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Nonsmooth Pseudolinearity

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Abstract. In this paper, we introduce the notion of generalized pseudolinearity for nondifferentiable and nonconvex but locally Lipschitz functions defined on a Banach space. We present some characterizations of generalized pseudolinear functions. The characterizations of the solution set of a convex and nondifferentiable but generalized pseudolinear program are obtained. The results of this paper extend various results for pseudolinear functions, pseudoinvex functions and pseudolinear functions, and also for pseudoinvex programs, pseudolinear programs and pseudolinear programs.

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1. Introduction

A vector $\xi \in X^*$ is said to be a proximal subgradient of f at $x \in K$, if $(\xi, -1) \in N^P_{epif}(x, f(x))$, where $epif = \{(x, \alpha) : \alpha \geqslant f(x)\}$. The set of all proximal subgradient vectors of f at x is denoted by $\partial_P f(x)$. A vector $\xi \in X^*$ is a limiting subdifferential vector of f at $x \in K$, if there exist two sequences $\xi_i \in X^*$ and $x_i \in X$ such that $\xi_i \in \partial_P f(x_i), \xi_i \to \xi, x_i \to x$ and $f(x_i) \to f(x)$. The set of all limiting subdifferential vectors of f at x is denoted by $\partial_L f(x)$. The notion of the limiting subdifferential was

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first introduced, in the equivalent form, in [10]. One of the classes of functions whose set of limiting subdifferentials is nonempty is the class of locally Lipschitz functions. Considering this class, the following results are obtained and known in nonsmooth analysis (see [3,6,7]).

2. Preliminaries

Definition 2.1. *A Banach space X is an Asplund, or it has the Asplund property, if every convex continuous function $\varphi : U \to \mathbb{R}$ defined on an open convex subset U of X is Fréchet differentiable on a dense subset of U.*

Remark 2.2. *One of the most popular Asplund spaces is any reflexive Banach space [6].*

Theorem 2.3. [6] *Let X be a Asplund space and $\varphi : U \to \mathbb{R}$ proper and lower semicontinuous around $\bar{x} \in dom\varphi$, then*

$$\partial_L \varphi(\bar{x}) = \limsup_{x \to \bar{x}} \partial_F \varphi(x).$$

It is well known that

$$\partial_F f(x) \subseteq \partial_L f(x) \subseteq \partial_C f(x) \subseteq \partial_{CR} f(x).$$

Theorem 2.4. *Let f be locally Lipschitz at $x \in K$, then $\partial_L f(x)$ is closed. In fact, if $x_i \to x$, $\xi_i \in \partial_L f(x_i)$, and $\xi_i \to \xi$, then $\xi \in \partial_L f(x)$.*

Theorem 2.5. *f is locally Lipschitz, then the set of all limiting subdifferential vectors of f is uniformly bounded.*

Theorem 2.6.([7]) *Let f be locally Lipschitz on an open set containing $[x, y]$. Then*

$$f(y) - f(x) \leqslant \langle x^*, y - x \rangle$$

for some $c \in [x, y), x^* \in \partial_L f(c)$.

Throughout this paper, we suppose that $K \subseteq X$ be a nonempty set and $f : K \to \mathbb{R}$ be a function.

3. Characterization of Pseudolinear Functions

Definition 3.1. *The function f is said to be pseudoconvex if, for each $x, y \in K$ with $\langle \xi, y - x \rangle \geqslant 0$ for some $\xi \in \partial_L f(x)$, we have $f(y) \geqslant f(x)$.*

Definition 3.2. *The function f is said to be quasiinvex with respect to η if for each $x, y \in K$ and each $t \in [0,1]$, we have $f(x + t(y - x)) \leqslant \max(f(x), f(y))$. The function f is said to be strictly quasiinvex with respect to η if for each $x, y \in K$ and each $t \in [0,1]$, with $f(x) \neq f(y)$ we have $f(x + t(y - x)) < \max(f(x), f(y))$.*

A function f is said to be pseudoconcave on K if $-f$ is pseudoconvex.

