Journal of Mathematical Extension
Vol. 16, No. 4, (2022) (5)1-51
URL: https://doi.org/10.30495/JME.2022.1467
ISSN: 1735-8299
Original Research Paper

Cost Efficiency Estimation in Network DEA in the Case of Varying Input Prices

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Abstract. Data envelopment analysis (DEA) evaluates the cost efficiency of systems with multiple inputs and multiple outputs in two scenarios, one where the input prices are the same in all decision-making units (DMUs) (competitive space), and the other where the input prices differ from one DMU to another (non-competitive space). In many situations, the DMUs could have a multi-stage network structure with intermediate measures. Although a model has been presented for calculating the network cost efficiency in a competitive space in such cases, no method has been proposed so far to calculate the cost efficiency of such networks in a non-competitive space. The present article focuses on the concept of cost efficiency in DEA for DMUs with network structures and varying input prices. To this end, a cost-based production possibility set (PPS) is first introduced for the network systems and their sub-systems, and after calculating the network cost efficiency, it is decomposed into efficiencies such as technical, price, and allocative efficiencies, and the reasons behind the occurrence of extra costs due to various types of inefficiency are explained.

AMS Subject Classification: MSC code1; MSC code 2; more

Received: November 2019; Accepted: October 2020
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Keywords and Phrases: Network data envelopment analysis, cost efficiency, varying prices

1 Introduction

In conventional DEA approaches, only inputs and outputs are used to evaluate DMU performances, and hence, the internal structure of the DMU is ignored. Consequently, the impact that the efficiency of the internal stages has on the overall efficiency cannot be properly analyzed and evaluated. Therefore, it is necessary to use a network structure, as this way, the internal structure is taken into account, which will introduce systems with interdependent components, meaning that the outputs of some components can be used as inputs in other components.

Fare and Grosskopf [5, 6] and Fare et al. [7] developed several network DEA models. They calculated the efficiency of each stage independent of the overall efficiency within the network DEA framework. Tones and Tsutsui [23] introduced a slacks-based network DEA model called NSBM, through which the overall efficiency of the DMUs, in addition to the efficiency of each component, could be evaluated. Using a series and a parallel structure, Kao [12] presented a number of models for evaluating DMU networks, which were defined based on a multiplication of component efficiencies (component-wise multiplication). Chen et al. [7] obtained the overall efficiency using a weighted sum of the component efficiencies, and also outlined the relationship between their model and that of Kao. Similar to Chen et al. [3], Zhu [27] obtained the overall efficiency of the DMUs using individual component efficiencies. Liang et al. [14] extended the two-stage network DEA models by applying the game theory.

Lozano [15] introduced the production possibility sets of each stage and system individually. In addition, Lozano [15] also presented a model for finding the technical network efficiency of systems. However, this model is not able to determine the technical efficiency of each individual stage. Furthermore, Lozano [16] evaluated the overall network efficiency in a case where the stages produce both desirable and undesirable outputs, and using a simple slacks-based linear programming model, obtained the inefficiency of the overall system and its stages.

Using an axiomatic approach for the overall network structure, Boloori [2] resolved the issue of efficient objectives and interpreted the multiplier form of the dual of that model. Kao and Huang [13] determined the Malmquist productivity index for systems with two-stage processes. Wang et al. [25] discussed the decomposition weights and overall efficiency in a two-stage DEA model with shared resources. Amirteimoori et al. [1] proposed an additive model for two-stage processes with variable (flexible) intermediate measures and shared inputs. Wanke et al. [26] presented a dynamic network DEA model in 2019.

