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Original Research Paper

Controlled Fusion Frames in Hilbert Spaces and Their Dual

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Abstract. Controlled frames in Hilbert spaces have been introduced by Balazs, Antoine and Grybos to improve the numerical output of in relation to algorithms for inverting the frame operator. In this paper, we introduce the notion of controlled fusion frames on Hilbert spaces. It is shown that controlled fusion frames are a generalization of fusion frames giving a generalized way to obtain numerical advantage in the sense of preconditioning to check the fusion frame condition. For this end, we introduce the notion of Q-duality for Controlled fusion frames. Also, we survey the robustness of controlled fusion frames under some perturbations.

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1 Introduction

Frames, as an expansion of bases in Hilbert spaces, were first introduced by Duffn and Schaeffer during their study of nonharmonic Fourier series in 1952. They introduced frames. as an expansion of the bases in Hilbert spaces ([9]). Recently, frames play a fundamental role not only in mathematics but also in many aspects of applications and have been widely applied in filter bank theory, coding and communications, signal processing, system modeling (e.g. [2, 5, 10, 14, 15]). One of the newest generalization of frames is controlled frames. Controlled frames have been introduced to improve the numerical efficiency of interactive algorithms for inverting the frame operator on abstract Hilbert spaces (e.g. [1, 12, 13]).

This manuscript is organized as follows. In Section 2, we recall some definitions and Lemmas for frames and operators theory. In Section 3, we fix the notations of this paper, summarize known and prove some new results. In Section 4, we defined Q-duality and perturbation for controlled fusion frames and express some results about them.

Throughout this paper, H and K are separable Hilbert spaces, $\mathcal{B}(H,K)$ is the family of all the bounded linear operators on H into K and $GL(H)$ denotes the set of all bounded linear operators which have bounded inverses. Let $GL^+(H)$ be the set of all positive operators in $GL(H)$.

It is easy to check that if $C, C' \in GL(H)$, then C'^*, C'^{-1} and CC' are in $GL(H)$. We define π_W is the orthogonal projection onto W.

2 Preliminaries

In this section, some necessary definitions and lemmas are introduced.

Lemma 2.1 ([8]). *Let H_1, H_2 are Hilbert spaces and let $L_1 \in \mathcal{B}(H_1, H)$ and $L_2 \in \mathcal{B}(H_2, H)$. Then the following assertions are equivalent:*

(I) $\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2)$;

(II) $L_1 L_1^* \leq \lambda L_2 L_2^*$ *for some* $\lambda > 0$;

(III) there exists $U \in \mathcal{B}(H_1, H_2)$ *such that* $L_1 = L_2 U$.

Moreover, if those conditions are valid then there exists a unique operator U such that

(a) $\|U\|^2 = \inf\{\alpha > 0 \mid L_1L_1^* \leq \alpha L_2L_2^*\}$;

(b) $\ker L_1 = \ker U$;

(c) $\mathcal{R}(U) \subseteq \overline{\mathcal{R}(L_2^*)}$.

For the proof of the following lemma, we refer to [8].

Lemma 2.2. *Let $V \subseteq H$ be a closed subspace, and T be a linear bounded operator on H. Then*

$$\pi_V T^* = \pi_V T^* \pi_{\overline{TV}}.$$

If T is a unitary (i.e. $T^*T = Id_H$), then

$$\pi_{\overline{TV}} T = T\pi_V.$$

If an operator U has closed range, then there exists a right-inverse operator $U^\dagger$ (pseudo-inverse of U) in the following senses (see [7]).

Lemma 2.3. *Let $U \in \mathcal{B}(K, H)$ be a bounded operator with closed range $\mathcal{R}(U)$. Then there exist a bounded operator $U^\dagger \in \mathcal{B}(H, K)$ for which*

$$UU^\dagger x = x, \quad x \in \mathcal{R}(U),$$

and

$$(U^*)^\dagger = (U^\dagger)^*.$$

Definition 2.4. A sequence $\{f_i\}_{i\in I}$ in H is a frame if there exist constants $0 < A \leq B < \infty$ such that for all $f \in H$

$$A\|f\|^2 \leqslant \sum_{i\in I} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2.$$

Definition 2.5. Let $W := \{W_i\}_{i\in I}$ be a family of closed subspaces of H and $v := \{v_i\}_{i\in I}$ be a family of weights (i.e. $v_i > 0$ for any $i \in I$). We say that W is a fusion frame with respect to v for H if there exist $0 < A \leq B < \infty$ such that for each $f \in H$,

$$A\|f\|^2 \leq \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 \leq B\|f\|^2. \tag{1}$$

