a,y]^n v = [a,x]^n v = (-1)^n v$, hence we get the following contradiction

$$0 = (([a,x]y + x[a,y])^n - [a,x]^n[a,y]^n)v = -v.$$

So we conclude that $\{v, av\}$ are linearly C-dependent. Hence for each $v \in V$, $av = v\alpha_v$ for some $\alpha_v \in C$. Now we prove α_v is not depending on the choice of $v \in V$.
Since $\dim_C V \geqslant 2$ there exists $w \in V$ such that v and w are linearly independent over C. Now there exist $\alpha_v, \alpha_w, \alpha_{v+w} \in C$ such that

$$av = v\alpha_v, aw = w\alpha_w, a(v+w) = (v+w)\alpha_{(v+w)}.$$

Which implies

$$v(\alpha_v - \alpha_{(v+w)}) + w(\alpha_w - \alpha_{(v+w)}) = 0,$$

and since $\{v, w\}$ are linearly C-independent, it follows $\alpha_v = \alpha_{(v+w)} = \alpha_w$. Therefore there exists $\alpha \in C$ such that $av = v\alpha$ for all $v \in V$. Now let $r \in R$, $v \in V$. Since $av = v\alpha$,

$$[a, r]v = (ar)v - (ra)v = a(rv) - r(av) = (rv)\alpha - r(v\alpha) = 0,$$

that is $[a, r]V = 0$. Hence $[a, r] = 0$ for all $r \in R$, implying $a \in C$. $\square$

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Let R be not commutative. By the given hypothesis, R satisfies the generalized differential identity

$$(d(x)y + xd(y))^n = (d(x))^n(d(y))^n. \tag{1}$$

By Lemma 2.2, R and Q satisfy the same differential identities, thus Q satisfies (1). We divide the proof in two cases:

Case 1. d is a Q-inner derivation. In the case, there exists an element $a \in Q$ such that $d(x) = [a, x]$ and $d(y) = [a, y]$ for all $x, y \in Q$. Notice that Q satisfies the generalized polynomial identity $([a, x]y + x[a, y])^n = [a, x]^n[a, y]^n$. In this case the conclusion follows from Lemma 1. Thus we have $a \in C$ and so $d = 0$.

Case 2. d is not a Q-inner derivation. Applying Theorem 2.2, then (1) becomes

$$(zy + xw)^n - (z)^n(w)^n,$$

for all $x, y, z, w \in Q$. If $z = w$, then Q satisfies

$$(zy + xz)^n - z^{2n} = 0.$$

This is a polynomial identity. Hence there exists a field F such that $Q \subseteq M_k(F)$, the ring of $k \times k$ matrices over field F, where $k > 1$. Moreover Q and $M_k(F)$ satisfy the same polynomial identity [17, Lemma 1]. Choose

$$x = z = e_{ij}, \quad y = e_{ji},$$

for all $i \neq j$. This leads to the contradiction

$$0 = (zy + xz)^n - z^{2n} = e_{ii}.$$

This completes the proof. $\square$

The following example shows the hypothesis of primeness is essential in Theorem 1.1.

Example 2.5. Let S be any ring, and $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in S \right\}$.
Define $d : R \to R$ as follows:

$$d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $0 \neq d$ is a derivation of R such that $(d(xy))^n = (d(x))^n(d(y))^n$ for all $x, y \in R$, where $n \geqslant 1$ is a fixed integer, however R is not commutative.

Now let R be a semiprime ring.
We establish the following technical result required in the proof of Theorem 1.2.

Lemma 2.6. (see [2, Lemma 1 and Theorem 1] or [18, pages 31-32]). *Let R be a semiprime ring and P a maximal ideal of C. Then PQ is a prime ideal of Q invariant under all derivations of Q. Moreover*

$$\cap\{P|PQ \textit{ is maximal ideal of } C\} = 0.$$

Now we can prove Theorem 1.2.