Definition 3.3. *Let K be convex set. A function f is said to be pseudolinear If f is both pseudoconvex and pseudoconcave on K.*

Example 3.4. Let $X = \mathbb{R}$, $K = (-1, 1)$ and $f : X \to \mathbb{R}$ be defined as

$$f(x) := \begin{cases} 0 & \text{if } \ x < 0; \\ \sqrt{x} & \text{if } \ x \geqslant 0. \end{cases}$$

The limiting subdifferential mapping of f is

$$\partial_L f(x) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } \ x > 0; \\ [0, \infty) & \text{if } \ x = 0; \\ 0 & \text{if } \ x < 0. \end{cases}$$

Then we can see easily that the continuous (not locally Lipschitz) function f is pseudoconvex and quasiconvex.

Theorem 3.5. *Let X be an Asplund space and f be locally Lipschitz on $K \subset X$, an open convex set. If the function f is a pseudolinear. Then for all $x, y \in K$, $f(x) = f(y)$ if and only if there exists $\xi \in \partial_L f(x)$, such that $\langle \xi, y - x \rangle = 0$.*

Proof. Suppose that there exists $\xi \in \partial_L f(x)$, such that $\langle \xi, y - x \rangle = 0$. By the definition pseudoconvex function, we have:

$$\langle \xi, y - x \rangle \leqslant 0 \Rightarrow f(y) \leqslant f(x). \tag{1}$$

From the definition of pseudoincave function, we get

$$\langle \xi, y - x \rangle \geqslant 0 \Rightarrow f(y) \geqslant f(x). \tag{2}$$

Then, from (1) and (2), we obtain

$$\langle \xi, y - x \rangle = 0 \Rightarrow f(x) = f(y).$$

Conversely, For any $x, y \in K$, suppose that $f(x) = f(y)$. We need to prove that there exists $\zeta \in \partial_L f(y)$, such that $\langle \zeta, x - y) \rangle = 0$. We first show that for any $t \in (0, 1)$,

$$f(y + t(x - y)) = f(y). \tag{3}$$

If $f(y + t(x - y)) > f(y)$, by the pseudoconvexity of f, for any $\zeta \in \partial_L f(y + t(x - y))$, we have

$$\langle \zeta, -t(x - y) \rangle < 0.$$

Therefore,

$$\langle \zeta, (1 - t)(x - y) \rangle > 0.$$

By the pseudoconvexity of f, we have $f(x) \geqslant f(y + t(x - y))$ which contradicts to $f(y + t(x - y)) > f(y) = f(x)$.
Similarly, we can get that $f(y+t(x-y)) < f(y)$ leads to a contradiction.
From (3), for any $t \in (0, 1)$,

$$f(y + t(x - y)) - f(y) = 0.$$

By assumption of the theorem, f is Lipschitz on open set K containing $[y, y + t(x - y)]$ and hence there exists $x(t) \in [y, y + t(x - y)]$ and $\zeta_t \in f(x(t))$ by Theorem 2.4, such that

$$0 = f(y + t(x - y)) - f(y) \leqslant \langle \zeta_t, t(x - y) \rangle \tag{4}$$

and hence for $\zeta_t \in \partial_L f(x(t))$ we have

$$\langle \zeta_t, t(x - y) \rangle \geqslant 0,$$

Therefore

$$\langle \zeta_t, x - y \rangle \geqslant 0.$$

Since $\partial_L f$ is locally bounded at y, hence, there exists a neighborhood of y and a constant $k > 0$ such that for each z in this neighborhood and $\xi \in \partial_L f(z)$ we have $\|\xi\| \leqslant k$. Since $x(t) \to y$ when $t \to 0^+$, thus for $t > 0$ small enough $\|\zeta_t\| \leqslant k$. Without loss of generality, we may assume that $\zeta_t \to \zeta$ in w*-topology. Since the set-valued mapping $z \mapsto \partial_L f(z)$ has a closed graph, thus for each $x \in K$ there exists $\zeta \in \partial_L f(y)$ such that

$$\langle \zeta, x - y \rangle \geqslant 0. \tag{5}$$

Since $-\zeta_t \in \partial_L(-f)(x(t))$, from (4) and Theorem 2.4, for $(-f)$, we have

$$0 = (-f)(y + t(x - y) - (-f)(y) \leqslant \langle -\zeta_t, t(x - y)\rangle$$

and then for for some $\zeta_t \in \partial_L f(x(t))$ we have

$$\langle -\zeta_t, t(x - y)\rangle \geqslant 0,$$

that is

$$\langle \zeta, (x - y)\rangle \leqslant 0. \tag{6}$$

Hence, by (5) and (6), we obtain $\langle \zeta, x - y \rangle = 0$. □

Remark 3.6. *Obviously, Theorem 3.5, is a generalization of Proposition 1 in [1].*