The constants A, B are called the fusion frame bounds. The family W is called a tight fusion frame if $A = B$, it is a Parseval fusion frame if $A = B = 1$, and v-uniform if $v = v_i = v_j$ for all $i, j \in I$. If the right-hand of the inequality (1) holds, then we say that W is a Bessel fusion sequence with Bessel fusion bound B. Moreover we say that W is an orthonormal fusion basis for H if $H = \bigoplus_{i\in I} W_i$. If W is a Bessel fusion sequence then the following operators are bounded:

$$T_W : (\sum_{i\in I} \bigoplus W_i)_{\ell^2} \longmapsto H, \qquad T_W^* : H \longmapsto (\sum_{i\in I} \bigoplus W_i)_{\ell^2},$$
$$T_W(\{c_i\}_{i\in I}) = \sum_{i\in I} v_i c_i, \qquad T_W^* f = \{v_i \pi_{W_i} f\}.$$

These operators are called synthesis operator and analysis operator, respectively. Thus, the fusion frame operator is defined by:

$$S_W : H \longmapsto H,$$
$$S_W f = T_W T_W^* f = \sum_{i\in I} v_i^2 \pi_{W_i} f.$$

For each Bessel fusion sequence W of H, we define the representation space is $(\sum \oplus W_i)_{\ell^2}$, due the notations of [6].

$$\Big(\sum \oplus W_i\Big)_{\ell^2} = \Big\{\{c_i\}_{i\in I} : \ c_i \in \mathbb{C}, \ \sum_{i\in I} |c_i|^2 < \infty\Big\}$$

with inner product given by

$$\langle \{c_i\}_{i\in I}, \{d_i\}_{i\in I}\rangle = \sum_{i\in I} \langle c_i, d_i\rangle.$$

3 Controlled Fusion Frame

In thissection, we present the notion of controlled fusion frames in Hilbert spaces and discuss on some their properties. Our approach to define controlled fusion frames is a generalization of the idea in [13].

Definition 3.1. Let $\{W_i\}_{i\in I}$ be a collection of closed subspace in Hilbert space H, $\{v_i\}_{i\in I}$ be a family of weightsand $C, C' \in GL(H)$. The sequence $W = \{(W_i, v_i)\}_{i\in I}$ is called a fusion frame controlled by C

and C' or CC'-Controlled fusion frames for H if there exist constants $0 < A \leq B < \infty$ such that for all $f \in H$

$$A\|f\|^2 \leq \sum_{i\in I} v_i^2 \langle \pi_{W_i} C' f, \pi_{W_i} C f\rangle \leq B\|f\|^2.$$

Throughout this paper, $W = \{(W_i, v_i)\}_{i\in I}$ unless otherwise stated. W is called a tight controlled fusion frame, if the constants A, B can be chosen such that $A = B$, a Parseval fusion frame provided $A = B = 1$. We call W is a C^2-Controlled fusion frame if $C = C'$. If only the second inequality is required, We call W is a Controlled Bessel fusion sequence with bound B. If W is a CC'-controlled fusion frame and $C^*\pi_{W_i}C'$ is a positive operator for each $i \in I$, then $C^*\pi_{W_i}C' = C'^*\pi_{W_i}C$ and we have

$$A\|f\|^2 \leq \sum_{i\in I} v_i^2 \|(C^*\pi_{W_i}C')^{\frac{1}{2}} f\|^2 \leq B\|f\|^2.$$

We define the controlled analysis operator by (for more details, we refer to [13])

$$\begin{aligned} &T_W : H \to \mathcal{K}_{2,W} \\ &T_W(f) = \{v_i (C^*\pi_{W_i}C')^{\frac{1}{2}} f\}_{i\in I}, \end{aligned}$$

where

$$\mathcal{K}_{2,W} := \big\{\{v_i (C^*\pi_{W_i}C')^{\frac{1}{2}} f\}_{i\in I} \ : \ f \in H\big\} \subset (\bigoplus_{i\in I} H)_{l^2}.$$

It is easy to see that $\mathcal{K}_{2,W}$ is closed and T_W is well defined. Moreover T_W is a bounded linear operator with the adjoint operator T_W^* defined by

$$\begin{aligned} &T_W^* : \mathcal{K}_{2,W} \to H \\ &T_W^* \{v_i (C^*\pi_{W_i}C')^{\frac{1}{2}} f\}_{i\in I} = \sum_{i\in I} v_i^2 C^*\pi_{W_i}C' f. \end{aligned}$$

Therefore, we define the controlled fusion frame operator S_W on H by

$$S_W f = T_W^* T_W(f) = \sum_{i\in I} v_i^2 C^*\pi_{W_i}C' f.$$

Theorem 3.2. *W is a CC'-controlled fusion Bessel sequence for H with bound B if and only if the operator*

$$T_W^*: \mathcal{K}_{2,W} \to H$$
$$T_W^*\{v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\}_{i\in I} = \sum_{i\in I} v_i^2 C^*\pi_{W_i}C'f.$$

is well -defined and bounded operator with $\|T_W^*\| \leq \sqrt{B}$.

