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Original Research Paper

Maps Preserving the Difference of Minimum and Surjectivity Moduli of Self-adjoint Operators

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Abstract. Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space and $\mathcal{SA}(\mathcal{H})$ be the real Jordan algebra of all bounded self-adjoint operators acting on $\mathcal{H}$. In this paper, we study the general form of surjective non-linear maps $\xi : \mathcal{SA}(\mathcal{H}) \to \mathcal{SA}(\mathcal{H})$, that preserve the difference of minimum and surjectivity moduli of self-adjoint operators in both directions. It turns out that

$$\xi(P) = EPE^* + R, \qquad (P, R \in \mathcal{SA}(\mathcal{H}))$$

where $E : \mathcal{H} \to \mathcal{H}$, is either a bounded unitary or an anti-unitary operator.

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1 Introduction

Recently non-linear preserver problems have been investigated by many authors, see for instance [10, 14, 15, 18, 20]. In [5], authors characterized

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surjective maps preserving the spectral radius of the difference of matrices. In [19], Molnar studied maps preserving the spectrum of operator or matrix products. His results have been extended in several direction for uniform algebras and semisimple commutative Banach algebras, and a number of results is obtained on maps preserving several spectral and local spectral quantities of operator or matrix product, or Jordan product, or difference; see for instance [9, 13, 14, 15, 17] and the references therein. In [18], authors characterized maps of $\xi : \mathcal{B}(X) \to \mathcal{B}(X)$ that preserve difference minimum moduli of operators. They proved that this map has the form $\xi(P) = EP^\sharp F + R$ for E, F isometry operators and $P^\sharp$ denotes P, or P^*.

In this paper, we attempt to determine the general form of ξ when it is restricted to the real Jordan algebra $\mathcal{SA}(\mathcal{H})$.

Throughout this paper, $\mathcal{H}$ stands for an infinite dimensional separable complex Hilbert space, and $\mathcal{B}(\mathcal{H})$, $\mathcal{SA}(\mathcal{H})$ and $\mathcal{F}(\mathcal{H})$, $\mathcal{FS}(\mathcal{H})$ denote the space of all bounded operators, self-adjoint bounded operators, finite rank operator on $\mathcal{H}$, finite rank operator on $\mathcal{SA}(\mathcal{H})$, and $\mathcal{A}_s(\mathcal{H})$ the set of all algebraic operators on $\mathcal{SA}(\mathcal{H})$. In [15], Havlicek and Semrl showed a complete characterization of bijective maps ξ on $\mathcal{B}(\mathcal{H})$ satisfying the condition

$$\xi(Q) - \xi(P) \textit{ is invertible} \iff Q - P \textit{ is invertible.}$$

Theorem 1.1. *([15] Theorem 1.2) Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$. Let $\xi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be bijective map such that for every pair $M, W \in \mathcal{B}(\mathcal{H})$ the operator $M - W$ is invertible if and only if $\xi(M) - \xi(W)$ is invertible. Then there exist $R \in \mathcal{B}(\mathcal{H})$ and invertible $P, Q \in \mathcal{B}(\mathcal{H})$ such that either*

$$\xi(M) = PMQ + R \tag{1}$$

for every $M \in \mathcal{B}(\mathcal{H})$*,or*

$$\xi(M) = PM^tQ + R \tag{2}$$

for every $M \in \mathcal{B}(\mathcal{H})$ *,or*

$$\xi(M) = PM^*Q + R \tag{3}$$

for every $M \in \mathcal{B}(\mathcal{H})$, *or*

$$\xi(M) = P(M^t)^* Q + R \tag{4}$$

for every $M \in \mathcal{B}(\mathcal{H})$.

Here M^t *and* M^* *denote the transpose with respect to an arbitrary but fixed orthonormal basis, and the usual adjoint of* M *in the Hilbert space sense, respectively.*

Definition 1.2. ([23]) The minimum modulus of an operator $P \in \mathcal{B}(\mathcal{H})$ denoted by $\rho(P)$, is defined by $\rho(P) = \inf\{||Ph|| : \ h \in \mathcal{H} \ , \ ||h|| = 1\}$.
The surjectivity modulus of P is defined by
$\tau(P) = \sup\{\varepsilon \geqslant o : \varepsilon B(0,1) \subset P(B(0,1))\}$, where $B(0,1) = \{x \in H : ||x|| < 1\}$. Maximum modulus of P is defined by
$N(P) = \max\{\rho(P), \tau(P)\}$.

Definition 1.3. ([23]) A linear map $\xi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ preserves the minimum modulus(resp. surjectivity modulus), if for all $P \in \mathcal{B}(\mathcal{H})$,

$$\rho(\xi(P)) = \rho(P) \quad , \qquad \qquad (resp.\ \tau(\xi(P)) = \tau(P)).$$

Note that $\rho(P^*) = \tau(P)$ and $\tau(P^*) = \rho(P)$ *for all* $P \in \mathcal{B}(\mathcal{H})$.