Example 3.7. Consider $K = (-1, 1) \subseteq \mathbb{R}$. It is clear that K is convex set. Let $f : K \to \mathbb{R}$ be a function defined by

$$f(x) := \begin{cases} 0 & \text{if } x \in (-1, 0], \\ (-1/2)x & \text{if } x \in (0, 1). \end{cases}$$

Then f is locally Lipschitz with constant 1. Then, we claim that for all $x, y \in K$, we have $f(x) = f(y)$ if and only if there exists $\xi \in \partial_L f(x)$ such that $\langle \xi, x - y \rangle = 0$.

$$\partial_L f(0) = \{-1/2, 0\}$$

if $x = 0$ and $y \in (0, 1)$, thus for each $\xi \in \partial_L f(0)$ we have

$$\langle \xi, y \rangle \in [-1/2, 0]$$

but $f(y) < f(0)$, therefore f is not pseudoconvex and so f is not psedolinear.

Theorem 3.8. *Let K be an open convex set. Then f is $\eta-$pseudolinear on K if and only if there exists a function $p : K \times K \to \mathbb{R}^+$such that for all $x, y \in K$, there exists $\xi \in \partial_L f(x)$, such that*

$$f(y) = f(x) + p(x, y)\langle \xi, y - x\rangle. \tag{7}$$

Proof. Let f be pseudolinear. We have to construct a function $p : K \times K \to \mathbb{R}^+$ such that for all $x, y \in K$, for some $\xi \in \partial_L f(x)$,
$f(x) = f(y) + p(x, y)\langle \xi, y - x\rangle$.
If $\langle \xi, y - x\rangle = 0$ for any $x, y \in K$ and for some $\xi \in \partial_L f(x)$ we define $p(x, y) = 1$. By Theorem 3.5, we have $f(y) = f(x)$, thus (3.7) holds.
If $\langle \xi, y - x\rangle \neq 0$ for any $x, y \in K$ for some $\xi \in \partial_L f(x)$, we define

$$p(x; y) = \frac{f(y) - f(x)}{\langle \xi, y - x\rangle}. \tag{8}$$

Then we have to show that $p(x, y) > 0$. If $f(y) > f(x)$, then by the pseudoconvexity of -f, we have $\langle \xi, y-x\rangle > 0$ for some $\xi \in \partial_L f(x)$. From (8), we get $p(x, y) > 0$. Similarly, if $f(y) < f(x)$, we can get $p(x, y) > 0$ by using pseudoconvexity of f.

Conversely, suppose that for any $x, y \in K$, there exist a function $p : K \times K \to \mathbb{R}^+$ such that (7) holds for all $x, y \in K$ and for some $\xi \in \partial_L f(x)$.
If $\langle \xi, y - x\rangle \geqslant 0$, then from (7), we have

$$f(y) - f(x) = p(x, y)\langle \xi, y - x\rangle \geqslant 0.$$

Hence f is pseudoconvex. Likewise, if $\langle \xi, y - x\rangle \leqslant 0$, we can prove that f is pseudoconcave. Hence, f is pseudolinear. □

Remark 3.9. *It is clear that Theorem 3.8, is a generalization of Proposition 2 in [1].*

Lemma 3.10. *Let K be convex set. f is locally Lipschitz and pseudoconvex. Then f is quasiconvex on K.*