When some pricing information is available, the concepts of cost, revenue, and profit efficiency come into play.The concept of cost efficiency was first introduced by Farrell et al. [9]. Fare et al. [8] extended Farrell's concept of cost efficiency, and by presenting a linear programming model, were able to calculate the cost efficiency. Comanho and Dyson [4] were able to find the upper and lower bounds of cost efficiency in cases with imprecise input prices. Jahanshahloo et al. [11] focused on a commonly used cost efficiency model. In this respect, they reduced the number of restrictions (limitations) and variables in the model, which resulted in a considerable reduction in computational operations. Mostafaee and Saljooghi [18] presented a model for evaluating the upper and lower bounds of the cost efficiency value in cases with a chance of imprecise input and output data. Tone [24] showed that varying DMU prices would yield incorrect results with regard to cost efficiency and using Farrell's cost efficiency may cause problems. To overcome this limitation, Tone [24] abandoned the idea of evaluation with fixed prices and proposed evaluating the DMUs in a cost space. Sahoo et al. [22] presented a non-parametric measure of economic efficiency in non-competitive spaces with unknown prices. Puri and Yadaf [20] discussed cost and revenue efficiency in fuzzy spaces where the input and output data and their respective prices are not precisely known. Ghiyasi [10] conducted a study on cost and revenue efficiency in inverse DEA. Mozaffari et al. [19] used the DEA-R models in cost efficiency measurement. By introducing a production possibility set in which the DMUs are evaluated based on their own prices and those of other DMUs in a non-competitive space, Fallahnejad et al. [21] presented a novel method for evaluating the cost, revenue, and profit efficiencies in a non-competitive space.

Since there are units in the real world that have a network structure, the present article discusses the cost efficiency of such decision-making units both in a competitive (input prices are the same for all DMUs) and a non-competitive space (input prices differ from one DMU to another). In this regard, Lozano [15] obtained the cost efficiency for DMUs with a network structure in cases where the input prices are the same in all units and called it the network cost efficiency. However, there is a flaw to Lozano's cost efficiency model in cases where the DMUs have varying input prices (non-competitive space). In this respect, if in a non-competitive space, among DMUs with network structures, there are two DMUs with the same inputs and outputs where the input price of one unit is twice that of the other unit, Lozano's method would produce the same cost efficiency for both DMUs. Due to this reason, after introducing Lozano's network cost efficiency in Section 2 of the current article, an attempt is made to resolve the issue with this model in Section 3. In this respect, putting aside the assumption of identical prices among all units, the units are evaluated in a cost space in which different DMUs have different input prices. We will define the cost-based production possibility set for each component and for the entire system separately. The DMUs are evaluated in a cost space where the network cost efficiency of the system is decomposed into technical, price, and allocative efficiencies. A numerical example is provided in Section 4, and finally, Section 5 presents the conclusion.

2 Current Network Cost Efficiency

Let n be the number of DMU_js that we have. Each DMU_j consists of $k(k=1,\ldots,K)$ components. The input and output matrices are defined as $X^K=(x_1^k,x_2^k,\ \ldots,\ x_n^k)$ and $Y^K=(y_1^k,y_2^k,\ \ldots,\ y_n^k)$, respectively, where $x_j^k=(x_{1j}^k,\ x_{2j}^k,\ \ldots,\ x_{m_kj}^k)$, $x_j^k\in R^{m_k}$, $y_j^k=(y_{1j}^k,\ y_{2j}^k,\ \ldots,\ y_{r_kj}^k)$, $y_j^k\in R^{r_k}$, m_k, and r_k indicate the number of inputs and outputs in each component, respectively. Also, x_{ij}^k is the value of the i^{th} observed exogenous input that is consumed by the k^{th} component of DMU_j, and y_{rj}^k is the value of the r^{th} observed final output that is produced by the k^{th} component of DMU_j. $x_{ij}=\sum_{k\in P_I(i)}^K x_{ij}^k$ (assuming that $P_I(i)$ is a set consisting of components that consume the i^{th} exogenous input) is the sum of the values of the i^{th} exogenous input consumed by every component in DMU_j. In a similar way, $y_{rj} = \sum_{k \in P_O(r)}^{K} y_{rj}^k$ (assuming that $P_0(r)$ is a set consisting of components that produce the r^{th} final output) is the sum of the values of the r^{th} final output produced by every component in DMU_j.