Theorem 1.4. *([23] Theorem 3.5) Let* $\xi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *be a surjective linear map. The following are equivalent:*

1. $\rho(\xi(P)) = \rho(P)$, *for all* $P \in \mathcal{B}(\mathcal{H})$.
2. $\tau(\xi(P)) = \tau(P)$, *for all* $P \in \mathcal{B}(\mathcal{H})$.
3. *There exist two unitary operators* $E \in \mathcal{B}(\mathcal{H})$ *and* $F \in \mathcal{B}(\mathcal{H})$ *such that* $\xi(P) = EPF$ *for every* $P \in \mathcal{B}(\mathcal{H})$.

In this paper, we let $\nu(.)$ stand for either $\rho(.)$ *or* $\tau(.)$ *or* $N(.)$. We stablish a similar result to Theorem 1.1 of characterizing maps from $\mathcal{SA}(\mathcal{H})$ onto $\mathcal{SA}(\mathcal{H})$ preserving any of the surjectivity, the injectivity, and the boundedness from below of the difference and sum of operators. We show the adjacency of operators in term of any of the previous mentioned spectral quantities and use such a description to show that if a map ξ from $\mathcal{SA}(\mathcal{H})$ onto $\mathcal{SA}(\mathcal{H})$ preserves operator pairs difference is invertible and thus Theorem 1.1 ensures that such a map ξ takes either (1) or (3). Then we describe maps ξ from $\mathcal{SA}(\mathcal{H})$ onto $\mathcal{SA}(\mathcal{H})$ satisfying

$$\nu(\xi(M) - \xi(W)) = \nu(M - W) \tag{5}$$

$$\nu(\xi(M) + \xi(W)) = \nu(M + W) \tag{6}$$

for all $M, W \in \mathcal{SA}(\mathcal{H})$.

For $g, h \in \mathcal{H}$, $\langle g, h\rangle$ stands for the inner product of g and h. For every $P \in \mathcal{B}(\mathcal{H})$, we use the notations $\text{rank}(P)$, $\ker(P)$, $\text{ran}(P)$ and $\sigma(P)$ for the rank, kernel, range and the spectrum of P, respectively. A conjugate linear bijective operator E on $\mathcal{H}$ is called anti-unitary, provided that $\langle Ex, Ey\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. The identity operator on $\mathcal{H}$ will be denoted by I. Two operators Q, P in $\mathcal{SA}(\mathcal{H})$ are called adjacent, provided that $Q - P$ is a rank one operator. It is said that a surjective map $\xi : \mathcal{SA}(\mathcal{H}) \longrightarrow \mathcal{SA}(\mathcal{H})$ preserves adjacency of operators in both directions, if it preserves adjacent operators in both directions. Recall that we say that an operator is a rank one operator on $\mathcal{H}$, if there exist $u, v \in \mathcal{H}$ so that $Tx = \langle x, v\rangle u$ for all $x \in \mathcal{H}$. We use the notation $T = u \otimes v$. Every self-adjoint rank one operator on $\mathcal{H}$ is of the form $\lambda b \otimes b$ for some non-zero $b \in \mathcal{H}$ and $\lambda \in \mathbb{R}$.

2 Main results

In this part, we recall some important lemmas that will be used in the proof of our results. Recall that the spectral radius of an operator $P \in \mathcal{B}(\mathcal{H})$ is
$r(P) = \lim_{n\to\infty} ||P^n||^{1/n}$, and coincides with the maximum modulus of $\sigma(P)$, the spectrum of P.

Lemma 2.1. *([12]) Let $A \in \mathcal{B}(\mathcal{H})$. Then $A = 0$ if and only if $r(A + P) = 0$, for all nilpotent operators $P \in \mathcal{B}(\mathcal{H})$ of rank at most one.*

Lemma 2.2. *([23]) Let $C, D \in \mathcal{B}(\mathcal{H})$ be two invertible operators. The following are equivalent:*

(1) $\rho(CPD) = \rho(P) \quad for\ all\ P \in \mathcal{B}(\mathcal{H}),$

(2) $\tau(CPD) = \tau(P)$ *for all* $P \in \mathcal{B}(\mathcal{H})$,

(3) *there are two unitary operators* $E \in \mathcal{B}(\mathcal{H})$ *and* $F \in \mathcal{B}(\mathcal{H})$ *such that* $C = \alpha E$, $D = \beta F$ *where* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ *with* $|\alpha\beta| = 1$.

Let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices, and note that $\rho(P) = \tau(P) = N(P)$ for all matrices $P \in M_n(\mathbb{C})$.

Theorem 2.3. *([15]) A surjective map* ξ *on* $M_n(\mathbb{C})$ *satisfies*

$$\rho(\xi(Q) - \xi(P)) = \rho(Q - P), \qquad (Q, P \in M_n(\mathbb{C}))$$

if and only if there are $E, F, R \in M_n(\mathbb{C})$ *with* E *and* F *unitary matrices such that*

$$\xi(P) = EP^\sharp F + R\ , \qquad (P \in M_n(\mathbb{C}))$$

where $P^\sharp$ *uses for* P *or* P^{tr} *or* P^* *or* $\bar{P}$*, the complex conjugation of* P.