Proof. Suppose that f is pseudoconvex. Assume that f is not quasiconvex. Then, there exist $x, y \in K$ such that $f(x) \leqslant f(y)$ and a $t_0 \in (0,1)$, such that

$$f(x) \leqslant f(y) < f(x_0). \tag{9}$$

where $x_0 = y + t_0(x-y)$. Let $\varphi(t) = f(y + t(x-y))$. Since f is locally Lipschitz function,$\varphi(t)$ is continuous function. It follows that $\varphi(t)$ attains its maximum. From (9) and $x_0 = y + t_0(x-y) \in K$, we have $\varphi(0) = f(y) < f(x_0)$. So $t = 0$ is not a maximum point. By (9), we have $f(x) = \varphi(1) \leqslant \max\{f(x), f(y)\} = f(y) < f(x_0)$ which leads to $t = 1$ is not a maximum point. Hence, there exits $t^* \in (0,1)$ such that $f(y^*) = \max_{t\in[0,1]} f(y + t(x-y))$, where $y^* = y + t^*(x-y)$. Thus, we have $0 \in \partial_L f(y^*)$. Now $\xi = 0$, we can obtain that $\langle \xi, x - y^* \rangle = 0$. Since f is pseudoconvex, we obtain $f(x) \geqslant f(y^*)$ which contradicts (9). The proof is complete. $\square$

Remark 3.11. *It is clear that Lemma 3.10, is a generalization of Lemma 4.1, in [12].*

Theorem 3.12. *Let K be convex. If f is pseudolinear. Then for any $x, y \in K$, there exists $\zeta \in \partial_L f(y)$, such that*

$$\langle \zeta, x - y \rangle = 0 \Rightarrow f(z_t) = f(y) = f(x),$$

where $z_t = y + t(x-y), t \in [0,1]$.

Proof. For any $x, y \in K$ and $t \in [0,1]$, let $z_t = y + t(x-y)$. Suppose that there exists $\zeta \in \partial_L f(y)$, such that

$$\langle \zeta, \eta(x,y) \rangle = 0. \tag{10}$$

Since f is pseudoconvex, we have

$$f(x) \geqslant f(y). \tag{11}$$

By the quasiconvexity of f and Lemma 3.10, we get

$$f(z_t) \leqslant \max\{f(x), f(y)\} = f(x). \tag{12}$$

Since $-f$ is pseudoconvex and from (10), we have

$$f(x) \leqslant f(y). \tag{13}$$

Again by the quasiconvexity of $-f$ and Lemma 3.10, we get

$$f(z_t) \geqslant \min\{f(x), f(y)\} = f(x). \tag{14}$$

From (12) and (14), we can get $f(z_t) = f(x)$. From (13) and (14), we can obtain $f(y) = f(x)$. The proof is completed. $\square$

Remark 3.13. *It is clear that Theorem 3.12, is a generalization of Theorem 3.3, in [13].*

4. Characterizations of the Solution Sets of Optimization Problems

Consider the following optimization problem:

$$(VOP) \qquad \min f(x) \quad \text{subject to } x \in K,$$

where $K \subseteq X$ is a nonempty convex set and $f : K \to \mathbb{R}$. Throughout this section, we assume that the solution set

$$S = \arg\min_{x \in K} f(x)$$

of the (VOP) is nonempty.

Proposition 4.1. *The solution set S of (OP) is convex with respect to if $f : K \to \mathbb{R}$ is locally Lipschitz and pseudolinear.*

Proof. Suppose that $x_1, x_2 \in S$. Then $f(x_1) \leqslant f(y)$ and $f(x_2) \leqslant f(y)$ for all $y \in K$. By pseudolinearity of f for some $\xi_1 \in \partial_L f(x_1)$,$\langle \xi_1, x_2 -$

$x_1\rangle = 0$. Since $\langle \xi_1, x_1 - x_2\rangle = 0$, and so, $-t\langle \xi_1, x_1 - x_2\rangle = 0$ for all $t \in [0,1]$. Therefore, we get

$$\langle \xi_1, t(x_2 - x_1)\rangle = -t\langle \xi_1, x_1 - x_2\rangle = 0.$$

Since f is pseudolinear, by Theorem 3.12, $f(x_1 + t(x_2 - x_1)) = f(x_1)$, and therefore, $x_1 + t(x_2 - x_1)$ is also a solution of (VOP), and thus, the solution set of optimization problem is convex. □

We give some characterizations of the solution set of a pseudoconvex program in terms of any of its solutions.