Proof. The necessary condition follows from the definition of CC'-controlled fusion Bessel sequence. We only need to prove that the sufficient condition holds. Suppose that T_W^* is well-defined and bounded operator with $\|T_W^*\| \leq \sqrt{B}$. For any $f \in H$, we have

$$\begin{aligned}
\Big(\sum_{i\in I} v_i^2\langle \pi_{W_i}C'f, \pi_{W_i}Cf\rangle\Big)^2 &= \Big(\sum_{i\in I} v_i^2\langle C^*\pi_{W_i}C'f, f\rangle\Big)^2 \\
&= \Big(\big\langle T_W^*\{v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\}_{i\in I}, f\big\rangle\Big)^2 \\
&\leq \|T_W^*\|^2\|\{v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\}_{i\in I}\|^2\|f\|^2.
\end{aligned}$$

But

$$\|\{v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\}_{i\in I}\|_2^2 = \sum_{i\in I} v_i^2\langle \pi_{W_i}C'f, \pi_{W_i}Cf\rangle.$$

It follows that

$$\sum_{i\in I} v_i^2\langle \pi_{W_i}C'f, \pi_{W_i}Cf\rangle \leq B\|f\|^2.$$

This means that W is a CC'-controlled fusion Bessel sequence for H. □

Theorem 3.3. *W is a CC'-Controlled fusion frame for H if and only if*

$$T_W^*: \mathcal{K}_{2,W} \to H$$
$$T_W^*\{v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\}_{i\in I} = \sum_{i\in I} v_i^2 C^*\pi_{W_i}C'f$$

is a well-defined, bounded and surjective.

Proof. If W is a CC'-controlled fusion frame for H, the operator S_W is invertible. Thus, T_W^* is surjective.

Conversely, let T_W^* be a well-defined, bounded and surjective. Then, by Theorem 3.2, W is a CC'-controlled Bessel fusion sequence for H. So, $T_W(f) = \{v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\}_{i\in I}$ for all $f \in H$. Since T_W^* is surjective, by Lemma 2.3, there exists the operator $T_W^{*\dagger} : H \to \mathcal{K}_{2,W}$ such that $T_W^\dagger T_W = I_H$. Now, for each $f \in H$ we have

$$\|f\|^2 \leq \|T_W^\dagger\|^2.\|T_w f\|^2 = \|T_W^\dagger\|^2.\sum_{i\in I} v_i^2\langle \pi_{W_i}C'f, \pi_{W_i}Cf\rangle^2.$$

Therefore, W is a CC'-controlled fusion frame for H with the lower controlled fusion frame bound $\|T_W^\dagger\|^{-2}$ and the upper controlled fusion frame $\|T_W^*\|^2$. □

Theorem 3.4. *Let W be a C^2-controlled fusion frame with frame bounds A and B. If $U \in \mathcal{B}(H)$ is an invertible operator such that $U^*C = CU^*$, then $\{(UW_i, v_i)\}_{i\in I}$ is a C^2-controlled fusion frame for H.*

Proof. Let $f \in H$ and by Lemma 2.2, we have

$$\|\pi_{W_i}CU^*f\| = \|\pi_{W_i}U^*Cf\| = \|\pi_{W_i}U^*\pi_{UW_i}C^*f\| \leq \|U\|\|\pi_{UW_i}Cf\|.$$

Therefore,

$$A\|U^*f\|^2 \leq \sum_{i\in I}\|\pi_{W_i}CU^*f\|^2 \leq \|U\|^2\sum_{i\in I}\|\pi_{UW_i}Cf\|.$$

But,

$$\|f\|^2 \leq \|(U^{-1})^*U^*f\|^2 \leq \|U^{-1}\|^2\|U^*f\|^2.$$

Then,

$$A\|U^{-1}\|^{-2}\|U\|^{-2}\|f\|^2 \leq \sum_{i\in I}\|\pi_{UW_i}Cf\|.$$

On the other hand, from lemma 2.2, we obtain, with U^{-1} instead of T:

$$\pi_{UW_i} = \pi_{UW_i}(U^*)^{-1}\pi_{W_i}U^*.$$

Thus,

$$\|\pi_{UW_i}Cf\| \leq \|U^{-1}\|\|\pi_{W_j}U^*Cf\|,$$

and it follows

$$\begin{aligned}\sum_{i\in I} v_i^2\|\pi_{UW_i}Cf\|^2 &\leq \|U^{-1}\|^2\sum_{i\in I} v_i^2\|\pi_{W_i}U^*Cf\|^2\\ &= \|U^{-1}\|^2\sum_{i\in I} v_i^2\|\pi_{W_i}CU^*f\|^2\\ &\leq B\|U^{-1}\|^2\|U\|^2\|f\|^2.\end{aligned}$$