We describe maps ξ from $\mathcal{SA}(\mathcal{H})$ onto $\mathcal{SA}(\mathcal{H})$ satisfying (5) and (6) and prove some details needed for the proof of the main results. These results generalize published results on linear or additive maps preserving the minimum and surjectivity moduli of operators. See for instance [6, 8, 23].

Theorem 2.4. *Let* $\xi : \mathcal{SA}(\mathcal{H}) \to \mathcal{SA}(\mathcal{H})$ *be a surjective map satisfying (5). Then following situation hold:*
There are $R \in \mathcal{B}(\mathcal{H})$ *and unitary or anti-unitary operator* $E : \mathcal{H} \to \mathcal{H}$ *such that*

$$\xi(P) = EPE^* + R\ , \quad P \in \mathcal{SA}(\mathcal{H}). \tag{7}$$

If ξ *satisfies* (7)*, then* (5) *is satisfied.*

For our aims, we show a similar result to Havlicek and Semrl [15] and Hou and Huang [16] by replacing the invertibility by surjectivity and boundedness below. This result plays an important role in the proof of the Theorem 2.4. Let $\Delta_{Inj}(\mathcal{H}), \Delta_{Surj}(\mathcal{H}), \Delta_{LB}(\mathcal{H}), \Delta_{Inj-or-Surj}(\mathcal{H})$ and $\Delta_{LB-or-Surj}(\mathcal{H})$ be respectively the subsets of all non-injective, non-surjective, lower bounded, non-injective or non-surjective, and lower bounded or non-surjective operators on $\mathcal{SA}(\mathcal{H})$. Assume that $\Delta(\mathcal{H})$

denotes any of these sets, and note that every operator $P \in \Delta(\mathcal{H})$ is either non-injective or non-surjective and that

$$P.\Delta(\mathcal{H}) = \Delta(\mathcal{H}).P = \Delta(\mathcal{H})$$

for all invertible operators $P \in \mathcal{SA}(\mathcal{H})$.

Theorem 2.5. *If a surjective map $\xi : \mathcal{SA}(\mathcal{H}) \longrightarrow \mathcal{SA}(\mathcal{H})$ satisfies*

$$\xi(Q) - \xi(P) \in \Delta(\mathcal{H}) \Longleftrightarrow Q - P \in \Delta(\mathcal{H}), \tag{8}$$

there is an operator $R \in \mathcal{SA}(\mathcal{H})$ such that ξ is of the form

$$\xi(P) = APA^* + R \quad , \quad P \in \mathcal{SA}(\mathcal{H})$$

for some bijective continuous mapping $A : \mathcal{H} \to \mathcal{H}$.

For proving of this theorem, we show that if ξ satisfies (8), then ξ is a bijective map that preserves the invertibility of the difference of operators. But to show this, we first specify the adjacency of operators in term of operators in $\Delta(\mathcal{H})$, similar to following lemma.

Lemma 2.6. *Let Q, P be two different operators in $\mathcal{SA}(\mathcal{H})$. Then Q, P are adjacent, if and only if there exists $R \in \mathcal{SA}(\mathcal{H}) \setminus \{Q, P\}$ such that $R - P \in \Delta(\mathcal{H})$ and for every $Y \in \Delta(\mathcal{H}), Y - R, Y - P \in \Delta(\mathcal{H})$ imply $Y - Q \in \Delta(\mathcal{H})$.*

Proof. We can restrict ourselves to the case where $P = 0$. Assume Q is a rank one operator. Hence $Q = \lambda h \otimes h$, where $h \in \mathcal{H}$ is a unit vector and λ is non-zero real scalar. Let $R = -Q$. Then $R \in \Delta(\mathcal{H}) \setminus \{Q, 0\}$. Let $Y \in \Delta(\mathcal{H})$ be such that $Y - R \in \Delta(\mathcal{H})$. We claim $Y - Q$ is non-invertible(non-injective or non-surjective). For this, we consider two cases: if $\ker(Y) \cap \{h\}^\perp \neq \{0\}$, then $\ker(Y - Q) \neq \{0\}$ and consequently $Y - Q$ is non-invertible. Assume $\ker(Y) \cap \{h\}^\perp = \{0\}$, then $Y + Q$ is non-injective, as $Y - R = Y + Q \in \Delta(\mathcal{H})$. Let $k \in \ker(Y + Q)$ be a non-zero unit vector. Then $Yk = -\lambda\langle k, h\rangle h$. Hence $k \notin \{h\}^\perp$. As $\mathcal{H} = \{h\}^\perp \oplus \mathbb{C}h$, it follows that $k = \beta h + g$, for some non-zero scalars $\beta \in \mathbb{C}$ and $g \in \{h\}^\perp$. Since $\langle g, h\rangle = 0$, we have $Yk = -\lambda\beta h$. Consequently, as $Y - Q = Y(I + h \otimes h)$, applying the facts that Y is non-invertible and $I + h \otimes h$ is invertible, it follows that $Y - Q$ is not invertible. So $Y - Q \in \Delta(\mathcal{H})$.