Theorem 4.2. *Suppose K is an convex set. Let $f : X \to \mathbb{R}$ be a non-differentiable, locally Locally Lipschitz pseudolinear and $\bar{x} \in S$. Then $S = S_1 = S_2$, where*

$$S_1 = \{x \in K : \langle \xi, \bar{x} - x\rangle = 0 \text{ for some } \xi \in \partial_L f(x)\}$$

$$S_2 = \{x \in K : \langle \xi, \bar{x} - x\rangle \geqslant 0 \text{ for some } \zeta \in \partial_L f(\bar{x})\}.$$

Proof. The point $x \in S$ if and only if $f(x) = f(\bar{x})$. Then from Theorem 3.1, we have $f(x) = f(\bar{x})$ if and only if $\langle \xi, \bar{x} - x\rangle = 0$ for some $\xi \in \partial_L f(x)$. Also, $f(\bar{x}) = f(x)$ if and only if $\langle \zeta, x - \bar{x}\rangle = 0$ for some $\zeta \in \partial_L f(\bar{x})$. The latter is equivalent to $\langle \zeta, \bar{x} - x\rangle = 0$ for some $\zeta \in \partial_L f(\bar{x})$.
□

Corollary 4.3. *Let K, f be the same as in Theorem 4.2 and let $\bar{x} \in S$. Then $S = S_3 = S_4$, where*

$$S_3 = \{x \in K : \langle \xi, \bar{x} - x\rangle \geqslant 0 \text{ for some } \xi \in \partial_L f(x)\}$$

$$S_4 = \{x \in K : \langle \zeta, \bar{x} - x\rangle \geqslant 0 \text{ for some } \zeta \in \partial_L f(\bar{x})\}.$$

Proof. It is clear from Theorem 4.2 that $S \subset S_3$. We prove that $S_3 \subset S$. Assume that $x \in S_3$, that is, $x \in K$ such that $\langle \xi, \bar{x} - x\rangle \geqslant 0$ for some $\xi \in \partial_L f(x)$. In view of Theorem 3.8, there exists a real-valued function p defined on $K \times K$ such that $p(x, y) > 0$ and

$$f(\bar{x}) = f(x) + p(x, \bar{x})\langle \xi, \bar{x} - x\rangle \geqslant f(x).$$

This implies that $x \in S$, and hence, $S_3 \subset S$. Similarly, we can prove that $S = S_4$. $\square$

Remark 4.4. *Corollary 4.3, improves and generalizes Theorem 1, in [5] and Theorem 3.1, in [11] and [4].*

Theorem 4.5. *Let $K \subseteq X$ be an convex set. Let f be nondifferentiable and locally Lipschitz on K. If $f : K \to \mathbb{R}$ is pseudolinear and $\bar{x} \in S$, then $S = S_5 = S_7$, where*

$$S_5 = \{x \in K : \langle \zeta, x - \bar{x} \rangle = \langle \xi, \bar{x} - x \rangle \text{ for some } \xi \in \partial_L f(x),\ \zeta \in \partial_L f(\bar{x})\}, \tag{15}$$

$$S_7 = \{x \in K : \langle \zeta, x - \bar{x} \rangle \leqslant \langle \xi, \bar{x} - x \rangle \text{ for some } \xi \in \partial_L f(x),\ \zeta \in \partial_L f(\bar{x})\}. \tag{16}$$

Proof. Let $x \in S$. Then by Theorem 4.2, for some $\xi \in \partial_L f(x)$ and some $\zeta \in \partial_L f(\bar{x})$,

$$\langle \xi, \bar{x} - x \rangle = 0 = \langle \zeta, \bar{x} - x \rangle. \tag{17}$$

Since we have

$$\langle \xi, x - \bar{x} \rangle = 0 = \langle \zeta, \bar{x} - x \rangle. \tag{18}$$