In a network system, there are intermediate products g that are produced and consumed inside the system. Note that these intermediate products are different from exogenous inputs and final outputs. Let z_{gj}^k, $k \in R^{\text{out}}(k)$ (assuming that $R^{\text{out}}(k)$ is a set consisting of components that produce intermediate products) be the value of the intermediate product produced by the system and used as the input for another component. Also, assume that z_{gj}^k, $k \in R^{\text{in}}(k)$ (assuming that $R^{\text{in}}(k)$ is a set consisting of components that consume the intermediate product g) is the value of the intermediate product that is consumed by the k^{th} component and is the output of another component. It is assumed that the sum of the g^{th} intermediate products produced by the k^{th} component for use in other components $(\sum_{k \in R^{\text{in}}(k)} z_{gj}^k)$ is equal to the sum of the g^{th} intermediate products produced by other components and used in the k^{th} component $(\sum_{k \in R^{\text{out}}(k)} z_{gj}^k)$.

The real input cost of DMU_o, which is indicated by C_o, can be expressed as follows:

$$C_o = \sum_{i=1}^{m} c_{io} x_{io} = \sum_{i=1}^{m} c_{io}^k \sum_{k \in P_I(i)}^{K} x_{io}^k \tag{1}$$

where c_{io}^k is the price of the i^{th} input used in the k^{th} component of DMU_0. To calculate the technical efficiency of the network, we need a network DEA model.

Lozano defines the PPS for the component k as follows [15]:

$$T_k = \left\{ \begin{array}{c} \left(x_i^k, y_r^k, z_g^k\right): \ \exists \lambda_j^k \in \Lambda \quad \forall j \ x_i^k \geq \sum_j \lambda_j^k x_{ij}^k \qquad \forall i \in I(k) \\ , y_r^k \leq \sum_j \lambda_j^k y_{rj}^k \quad \forall r \in O(k) \\ z_g^k \geq \sum_j \lambda_j^k z_{gj}^k \ \forall g \in R^{\text{in}}(k) \quad z_g^k \leq \sum_j \lambda_j^k z_{gj}^k \ \forall g \in R^{\text{out}}(k) \end{array} \right\}$$

$I(k)$ and $O(k)$ are the sets of exogenous inputs and final outputs in the component k, respectively.

The system's PPS, which is a combination of component-wise PPSs, can be defined as follows:

$$T=\left\{\begin{array}{c}(x_i,y_r):\ \exists\left(x_i^k,y_r^k,z_g^k\right)\in T_k\quad\forall k\ \forall j\qquad x_i\geq\sum_{k\in p_I(i)}x_{ij}^k\quad\forall i\\ y_r\leq\sum_{k\in p_O(r)}y_{rj}^k\quad\forall r\\ \sum_{k\in R^{\text{out}}(k)}z_{gj}^k\ -\sum_{k\in R^{\text{in}}(k)}z_{gj}^k\ \geq 0\quad\forall g\end{array}\right\}$$

Based on the definition of T, Model (2) is presented for the technical efficiency of the network as follows:

$$\begin{aligned}\theta^{CCR*}=\min\quad&\theta^{CCR}\\ &\sum_{k\in p_I(i)}\sum_j\lambda_j^k x_{ij}^k\leq\theta^{CCR}x_{io}\quad\forall i\\ &\sum_{k\in p_O(r)}\sum_j\lambda_j^k y_{rj}^k\geq y_{ro}\quad\forall r\\ &\sum_{k\in R^{\text{out}}(k)}\sum_j\lambda_j^k z_{gj}^k-\sum_{k\in R^{\text{in}}(k)}\sum_j\lambda_j^k z_{gj}^k\geq 0\quad\forall g\\ &\lambda_j^k\geq 0,\quad\theta\text{ free}\end{aligned}\tag{2}$$