For the inverse direction, it is assumed dim ran$(Q) \geq 2$. We claim that for every $R \in \Delta(\mathcal{H}) \setminus \{Q, 0\}$, there exists $Y \in \Delta(\mathcal{H})$ such that $Y - R \in \Delta(\mathcal{H})$ and $Y - Q \notin \Delta(\mathcal{H})$. For this, let $R \in \Delta(\mathcal{H}) \setminus \{Q, 0\}$ be fixed. There are two cases: If $Q \notin \Delta(\mathcal{H})$, then it is enough to consider $Y = 0$. If $Q \in \Delta(\mathcal{H})$, then Q is not injective and there exist some $h \in \mathcal{H}$ such that $(R-Q)h \neq 0$, as $R \neq Q$. Applying the fact that dim ran$(Q) \geq 2$, it follows that there exist some $k \in \mathcal{H}$ such that the vectors $\{(R-Q)h, Qk\}$ are linearly independent. By replacing k with $k + \theta$, for some $\theta \in \ker(Q)$ if necessary, we may assume $\{h, k\}$ are linearly independent. Let $V = \operatorname{span}\{h, k, (R-Q)h, Qk\}$. Then we can represent the operators Q and R with respect to the decomposition of $\mathcal{H} = V \oplus V^{\perp}$ as follows:

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^* & Q_3 \end{bmatrix}, R = \begin{bmatrix} R_1 & R_2 \\ R_2^* & R_3 \end{bmatrix}.$$

Let $Y \in \mathcal{SA}(\mathcal{H})$ be the operator that with respect to the decomposition $\mathcal{H} = V \oplus V^{\perp}$ is represented as

$$Y = \begin{bmatrix} P + Q_1 & Q_2 \\ Q_2^* & cI \end{bmatrix},$$

where $c \notin \sigma(Q_3)$ and $P \in \mathcal{SA}(V)$ is an invertible operator such that $Ph = (R_1 - Q_1)h$ and $Pk = -Q_1 k$. It follows that R, Y and $Y - R$ are algebraic operators. But as $Yk = (Y - R)h = 0$, hence $Y, Y - R \in \Delta(\mathcal{H})$. On the other hand, as

$$Y - Q = \begin{bmatrix} P & 0 \\ 0 & cI - Q_3 \end{bmatrix},$$

it follows that $Y - Q$ is invertible, thus $Y - Q \notin \Delta(\mathcal{H})$, which completes the proof. $\square$

Proposition 2.7. *Let $Q, P \in \mathcal{SA}(\mathcal{H})$ and for every invertible $Y \in \mathcal{SA}(\mathcal{H})$ we have*

$$Q - Y \in \Delta(\mathcal{H}) \Longleftrightarrow P - Y \in \Delta(\mathcal{H}).$$

Then $Q = P$.

Proof. Since $0 \in \Delta(\mathcal{H})$ by setting $Y = Q$, it follows $P - Q \in \Delta(\mathcal{H})$. As $\Delta(\mathcal{H})$ does not contain any invertible operator and $P - Q \in \Delta(\mathcal{H})$, for asserting that $P = Q$, it is enough to show that $P - Q$ is a scalar multiple of the identity. If this is not so, $P - Q \neq cI$ for every $c \in \mathbb{R}$, then there exists a unit vector $k \in \mathcal{H}$ such that $k, (P-Q)k$ are linearly independent. Let $V = \text{span}\{k, (P-Q)k\}$. Then $P - Q$ has the matrix representation

$$P - Q = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$$

$$where \quad A = \begin{bmatrix} 0 & 1 \\ 1 & \beta \end{bmatrix}.$$

Let

$$R = \begin{bmatrix} 0 & -B \\ -B^* & I - C \end{bmatrix}.$$

Then $R \in \mathcal{SA}(\mathcal{H})$ and since $Rk = 0$, it follows that R is not invertible. Since by assumption $P - Q \in \Delta(\mathcal{H})$, it follows that C and hence R are algebraic. Consequently, if we set $Y = Q - R$, then $Q - Y = R \in \Delta(\mathcal{H})$. But since

$$P - Y = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix},$$

is invertible, so $P - Y \notin \Delta(\mathcal{H})$. we get a contradiction. $\square$

Lemma 2.8 take from [21], and characterize non-linear maps that preserving finite rank operators and preserve the operator that difference of them is rank one. Recall that a map $B : \mathcal{H} \to \mathcal{H}$ is called semilinear if it is additive and there is an automorphism σ of $\mathbb{C}$ such that $B(\alpha x) = \sigma(\alpha)Bx$ for all $x \in \mathcal{H}$ *and* $\alpha \in \mathbb{C}$. Such a map is sometimes said σ-semilinear when the automorphism σ is specified.