Thus $x \in S_5$, and hence, $S \subset S_5$. $S_5 \subset S_6$ is obvious.
We now prove that $S_6 \subset S$. Assume that $x \in S_6$. Then for some $\xi \in \partial_L f(x)$ and some $\zeta \in \partial_L f(\bar{x})$,

$$\langle \zeta, \bar{x} - x \rangle \geqslant \langle \xi, (x - \bar{x} \rangle. \tag{19}$$

Suppose that $x \notin S$. Then $f(\bar{x}) < f(x)$. By the pseudoconcavity of f, we have

$$\langle \zeta, x - \bar{x} \rangle > 0.$$

It follows that

$$\langle \zeta, \bar{x} - x \rangle < 0.$$

By using (19), we have

$$\langle \xi, x - \bar{x} \rangle < 0 \quad \text{or} \quad \langle \xi, \bar{x} - x \rangle > 0.$$

By Theorem 3.8, there exists exists a function p defined on $K \times K$ such that $p(x, \bar{x}) > 0$ and for some $\xi_0 \in \partial_L f(x)$,

$$f(\bar{x}) = f(x) + p(x, \bar{x})\langle \xi_0, \bar{x} - x\rangle > f(x),$$

which contradict with of the fact that $f(\bar{x}) < f(x)$. Hence $x \in S$. $\square$

Theorem 4.6. *Let K be an open convex set. If f is pseudolinear on K and $\bar{x} \in S$. Then there exist $\xi \in \partial_L f(x), \zeta \in \partial_L f(\bar{x}), \xi_t \in \partial_L f(z_t)$,* where $z_t = x + t(\bar{x} - x)$, $t \in [0, 1]$,

$$S_1 = \{x \in K : \langle \xi, \bar{x} - x\rangle = 0\},$$

$$S_2 = \{x \in K : \langle \zeta, (x - \bar{x}\rangle = 0\},$$

$$S_3 = \{x \in K : \langle \xi_t, \bar{x} - x\rangle = 0\},$$

such that $S = S_1 = S_2 = S_3$.

Proof. $x \in S$ if and only if $f(x) = f(\bar{x})$. From Theorem 3.5, we have $x \in S$ if and only if there exists $\xi \in \partial_L f(x)$, such that $\langle \xi, \bar{x} - x\rangle = 0$.

Then, we have $S = S_1$. Next, we prove that $S_1 = S_2$. For any $x \in S_1$, there exists $\xi \in \partial_L f(x)$ such that $\langle \xi, \bar{x} - x\rangle = 0$. By Theorem 3.5, we can obtain $f(x) = f(\bar{x})$. Again by Theorem 3.5, there exists $\zeta \in \partial_L f(\bar{x})$ such that $\langle \zeta, x - \bar{x}\rangle = 0$. Thus, $x \in S_2$ and $S_1 \subset S_2$. By the similar method we have $S_2 \subset S_1$. Hence $S_1 = S_2$. Now, we prove that $S = S_3$. For any $x \in S, f(x) = f(\bar{x})$, by Theorem 3.5, we can get $\xi \in \partial_L f(x)$ such that $\langle \xi, \bar{x} - x\rangle = 0$. By Theorem 3.12, for any $t \in (0, 1], z_t = x + t(\bar{x} - x), f(z_t) = f(x)$. By Theorem 3.5, there exists $\xi_t \in \partial_L f(z_t)$ such that

$$\langle \xi_t, \bar{x} - x\rangle = 0. \tag{20}$$

We have

$$x - z_t = -t(\bar{x} - x). \tag{21}$$

From (20) and (21), $\langle \xi_t, \bar{x} - x\rangle = 0$. Thus, $x \in S_3$ i.e., $S \subset S_3$.
Let $x \in S_3$, then for any $t \in (0; 1], z_t = x + t(\bar{x} - x)$, there exists $\xi_t \in \partial_L f(z_t)$ such that

$$\langle \xi_t, \bar{x} - x \rangle = 0. \tag{22}$$

From (21) and (22), we can obtain that

$$\langle \xi_t, x - z_t \rangle = 0. \tag{23}$$

From (23) and Theorem 3.12, we have $f(x) = f(z_t)$. By Theorem 3.5, it follows that there exists $\xi \in \partial_L f(x)$, such that

$$\langle \xi, z_t - x \rangle = 0. \tag{24}$$

Since

$$z_t - x = t(\bar{x} - x). \tag{25}$$

From (24) and (25), it follows that $\langle \xi, \bar{x} - x \rangle = 0$. Therefore, by Theorem 3.5, we have $f(x) = f(\bar{x})$ and $S_3 \subset S$. The proof is completed. $\square$