□

Theorem 3.5. *Let $W = \{(W_i, v_i)\}_{i\in I}$ be a C^2-controlled fusion frame with frame bounds A and B. If $U \in \mathcal{B}(H)$ is an invertible and unitary operator such that $UC = CU$, then $\{(UW_i, v_i)\}_{i\in I}$ is a C^2-controlled fusion frame for H.*

Proof. Using Lemma 2.2, we have for any $f \in H$,

$$\begin{aligned}A\|f\|^2 &\leq \|U\|^2\|U^{-1}f\|^2\\ &\leq \|U\|^2\sum_{i\in I} v_i^2\|\pi_{W_i}U^{-1}Cf\|^2\\ &\leq \|U\|^2\sum_{i\in I} v_i^2\|U^{-1}\pi_{UW_i}Cf\|^2\\ &\leq \|U\|^2\|U^{-1}\|^2\sum_{i\in I} v_i^2\|\pi_{UW_i}Cf\|^2,\end{aligned}$$

and we obtain

$$\sum_{i\in I} v_i^2\|\pi_{UW_i}Cf\|^2 \geq \frac{A}{\|U\|^2\|U^{-1}\|^2}\|f\|^2.$$

On the other hand, from Lemma 2.2, we obtain

$$\begin{aligned}\sum_{i\in I} v_i^2\|\pi_{UW_i}Cf\|^2 &\leq \|U\|^2\sum_{i\in I} v_i^2\|\pi_{W_i}U^{-1}Cf\|^2\\ &= \|U\|^2\sum_{i\in I} v_i^2\|\pi_{W_i}CU^{-1}f\|^2\\ &\leq B\|U^{-1}\|^2\|U\|^2\|f\|^2.\end{aligned}$$

□

Theorem 3.6. *Let $W := \{(W_i, v_i)\}_{i\in I}$ and $Z := \{(Z_i, v_i)\}_{i\in I}$ be two CC'-controlled fusion Bessel sequence for H with bounds B_1 and B_2, respectively. Suppose that T_W^* and T_Z^* be their controlled synthesis operators for W and Z. Let a bounded linear operator $U : (\sum \oplus z_i)_{\ell^2} \to (\sum \oplus w_i)_{\ell^2}$ exist such that $T_w^* U T_z = I_H$. Then, both W and Z are CC'-controlled fusion frames for H.*

Proof. For each $f \in H$, we have

$$\begin{aligned}
\|f\|^4 &= |\langle f, f\rangle|^2 \\
&= |\langle UT_z f, T_w f\rangle|^2 \\
&\leq \|U\|^2 \cdot \|T_z f\|^2 \cdot \|T_w f\|^2 \\
&= \|U\|^2 \cdot \left(\sum_{i\in I} \langle \pi_{z_i} c' f, \pi_{z_i} cd\rangle\right) \left(\sum_{i\in I} v_i^2 \langle \pi_{w_i} c' f, \pi_{w_i} cf\rangle\right) \\
&\leq \|U\|^2 \cdot B_2 \cdot \|f\|^2 \cdot \left(\sum_{i\in I} v_i^2 \langle \pi_{w_i} c' f, \pi_{w_i} cf\rangle\right).
\end{aligned}$$

Thus,

$$\|U\|^{-2} \cdot B_2^{-1} \leq \left(\sum_{i\in I} v_i^2 \langle \pi_{w_i} c' f, \pi_{w_i} cf\rangle\right),$$

and W is a CC'-controlled fusion frame for H. Similarly, Z is a CC'-controlled fusion frame with the lower bound $\|U\|^{-2} \cdot B_1^{-1}$. □

Theorem 3.7. *Let $W = \{(W_i, v_i)\}_{i\in I}$ and $Z = \{(Z_i, v_i)\}_{i\in I}$ be CC'-Controlled Bessel fusion sequences for H with boundesB_1 and B_2. Suppose that T_z be the analysis and T_w^* be synthesis operators for Z and W. Let there exist $0 < \epsilon < 1$ and a bounded linear operators $U : \left(\sum_{i\in I} \oplus Z_i\right) \to \left(\sum_{i\in I} \oplus W_i\right)$ such that*

$$\|f - T_w^* U T_z f\| \leq \epsilon \|f\|.$$

Then W and Z are $CC'-$controlled fusion frames for H.