Lemma 2.8. *(Petek-Semrl [21]) Assume that X and Y are Banach spaces of dimensions at least 2, and let ξ be a bijective map from $\mathcal{F}(X)$ into $\mathcal{F}(Y)$ such that whenever C, D are operators in $\mathcal{F}(X)$ that satisfies*

$$C - D \text{ has rank one} \iff \xi(C) - \xi(D) \text{ has rank one},$$

then one of the following properties is satisfied:

(1) *There is an automorphism* σ *of* $\mathbb{C}$, $R \in \mathcal{B}(Y)$, *and bijective* σ-*semilinear maps* $S : X \to Y$ *and* $T : X^* \to Y^*$ *such that* $D \mapsto \xi(D) - R$ *is an additive map defined by* $\xi(x \otimes f) - R = Sx \otimes Tf, \qquad (x \in X, f \in X^*)$.

(2) *There is an automorphism* σ *of* $\mathbb{C}$, $R \in \mathcal{B}(Y)$, *and bijective* σ-*semilinear maps* $S : X \to Y^*$ *and* $T : X^* \to Y$ *such that* $D \mapsto \xi(D) - R$ *is an additive map defined by* $\xi(x \otimes f) - R = Tf \otimes Sx, \qquad (x \in X, f \in X^*)$.

This lemma is true for $X = Y = \mathcal{H}$ and $\mathcal{SA}(\mathcal{H})$.

Let $\mathcal{AI}_s(\mathcal{H})$ be the set of all invertible operators in $\mathcal{SA}(\mathcal{H})$.

Proposition 2.9. *Let* $Q, P \in \mathcal{SA}(\mathcal{H})$ *and for every* $N \in \mathcal{AI}_s(\mathcal{H})$ *we have*

$$Q - N \in \mathcal{AI}_s(\mathcal{H}) \Longleftrightarrow P - N \in \mathcal{AI}_s(\mathcal{H}).$$

Then $Q = P$.

Proof. First we claim that $\sigma(Q) = \sigma(P)$. For this note that for every scalar $\lambda \in \mathbb{R}$, $\lambda \in \sigma(Q)$ if and only if $Q - \lambda I$ is not invertible, but by assumption, this holds precisely when $P - \lambda I$ is not invertible or equivalently $\lambda \in \sigma(P)$.

For the rest of proof, it is enough to show that $Q - P$ is a scalar operator, since from $Q = P + \lambda I$ and the fact that $\sigma(Q) = \sigma(P)$, it follows that $\lambda = 0$. If $Q - P$ is not a scalar operator, then there exists $k \in \mathcal{H}$ such that the vectors k and $(Q - P)k$ are linearly independent. There are two cases: either $\{k, Pk\}$ or $\{k, Qk\}$ is a linearly independent set. It is enough to consider the first case, since the other one is similar. Put $V = \text{span}\{k, Qk, Pk\}$. Then the operator Q can be represented as

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^* & Q_3 \end{bmatrix}$$

regarding the decomposition of $\mathcal{H} = V \bigoplus V^\perp$. Before proceeding further, we make a claim:

Claim. There exists an invertible $A \in \mathcal{SA}(V)$ such that $Ak = Pk$ and $Q_1 - A$ is invertible.

For this we consider two cases:

Case1. Assume $\dim(V) = 3$. Let

$$Q_1 = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12}^* & q_{22} & q_{23} \\ q_{13}^* & q_{23}^* & q_{33} \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12}^* & a_{22} & a_{23} \\ a_{13}^* & a_{23}^* & a_{33} \end{bmatrix}$$

be the representations of Q_1 and A regarding the decomposition of $V = \{k\} \bigoplus \{Pk\} \bigoplus \{Qk\}$. Since $Q_1 k = Q_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, it follows that $q_{11} = q_{12} = 0$ and $q_{13} = 1$. Similarly, since $Ak = A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, it follows that $a_{11} = a_{13} = 0$ and $a_{12} = 1$. Hence, it is enough to consider

$$A = \begin{bmatrix} 0 & 1 & o \\ 1 & q_{22} - 1 & q_{23} \\ 0 & q_{23}^* & q_{33} - 1 \end{bmatrix}.$$

Then A is invertible and satisfies $Ak = Pk$.