Proof. Since any derivation d can be uniquely extended to a derivation in Q, and R, Q satisfy the same differential identities [18, Theorem 3], we have

$$(d(xy))^n = (d(x))^n(d(y))^n,$$

for all $x, y \in Q$. Let P be any maximal ideal of C by Lemma 2.6, PQ is prime ideal of Q invariant under d. Set $\overline{Q} = Q/PQ$. Then derivation d canonically induces a derivation $\overline{d}$ on $\overline{Q}$ defined by $\bar{d}(\bar{x}) = \overline{d(x)}$ for all $x \in Q$. Therefore,

$$(\bar{d}(\overline{xy}))^n = (\bar{d}(\bar{x}))^n(\bar{d}(\bar{y}))^n,$$

for all $\bar{x}, \bar{y} \in \overline{Q}$. By Theorem 1.1 $d(Q) \subseteq PQ$ or $[Q, Q] \subseteq PQ$. Hence $d(Q)[Q, Q] \subseteq PQ$ for any maximal ideal P of C. By Lemma 2.6, $d(Q)[Q, Q] = 0$. Without loss of generality we have $d(R)[R, R] = 0$. This implies that

$$d(R^2)[R, R] = d(R)R[R, R].$$

Therefore

$$[R, d(R)]R[R, d(R)] = 0.$$

By semiprimeness of R, we have $[R, d(R)] = 0$. This complete the proof. □

Now let R be a semiprime orthogonally complete ring with extended centeroid C. The notations $B = B(C)$ and spec(B) denotes Boolian ring of C and the set of all maximal ideal of B, respectively. It is well known that if $M \in$ spec(B) then $R_M = R/RM$ is prime [3, Theorem 3.2.7]. We use the notations Ω-Δ-ring, Horn formulas and Hereditary formulas. We refer the reader to [3, pages 37, 38, 43, 120] for the definitions and the related properties of these objects.

We establish the following technical result required in the proof of Theorem 1.3.

Lemma 2.7. [3, Theorem 3.2.18]. *Let R be an orthogonally complete Ω-Δ-ring with extended centroid C, $\Psi_i(x_1, x_2, \ldots, x_n)$ Horn formulas of signature Ω-Δ, $i = 1, 2, \ldots$ and $\Phi(y_1, y_2, \ldots, y_m)$ a Hereditary first order formula such that $\neg\Phi$ is a Horn formula. Further, let $\vec{a} = (a_1, a_2, \ldots, a_n) \in R^{(n)}$, $\vec{c} = (c_1, c_2, \ldots, c_m) \in R^{(m)}$. Suppose $R \models \Phi(\vec{c})$ and for every $M \in$ spec (B) there exists a natural number $i = i(M) > 0$ such that*

$$R_M \models \Phi(\phi_M(\vec{c})) \Longrightarrow \Psi_i(\phi_M(\vec{a})),$$

where $\phi_M : R \to R_M = R/RM$ is the canonical projection. Then there exists a natural number $k > 0$ and pairwise orthogonal idempotents

$e_1, e_2, \ldots, e_k \in B$ *such that* $e_1 + e_2 + \ldots + e_k = 1$ *and* $e_iR \models \Psi_i(e_i\vec{a})$ *for all* $e_i \neq 0$.

We denote $O(R)$ the orthogonal completion of R which is defined as the intersection of all orthogonally complete subset of Q containing R.

Now we can prove Theorem 1.3.

Proof. By assumption we have R satisfies

$$(d(xy))^n = (d(x))^n(d(y))^n.$$

According to [3, Theorem 3.1.16] $d(A) \subseteq A$ and $d(e) = 0$ for all $e \in B$. Therefore, A is an orthogonally complete Ω-Δ-ring, where $\Omega = \{o, +, -, \cdot, d\}$. Consider formulas

$$\Phi = (\forall x)(\forall y)\|(d(xy))^n = (d(x))^n(d(y))^n\|,$$

$$\Psi_1 = (\forall x)\|d(x) = 0\|,$$

$$\Psi_2 = (\forall x)(\forall y)\|xy = yx\|.$$

One can easily check that Φ is a hereditary first order formula and $\neg\Phi$, Ψ_1, Ψ_2 are Horn formulas. So using Theorem 1.1 shows that all conditions of Lemma 2.7 are fulfilled. Hence there exist two orthogonal idempotent e_1 and e_2 such that $e_1 + e_2 = 1$ and if $e_i \neq 0$, then $e_iA \models \Psi_i$, $i = 1, 2$. The proof is complete. $\square$

Acknowledgment
This paper is extracted from P.h.d Project that is done in Islamic Azad University Tehran Central Branch (IAUCTB). Authors want to thank authority of IAUCTB for their support to complete this research.

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