Proof. For each $f \in H$, we have

$$\|T_W^* U T_Z f\| \geq \|f\| - \|f - T_W^* U T_Z f\| \geq (1 - \epsilon)\|f\|.$$

Therefore

$$\begin{aligned}(1-\epsilon)\|f\| \leq \|T_W^* U T_Z f\| &= \sup_{\|g\|=1} |\langle T_W^* U T_Z f, g\rangle| \\ &= \sup_{\|g\|=1} |\langle U T_Z f, T_W g\rangle| \\ &\leq \sup_{\|g\|=1} \|U\| \cdot \|T_Z f\| \cdot \|T_W g\| \\ &\leq \|U\| \cdot \sqrt{B_1} (\sum_{i\in I} v_i^2 \langle \pi_{Z_i} C' f, \pi_{Z_i} C f\rangle)^{\frac{1}{2}},\end{aligned}$$

where B_1 is a controlled Bessel bound for W. Hence,

$$\frac{(1-\epsilon)^2}{B} . \|f\|^2 \leq (\sum_{i\in I} v_i^2 \langle \pi_{Z_i} C' f, \pi_{Z_i} C f\rangle).$$

Therefore, W is a CC'-controlled fusion frame for H. Similarly, we can show that Z is also a CC'-Controlled fusion frame for H. $\square$

4 $Q-$Dual and Perturbation on Controlled Fusion Frames

In this section, we introduce the duality of CC'-controlled fusion frames and we characterize their fundamental properties. Finally, perturbation of CC'-controlled frames will be discussed.

Definition 4.1. Assume that W is a CC'-controlled fusion frame for H. We call a CC'-controlled fusion Bessel sequence as $\widetilde{W} := \{(\widetilde{W}_i, z_i)\}_{i\in I}$ a Q-dual CC'-controlled fusion frame of W, if there exists a bounded linear operator $Q : \mathcal{K}_{2,\widetilde{W}} \longrightarrow \mathcal{K}_{2,W}$ such that

$$T_W^* Q T_{\widetilde{W}} = CC'.$$

Theorem 4.2. *Let $\widetilde{W}$ be a Q-dual CC'-controlled fusion frame for W and $Q : \mathcal{K}_{2,W} \longrightarrow \mathcal{K}_{2,\widetilde{W}}$. Then, the following conditions are equivalent.*

1. $T_W^* Q T_{\widetilde{W}} = CC'$;

2. $T^*_{\widetilde{W}}Q^*T_W = C'^*C^*$;

3. $\langle C'f, C^*g\rangle = \langle QT_{\widetilde{W}}f, T_Wg\rangle = \langle T_{\widetilde{W}}f, Q^*T_Wg\rangle$ *for all* $f, g \in H$.

Proof. Straightforward. □

Theorem 4.3. *If* $\widetilde{W} = \{(\widetilde{W}_i, z_i)\}_{i\in I}$ *is a* Q*-dual for* $W = \{(W_i, v_i)\}_{i\in I}$*, then* $\widetilde{W}$ *is a* CC'*-controlled fusion frame for* H*.*

Proof. Let $f \in H$ and by definition (4.1), W is a $CC'-$controlled fusion frame for H and we suppose that B is the upper bound of W. Therefore

$$\begin{aligned}\|f\|^4 &= |\langle f, f\rangle|^2\\ &= |\langle C'f, C^*(C^*)^{-1}(C'^*)^{-1}f\rangle|^2\\ &= |\langle QT_{\widetilde{W}}f, T_W(C^*)^{-1}(C'^*)^{-1}f\rangle|\\ &\leq \|T_{\widetilde{W}}f\|^2\|Q\|^2\|T_W\|^2\|C^{-1}\|^2\|C'^{-1}\|^2\|f\|^2\\ &\leq \|T_{\widetilde{W}}f\|^2\|Q\|^2B\|C^{-1}\|^2\|C'^{-1}\|^2\|f\|^2.\end{aligned}$$