Case2. Assume $\dim(V) = 2$. Let $Qk = \lambda k + \mu Pk$, for $\lambda, \mu \in \mathbb{R}$. Since $\{k, (P - Q)k\}$ is linearly independent, $\mu \neq 1$. Let

$$Q_1 = \begin{bmatrix} q_{11} & q_{12} \\ q_{12}^* & q_{22} \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{bmatrix}$$

be the representations of Q_1 and A regarding the decomposition of $V = \{k\} \bigoplus \{Pk\}$. Since $Q_1 k = Q_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda k + \mu Pk = \begin{bmatrix} \lambda \\ \mu \end{bmatrix}$, it follows that $q_{11} = \lambda$ ans $q_{12} = \mu$. Similarly, since $Ak = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, it follows that $a_{11} = 0$ and $a_{12} = 1$. Hence, it is enough to consider

$$A = \begin{bmatrix} 0 & 1 \\ 1 & q_{22} \end{bmatrix}.$$

Then since $\mu \neq 1$, A is invertible and satisfies $Ak = Pk$.

Consider the operator $N \in \mathcal{SA}(\mathcal{H})$ such that regarding the decomposition of $\mathcal{H} = V \bigoplus V^{\perp}$ has the matrix representation

$$N = \begin{bmatrix} A & Q_2 \\ Q_2^* & \lambda I \end{bmatrix},$$

where $\lambda \in \mathbb{R} \setminus \sigma(Q_3)$. Since A and $Q_1 - A$ are invertible, it follows from Lemma 2.6 that N and $Q - N$ are algebraicly invertible operators. But since $(P - N)k = 0$, we conclude that $P - N \notin \mathcal{AI}_s(\mathcal{H})$, which is a contradiction. □

Now, we can prove Theorem 2.5.

Proof.(Proof of Theorem 2.5) Assume that ξ satisfies in following relation

$$\xi(Q) - \xi(P) \in \Delta(\mathcal{H}) \Longleftrightarrow Q - P \in \Delta(\mathcal{H})$$

and note that $\xi - R$ satisfies to this relation. Thus, after replacing ξ_1 by $\xi - R$, we may assume that $\xi_1(0) = 0$. We do this through a few step.

Step 1. ξ_1 is bijective and preserves adjacency of operators in both directions.

Let $Q_1, Q_2 \in \mathcal{SA}(\mathcal{H})$ be such that $\xi_1(Q_1) = \xi_1(Q_2)$. Then by assumption, for every operator we have

$$\begin{aligned} Q_1 - N \in \Delta(\mathcal{H}) \Longleftrightarrow & \quad \xi_1(Q_1) - \xi_1(N) \in \Delta(\mathcal{H}) \\ \Longleftrightarrow & \quad \xi_1(Q_2) - \xi_1(N) \in \Delta(\mathcal{H}) \\ \Longleftrightarrow & \quad Q_2 - N \in \Delta(\mathcal{H}) \end{aligned}$$

for every $N \in \mathcal{SA}(\mathcal{H})$. Therefore, it follows from Proposition 2.9 that $Q_1 = Q_2$ and consequently ξ_1 is injective. It is also bijective because it is supposed to be surjective.

Now, let $A, B \in \mathcal{SA}(\mathcal{H})$ be such that $A - B$ has rank one. Then by Lemma 2.6 it follows that there exists $R \in \mathcal{SA}(\mathcal{H}) \setminus \{A, B\}$ such that $R - B \in \Delta(\mathcal{H})$ and for every $P \in \mathcal{SA}(\mathcal{H})$, the relations $P - R \in \Delta(\mathcal{H})$ and $P - B \in \Delta(\mathcal{H})$ yield that $P - A \in \Delta(\mathcal{H})$. As ξ_1 is injective, we get $\xi_1(R) \in \mathcal{SA}(\mathcal{H}) \setminus \{\xi_1(A), \xi_1(B)\}$. By assumption $\xi_1(R) - \xi_1(B) \in \Delta(\mathcal{H})$. Suppose $Q \in \mathcal{SA}(\mathcal{H})$ such that $Q - \xi_1(R) \in \Delta(\mathcal{H})$ and $Q - \xi_1(B) \in \Delta(\mathcal{H})$. There exists $P \in \mathcal{SA}(\mathcal{H})$ that $\xi_1(P) = Q$, as ξ_1 is surjective.

Thus $\xi_1(P)-\xi_1(R)\in\Delta(\mathcal{H})$ and $\xi_1(P)-\xi_1(B)\in\Delta(\mathcal{H})$, which implies $P-R\in\Delta(\mathcal{H})$ and $P-B\in\Delta(\mathcal{H})$. Hence, we have $P-A\in\Delta(\mathcal{H})$ and consequently $Q-\xi_1(A)\in\Delta(\mathcal{H})$. Now applying Lemma 2.6 it follows that $\xi_1(A)-\xi_1(B)$ has rank one. On the other hand, as ξ_1 is bijective and ξ_1^{-1} satisfies the same properties as ξ_1, it follows that ξ_1 preserves adjacency in both directions.

Step 2. ξ_1 maps rank one operators onto rank one operators and maps $\mathcal{FS}(\mathcal{H})$ onto itself.