Remark 4.7. *Theorem 4.5, extends Theorem 2 [1], to nondifferentiable and generalized pesudolinear functions. Hence Theorems 4.2 and 4.5, extend the results of Jeyakumar and Yang [4].*

Theorem 4.8. *Let K be an open convex set. If f is pseudolinear on K and $\bar{x} \in S$. Then, there exist $\xi \in \partial_L f(x), \zeta \in \partial_L f(\bar{x}), \xi_t \in \partial_L f(z_t)$,* where $z_t = x + t(\bar{x} - x)$, $t \in [0, 1]$,

$$S_4 = \{x \in K : \langle \xi, \bar{x} - x \rangle \geqslant 0\},$$

$$S_5 = \{x \in K : \langle \zeta, x - \bar{x} \rangle \geqslant 0\},$$

$$S_6 = \{x \in K : \langle \xi_t, \bar{x} - x) \rangle \geqslant 0\},$$

such that $S = S_4 = S_5 = S_6$.
The proof can obtain by some modifications from the proof of Theorem 4.6, hence it is omitted.

Remark 4.9. *Theorem 4.5, improves Corollary 3.1, in [4] and Corollary 4.2, in [13].*

Theorem 4.10. *Let K be an open convex set on K. If f is pseudolinear on K and $\bar{x} \in S$. Then $S = S_7 = S_8$, where*

$$S_7 = \{x \in K : \exists \xi \in \partial_L f(x), \zeta \in \partial_L f(y), \langle \xi, \bar{x} - x \rangle = \langle \zeta, x - y \rangle\},$$

$$S_8 = \{x \in K : \exists \xi \in \partial_L f(x), \zeta \in \partial_L f(\bar{x}), \langle \xi, \bar{x} - x \rangle \geqslant \langle \zeta, x - \bar{x} \rangle\}.$$

Proof. For any $x \in S, f(x) = f(\bar{x})$. By Theorem 3.5, there exist $\xi \in \partial_L f(x), \zeta \in \partial_L f(y)$ such that

$$\langle \xi, \bar{x} - x \rangle = 0 \tag{26}$$

$$\langle \zeta, x - \bar{x} \rangle = 0. \tag{27}$$

It implies $x \in S_7$ and $S \subset S_7$.
$S_7 \subset S_8$ is obvious.

Now, we prove that $S_8 \subset S$. For any $x \in S_8$, there exist $\xi \in \partial_L f(x)$ and $\zeta \in \partial_L f(\bar{x})$ such that

$$\langle \xi, \bar{x} - x \rangle \geqslant \langle \zeta, x - \bar{x} \rangle \tag{28}$$

Suppose that $x \notin S$, we get that $f(\bar{x}) < f(x)$. By the pseudoconvexity of $-f$, we have

$$\langle \zeta, x - \bar{x} \rangle > 0. \tag{29}$$

From (28) and (29), it follows that

$$\langle \xi, \bar{x} - x \rangle > 0. \tag{30}$$

By Theorem 3.8, there exists $\xi_0 \in \partial_L f(x)$, such that

$$f(\bar{x}) = f(x) + p(y, x)\langle \xi_0, \bar{x} - x \rangle < f(x).$$

Therefore,

$$\langle \xi_0, \bar{x} - x \rangle < 0. \tag{31}$$

From (30) and (31), we can get that there exists $\xi_{00} \in \partial_L f(x)$, such that $\langle \xi_{00}, \bar{x} - x \rangle = 0$. Again by Theorem 3.5, it follows that $f(x) = f(\bar{x})$, which is a contradiction. $\square$

Remark 4.11. *Theorem 4.6, generalizes Theorem 3.2, in [4] and Theorem 4.3, in [13].*

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