Hence,

$$B^{-1}\|C^{-1}\|^{-2}\|C'^{-1}\|^{-2}\|Q\|^{-2}\|f\|^2 \leq \sum_{i\in I} z_i^2\langle \pi_{\widetilde{W}_i}C'f, \pi_{\widetilde{W}_i}Cf\rangle^2,$$

and this completes the proof. □

Corollary 4.4. *If* E_{op} *and* F_{op} *are the optimal bounds of* $\widetilde{W}$*, then*

$$E_{op} \geq B_{op}^{-1}\|Q\|^{-2}\|C^{-1}\|^{-2}\|C'^{-1}\|^{-2} \quad \text{and} \quad F_{op} \geq A_{op}^{-1}\|Q\|^{-2}\|C^{-1}\|^{-2}\|C'^{-1}\|^{-2}$$

where A_{op} *and* B_{op} *are the optimal bounds of* W*, respectively.*

Consider a C^2-controlled fusion frame $W = \{(W_i, v_i)\}_{i\in I}$ for H. Applying Lemma 2.1, there exists an operator $X \in \mathcal{B}(H, \mathcal{K}_{2,W})$ such that $T^*_W X = I$. We denote the i-th component of Xf by $X_if = (Xf)_i$ and clearly $X_i \in \mathcal{B}(H, C^*(W_i))$. In the last result we note that these operators construct some Q-dual for W.

Theorem 4.5. *Let* $W := \{(W_i, v_i)\}_{i\in I}$ *be a* CC'*-controlled fusion frame, and* X *be an operator such that* $T^*_W X = I$*. Suppose that* $\widetilde{W} := \{(\widetilde{W}_i, v_i)\}_{i\in I}$*, where* $\widetilde{W}_i = C^*X_i^*C^*(W_i)$*, is a* CC'*-controlled fusion Bessel sequence. Then* $\widetilde{W}$ *is a* Q*-dual* CC'*-controlled fusion frame for* W*.*

Proof. Define the mapping

$$U_0 : R(T_{\widetilde{W}}) \to \mathcal{K}_{2,W},$$
$$U_0(T_{\widetilde{W}} f) = XCC'f.$$

Then U_0 is well-defined and bounded. Indeed, if $T_{\widetilde{W}} f_1 = T_{\widetilde{W}} f_2$, so

$$C^* \pi_{\widetilde{W}_i} C'(f_1 - f_2) = 0,$$

for any $i \in I$. Therefore,

$$C'(f_1 - f_2) \in (\widetilde{W_i})^\perp = \mathcal{R}(C^* X_i^*)^\perp = \ker(X_i C),$$

and hence $XCC'(f_1 - f_2) = 0$. Moreover,

$$\begin{aligned}
\|U_0\{C^* \pi_{\widetilde{W}_i} C')^{\frac{1}{2}} f\}\|^2 &= \|XCC'f\|^2 \\
&= \sum_{i\in I} \|\pi_{C^* W_i} X_i CC'f\|^2 \\
&= \sum_{i\in I} \|\pi_{C^* W_i} X_i C \pi_{C^* X_i^* C^* W_i} C'f\|^2 \\
&\leq \|X\|^2 \sum_{i\in I} \|\pi_{\widetilde{W}_i} C'f\|^2 \cdot \|C\|^2 \\
&= \|X\|^2 \sum_{i\in I} \|C^{*-1} C^* \pi_{\widetilde{W}_i} C'f\|^2 \cdot \|C\|^2 \\
&\leq \|X\|^2 \|C^{-1}\|^2 \sum_{i\in I} \|(C^* \pi_{\widetilde{W}} C')^{\frac{1}{2}} (C^* \pi_{\widetilde{W}} C')^{\frac{1}{2}} f\|^2 \cdot \|C\|^2 \\
&\leq \|X\|^2 \|C^{-1}\|^2 \|\|C\|\|C'\| \sum_{i\in I} \|(C^* \pi_{\widetilde{W}_i} C)^{\frac{1}{2}} f\|^2 \cdot \|C\|^2 \\
&= \|X\|^2 \|C^{-1}\|^2 \|\|C\|\|C'\|\|\{C^* \pi_{\widetilde{W}_i} C')^{\frac{1}{2}} f\}\|^2 \cdot \|C\|^2.
\end{aligned}$$

Assume that,

$$U = \begin{cases} U_0, & \text{on } \overline{R(T_{\widetilde{W}})}, \\ 0, & \text{on } \overline{R(T_{\widetilde{W}})}^\perp. \end{cases}$$

Hence, U is well-defined and bounded. If we ler $Q = U$, then we have $Q \in \mathcal{B}(\mathcal{K}_{2,\widetilde{W}}, \mathcal{K}_{2,W})$ and

$$T_W^* Q T_{\widetilde{W}} = T_W^* XCC' = CC'.$$

□

Example 4.6. Let $H = \mathbb{R}^3$ with the standard orthonormal basis $\{e_1, e_2, e_3\}$. We define

$$W_1 = \operatorname{span}\{e_1, e_2\}, \quad W_2 = \operatorname{span}\{e_2, e_3\}, \quad W_3 = \operatorname{span}\{e_3\},$$

and

$$C(x_1, x_2, x_3) = (ax_1, bx_2, cx_3), \qquad C'(x_1, x_2, x_3) = (\alpha x_1, \beta x_2, \gamma x_3),$$

where $a, b, c, \alpha, \beta, \gamma > 0$. It is easy to check that $C, C' \in GL^+(\mathbb{R}^3)$ and also $W := \{(W_i, 1)\}_{i=1,2,3}$ is a CC'-controlled fusion frame with bounds

$$\min\{a\alpha, 2b\beta, 2c\gamma\}, \quad \max\{a\alpha, 2b\beta, 2c\gamma\}.$$