Let F be a rank one operators in $\mathcal{SA}(\mathcal{H})$. Then F is adjacent to 0. By step 1, $\xi_1(F)$ is adjacent to $\xi_1(0)=0$. Hence $\xi_1(F)$ is a rank one operator. On the other hand as every rank 2 operator is adjacent to a rank one operator, hence, if $\text{rank}(E)=2$, then $\text{rank}(\xi_1(E))<\infty$. Similarly, it follows that ξ_1 maps $\mathcal{FS}(\mathcal{H})$ onto itself.

Step 3. ξ_1 preserves projections of rank one and there exists either a bijective linear or conjugate-linear operator $A:\mathcal{H}\to\mathcal{H}$ such that for every $P\in\mathcal{FS}(\mathcal{H})$

$$\xi_1(P)=\lambda APA^*.$$

As $\xi_1:\mathcal{FS}(\mathcal{H})\to\mathcal{FS}(\mathcal{H})$ preserves adjacency and satisfies $\xi_1(0)=0$, it follows [22, Theorem 2.1] that either

- There exists a rank one operator $R\in\mathcal{SA}(\mathcal{H})$ such that the range of ξ_1 is contained in the linear span of R; or
- There exists an injective linear or conjugate-linear map $A:\mathcal{H}\to\mathcal{H}$ such that

$$\xi_1(\sum_{j=1}^{k} n_j y_j\otimes y_j)=\sum_{j=1}^{k} n_j A(y_j\otimes y_j)A^*$$

  for every $\sum_{j=1}^{k} n_j y_j\otimes y_j\in\mathcal{FS}(\mathcal{H})$; or
- There exists an injective linear or conjugate-linear map $A:\mathcal{H}\to\mathcal{H}$ such that

$$\xi_1(\sum_{j=1}^{k} n_j y_j\otimes y_j)=-\sum_{j=1}^{k} n_j A(y_j\otimes y_j)A^*$$

  for every $\sum_{j=1}^{k} n_j y_j\otimes y_j\in\mathcal{FS}(\mathcal{H})$.

As ξ_1 is bijective, the first case doesnot happen. Since both ξ_1 and ξ_1^{-1} have the same properties, from above discussion it follows that there exists either an invertible linear or conjugate-linear operator $A : \mathcal{H} \to \mathcal{H}$ such that for every $P \in \mathcal{FS}(\mathcal{H})$

$$\xi_1(P) = \lambda APA^*,$$

where $\lambda \in \{-1, 1\}$. Replacing ξ_1 by $P \longmapsto (\xi_1(I))^{-\frac{1}{2}}(\xi_1(P))(\xi_1(I))^{-\frac{1}{2}}$, we may assume without loss of generality that ξ_1 is unital, that is, $\xi_1(I) = I$. Note that for an arbitrary unit vector $f \in \mathcal{H}$, $I - f \otimes f \in \Delta(H)$. Hence, by assumption we should have

$$\xi_1(I) - \xi_1(f \otimes f) = I - \lambda Af \otimes Af \in \Delta(\mathcal{H}).$$

But this happens precisely when $\lambda = 1$. Now, consider an arbitrary vector $b \in \mathcal{H}$. Then

$$\begin{aligned} \langle b, b\rangle = 1 &\iff I - b \otimes b \in \Delta(\mathcal{H}) \\ &\iff I - Ab \otimes bA^* \in \Delta(\mathcal{H}) \\ &\iff \langle Ab, Ab\rangle = 1. \end{aligned}$$

Hence ξ_1 preserves projections of rank one.

By replacing ξ_1 with $\xi_2 = A^*\xi_1 A$, in the sequel we may assume $\xi_2(F) = F$, for every $F \in \mathcal{FS}(\mathcal{H})$.

Step 4. ξ_2 preserves the difference of $\mathcal{AI}_s(\mathcal{H})$ in both directions, that is, for every $Q, P \in \mathcal{SA}(\mathcal{H})$

$$Q - P \in \mathcal{AI}_s(\mathcal{H}) \iff \xi_2(Q) - \xi_2(P) \in \mathcal{AI}_s(\mathcal{H}).$$

Let $Q, P \in \mathcal{SA}(\mathcal{H})$ be such that $Q - P \in \mathcal{AI}_s(\mathcal{H})$. Then for some unit vectors $b \in \mathcal{H}$, we have

$$\langle b, (Q - P)^{-1}b\rangle = 1.$$

Set $F = b \otimes b$. It follows that $(Q - P) - F$ is not invertible. Hence $Q - (P + F) \in \Delta(\mathcal{H})$ which implies $\xi_2(Q) - \xi_2(P + F) \in \Delta(\mathcal{H})$. On the other hand, since $(P + F) - P$ is rank one then so is $\xi_2(P + F) - \xi_2(P)$. Therefore, since

$$\xi_2(Q) - \xi_2(P) = \xi_2(Q) - \xi_2(P + F) + (\xi_2(P + F) - \xi_2(P)),$$

it follows that $\xi_2(Q) - \xi_2(P) \in \mathcal{A}_s(\mathcal{H})$. But since by assumption $Q - P$ is invertible, $Q - P \notin \Delta(\mathcal{H})$, which implies $\xi_2(Q) - \xi_2(P) \notin \Delta(\mathcal{H})$. So $\xi_2(Q) - \xi_2(P) \in \mathcal{AI}_s(\mathcal{H})$. Similarly, let $Q, P \in \mathcal{SA}(\mathcal{H})$ be such that $\xi_2(Q) - \xi_2(P) \in \mathcal{AI}_s(\mathcal{H})$. Since ξ_2^{-1} satisfies the same properties as ξ_2, we conclude $Q - P \in \mathcal{AI}_s(\mathcal{H})$.
Step 5. $\xi_2(P) = P$ for every $P \in \mathcal{AI}_s(\mathcal{H})$.

Let $P \in \mathcal{AI}_s(\mathcal{H})$. Since $P - 0 \in \mathcal{AI}_s(\mathcal{H})$, it follows from step 4 that $\xi_2(P) = \xi_2(P) - \xi_2(0) \in \mathcal{AI}_s(\mathcal{H})$. If $\xi_2(P) \neq P$, then there exists a unit vector $e \in \mathcal{H}$ such that $P^{-1}e \neq \xi_2(P)^{-1}e$, $\langle e, P^{-1}e\rangle = 1$ while $\langle e, \xi_2(P)^{-1}e\rangle \neq 1$. It shows that $P - e \otimes e \notin \mathcal{AI}_s(\mathcal{H})$ but $\xi_2(P) - e\otimes e = \xi_2(P) - \xi_2(e\otimes e) \in \mathcal{AI}_s(\mathcal{H})$, there appears a contradiction. This contradiction shows that $\xi_2(P) = P$.
Step 6. $\xi_2(P) = P$ for every $P \in \Delta(\mathcal{H})$.

Let $P \in \Delta(\mathcal{H})$. Then $\xi_2(P) = \xi_2(P) - \xi_2(0) \in \Delta(\mathcal{H})$. For every $N \in \mathcal{AI}_s(\mathcal{H})$, from step 5 we have $\xi_2(N) = N$ and

$$P - N \in \mathcal{AI}_s(\mathcal{H}) \Longleftrightarrow \xi_2(P) - N \in \mathcal{AI}_s(\mathcal{H}).$$

Hence, from Proposition 2.9 it follows that $\xi_2(P) = P$.
Step 7. $\xi_2(P) = P$ for every $P \in \mathcal{SA}(\mathcal{H})$.

$\xi_2 \mid_{\Delta(\mathcal{H})}$ is additive, then the desired result follows from step 6.
Let $P_1, P_2 \in \Delta(\mathcal{H})$ be fixed and consider the map $\xi : \mathcal{SA}(\mathcal{H}) \to \mathcal{SA}(\mathcal{H})$ that for every $P \in \mathcal{SA}(\mathcal{H})$ is defined by

$$\xi(P) := \xi_2(P + P_2) - P_2.$$

Then it follows from previous steps that ξ is bijective, preserves the difference of $\Delta(\mathcal{H})$ in both directions, $\xi(I) = I$ and $\xi(F) = F$ for all finite rank operator $F \in \mathcal{SA}(\mathcal{H})$. Hence, for every $P \in \Delta(\mathcal{H})$, $\xi(P) = P$. In particular, we have

$$P_1 = \xi(P_1) = \xi_2(P_1 + P_2) - P_2,$$

that follows $\xi_2(P_1 + P_2) = P_1 + P_2$. Hence $\xi_2 \mid_{\Delta(\mathcal{H})}$ is additive.

From step 7, it follows that for every $P \in \mathcal{SA}(\mathcal{H})$, $\xi_2(P) = P$. But from this we get

$$P = \xi_2(P) = A^*\xi_1(P)A = A^*(\xi(P) - R)A.$$

Hence for every $P \in \mathcal{SA}(\mathcal{H})$

$$\xi(P) = APA^* + R,$$

which is the desired result and finishes the proof. $\square$

Proof. (Proof of Theorem 2.4) Assume that ξ is a map satisfying

$$\nu(\xi(Q) - \xi(P)) = \nu(Q - P),$$

for all $Q, P \in \mathcal{SA}(\mathcal{H})$, and note that ξ is a bijective map satisfying (8), so ξ is of the form

$$\xi(P) = APA^* + R,$$

for all operators $P \in \mathcal{SA}(\mathcal{H})$. So we have

$$\nu(AQA^* + R - APA^* - R) = \nu(Q - P) \Longrightarrow \nu(A(Q - P)A^*) = \nu(Q - P),$$

and since ν is ρ or τ, by Lemma 2.2 there is an unitary operator $E : \mathcal{H} \to \mathcal{H}$ and scalar λ such that $A = \lambda E$ and $|\lambda\lambda^*| = 1$. Thus

$$\xi(P) = \lambda EP(\lambda E)^* + R,$$

so we have

$$\xi(P) = EPE^* + R. \quad \square$$

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