It is obvious that

$$\mathcal{K}_{2,W} = \{(\sqrt{a\alpha}x_1, \sqrt{b\beta}x_2, 0), (0, \sqrt{b\beta}x_2, \sqrt{c\gamma}x_3), (0, 0, \sqrt{c\gamma}x_3), \quad (x_1, x_2, x_3) \in \mathbb{R}^3\}.$$

Hence, we can get

$$Xf = \{(x_1, \frac{\sqrt{2}}{2}x_2, 0), (0, \frac{\sqrt{2}}{2}x_2, \frac{\sqrt{2}}{2}x_3), (0, 0, \frac{\sqrt{2}}{2}x_3)\},$$

for each $f = (x_1, x_2, x_3) \in \mathbb{R}^3$ and it is clear that all of X_i are adjoint. Now, by Theorem 4.5, if we define

$$\widetilde{W}_1 = \{a^2 e_1, \frac{\sqrt{2}}{2}b^2 e_2\}, \quad \widetilde{W}_2 = \{\frac{\sqrt{2}}{2}b^2 e_2, \frac{\sqrt{2}}{2}c^2 e_3\}, \quad \widetilde{W}_1 = \{\frac{\sqrt{2}}{2}c^2 e_3\},$$

then, $\widetilde{W} := \{(\widetilde{W}_i, 1)\}_{i=1,2,3}$ is a Q-dual CC'-controlled fusion frame for W. We notice that $\widetilde{W}$ is a CC'-controlled fusion frame with bounds

$$\min\{a^5\alpha, \sqrt{2}b^5\beta, \sqrt{2}c^5\gamma\}, \quad \max\{a^5\alpha, \sqrt{2}b^5\beta, \sqrt{2}c^5\gamma\}.$$

Perturbation of frames have been discussed by Cazassa and Christensen in [4]. In this part, we aim to present it for controlled fusion frames.

Definition 4.7. Let $W := \{(W_i, v_i)\}_{i\in I}$ and $Z := \{(Z_i, v_i)\}_{i\in I}$ be CC'-controlled fusion frame for H where $C, C' \in GL(H)$ and $0 \leq \lambda_1, \lambda_2 < 1$

be real numbers. Suppose that $\beta := \{c_i\}_{i\in I} \in \ell^2(I)$ is a positive sequence of real numbers. If

$$\|v_i(C^*\pi_{W_i}C' - C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|_2 \leq \lambda_1\|v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\|_2 + \lambda_2\|v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|_2 + \\ + \|\beta\|_2\|f\|,$$

then, we say that $Z := \{(Z_i, v_i)\}_{i\in I}$ is a $(\lambda_1, \lambda_2, \beta, C, C')$-perturbation of $W = \{(W_i, v_i)\}_{i\in I}$.

Theorem 4.8. *Let $W := \{(W_i, v_i)\}_{i\in I}$ be a CC'-controlled fusion frame for H with frame bounds A, B, and $Z := \{(Z_i, v_i)\}_{i\in I}$ be a $(\lambda_1, \lambda_2, \beta, C, C')$-perturbation of $W := \{(W_i, v_i)\}_{i\in I}$. Then $Z := \{(Z_i, v_i)\}_{i\in I}$ is a CC'-controlled fusion frame for H with bounds:*

$$(\frac{(1-\lambda_1)\sqrt{A} - \|\beta\|_2}{1+\lambda_2})^2 \quad , \quad (\frac{(1+\lambda_1)\sqrt{B} + \|\beta\|_2}{1-\lambda_2})^2$$

Proof. Let $f \in H$. We have

$$\begin{aligned}\|v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|_2 &= \|v_i(C^*\pi_{Z_i}C' - C^*\pi_{W_i}C')^{\frac{1}{2}}f + v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\|_2 \\ &\leq \|v_i(C^*\pi_{Z_i}C' - C^*\pi_{W_i}C')^{\frac{1}{2}}f\|_2 + \|v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\|_2 \\ &\leq \lambda_1\|v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\|_2 + \lambda_2\|v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|_2 + \\ &\quad + \|\beta\|_2\|f\| + \|v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\|_2.\end{aligned}$$

Hence,

$$(1-\lambda_2)\|(v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|_2 \leq (1+\lambda_1)\|v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\|_2 + \|\beta\|_2\|f\|.$$

Since W is a CC'-controlled fusion frame with bounds A and B, then

$$\|v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\|^2 = \sum_{i\in I} v_i^2\langle \pi_{W_i}C'f, \pi_{W_i}Cf\rangle \leq B\|f\|^2.$$

So,

$$\begin{aligned}\|v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|_2 &\leq \frac{(1+\lambda_1)\|v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\|_2 + \|\beta\|_2\|f\|}{1-\lambda_2} \\ &\leq (\frac{(1+\lambda_1)\sqrt{B} + \|\beta\|_2}{1-\lambda_2}\|f\|).\end{aligned}$$

Thus

$$\sum_{i\in I} v_i^2\langle \pi_{Z_i}C'f, \pi_{Z_i}Cf\rangle = \|v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|_2^2 \leq (\frac{(1+\lambda_1)\sqrt{B}+\|\beta\|_2}{1-\lambda_2}\|f\|)^2.$$

Now, for the lower bound, we have

$$\begin{aligned}\|v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|_2 &= \|v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f - v_i(C^*\pi_{W_i}C' - C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|_2\\ &\geq \|v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\|_2 - \|v_i(C^*\pi_{Z_i}C' - C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|_2\\ &\geq \|v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\|_2 - \lambda_1\|v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\|_2\\ &\quad - \lambda_2\|v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|_2 - \|\beta\|_2\|f\|.\end{aligned}$$

Therefore,

$$(1+\lambda_2)\|v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|_2 \geq (1-\lambda_1)\|v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\|_2 - \|\beta\|_2\|f\|,$$

or

$$\|v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|_2 \geq \frac{(1-\lambda_1)\|v_i(C^*\pi_{W_i}C')f\|_2 - \|\beta\|_2\|f\|}{1+\lambda_2}.$$

Thus, we get

$$\|v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\|^2 = \sum_{i\in I} v_i^2\langle \pi_{W_i}C'f, \pi_{W_i}Cf\rangle \geq A\|f\|^2.$$

So,

$$\|v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|_2 \geq (\frac{(1-\lambda_1)\sqrt{A}-\|\beta\|_2}{1+\lambda_2}\|f\|).$$

Thus,

$$\begin{aligned}\sum_{i\in I} v_i^2\langle \pi_{Z_i}C'f, \pi_{Z_i}Cf\rangle &= \|v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|_2^2\\ &\geq (\frac{(1-\lambda_1)\sqrt{A}-\|\beta\|_2}{1+\lambda_2}\|f\|)^2\end{aligned}$$

and the proof is completed. □

Theorem 4.9. *Let W be a CC'-controlled fusion frame with bounds A, B for H. Also, let $Z := \{Z_i\}_{i\in I}$ be a family of closed subspaces in H and*

$$\|v_i(C^*\pi_{W_i}C' - C^*\pi_{Z_i}C')^{\frac{1}{2}}f\| \leq \epsilon\|f\|,$$

for some $0 < \epsilon < \sqrt{A}$. Then $Z := \{(Z_i, v_i)\}_{i\in I}$ is a CC'-controlled fusion frame with bounds $(A - \epsilon^2)$ and $(B + \epsilon^2)$.

Proof. For every $f \in H$, we can write

$$\begin{aligned}\|v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|^2 &\leq \|v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\|^2 + \|v_i(C^*\pi_{W_i}C' - C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|^2 \\ &\leq (B + \epsilon^2)\|f\|^2\end{aligned}$$

Thus,

$$\sum_{i\in I} v_i^2\langle \pi_{Z_i}C'f, \pi_{Z_i}Cf\rangle = \|v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|^2 \leq (B + \epsilon^2)\|f\|^2.$$

Therefore, $Z := \{(Z_i, z_i)\}_{i\in I}$ is a Controlled Bessel fusion sequence. On the other hand

$$\begin{aligned}\|v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|^2 &\geq \|v_i(C^*\pi_{W_i}C')^{\frac{1}{2}}f\|^2 - \|v_i(C^*\pi_{W_i}C' - C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|^2 \\ &\geq (A - \epsilon^2)\|f\|^2.\end{aligned}$$

Hence,

$$\sum_{i\in I} v_i^2\langle \pi_{Z_i}C'f, \pi_{Z_i}Cf\rangle = \|v_i(C^*\pi_{Z_i}C')^{\frac{1}{2}}f\|^2 \geq (A - \epsilon^2)\|f\|^2$$

and the proof is completed. □

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