The third Constraint of this model indicates that the amount of produced intermediate products is at least as much as the amount of consumed intermediate products. The target operation points for each component in the PPS can be obtained as follows:

$$\begin{cases}\hat{x}_i^k=\sum_j\lambda_j^{k*}x_{ij}^k & \forall i\in I\left(k\right)\\ \hat{y}_i^k=\sum_j\lambda_j^{k*}y_{rj}^k & \forall r\in O\left(k\right)\\ \hat{z}_i^k=\sum_j\lambda_j^{k*}z_{gj}^k & \forall g\in R^{\text{in}}\left(k\right)\cup R^{\text{out}}\left(k\right)\end{cases}\tag{3}$$

By solving Model (2) and Eq. (3), technically efficient input and output targets can be obtained for the system or the observed DMU(x_0,y_0).

$$\begin{cases}x_i^*=\ \sum_{k\in p_I(i)}\hat{x}_i^k\leq x_{io}=\sum_{k\in p_I(i)}x_{io}^k & \forall i\\ y_r^*=\ \sum_{k\in p_O(r)}\hat{y}_i^k\geq y_{ro}=\sum_{k\in p_O(r)}y_{ro}^k & \forall r\end{cases}\tag{4}$$

The technically efficient input cost for DMU_o in T is denoted by C_o^{CRS*} and calculated as follows:

$$C_o^{CRS*}=\sum_{i=1}^m c_{io}{x^*}_{io}=\sum_{i=1}^m c_{io}^k\sum_{k\in p_I(i)}^K\hat{x}_{io}^k$$

$$= \sum_{i=1}^{m} c_{io}^{k} \sum_{k \in p_I(i)}^{K} \sum_{j} \lambda_j^{k^*} x_{ij}^{k} \quad \leq \sum_{i=1}^{m} c_{io}^{k} \sum_{k \in p_I(i)}^{K} x_{io}^{k} = \ C_o \tag{5}$$

Based on Eq. (5), the amount of loss in input costs, which is due to technical inefficiency and denoted by $L_o^* = C_o - C_o^{CRS*}$, is always non-negative.

Now, if the Constraint $\sum_j \lambda_j^k = 1, \ \forall \ k$ is added to Model (2), a model will be resulted that can find the pure technical efficiency of the network, where C_o^{VRS*} is the cost of the pure-technically efficient input in DMU_0.

When the input prices are available, a question that arises is that how and to what degree can we use the inputs to achieve the minimum cost.

With the assumption that C is a specific and shared vector of input prices, Lozano [15] determined the minimum total cost of DMU_o by solving the following model:

$$\begin{array}{ll} \min & \sum\limits_i c_i x_i \\ & \sum\limits_{k \in p_I(i)} \sum\limits_j \lambda_j^k x_{ij}^k \leq x_i \quad \forall i \\ & \sum\limits_{k \in p_O(r)} \sum\limits_j \lambda_j^k y_{rj}^k \geq y_{ro} \quad \forall r \\ & \sum\limits_{k \in R^{\text{out}}(k)} \sum\limits_j \lambda_j^k z_{gj}^k - \sum\limits_{k \in R^{\text{in}}(k)} \sum\limits_j \lambda_j^k z_{gj}^k \geq 0 \quad \forall g \\ & \lambda_j^k \geq 0, \quad x_i \geq 0 \end{array} \tag{6}$$

In Model (6), λ_j^k and x_i are variables, and using the optimal solution of Model (6), we will have the following cost efficient target operation point:

$$\begin{cases} \widehat{x}_i = \ \sum_{k \in p_I(i)} \widehat{\widehat{x}}_i^k = \sum_k \sum_j \lambda_j^{k^{**}} x_{ij}^k & \forall i \\ \widehat{y}_r = \ \sum_{k \in p_O(r)} \widehat{\widehat{y}}_r^k = \sum_k \sum_j \lambda_j^{k^{**}} y_{rj}^k & \forall r \end{cases} \tag{7}$$

The variable $\lambda_j^{k^{**}}$ in (7) is obtained by solving Model (6).

The network cost efficiency can be obtained as follows:

$$C_o^* = \frac{\sum_i c_i \widehat{\widehat{x}}_i}{\sum_i c_i x_{io}} \tag{8}$$

$\sum_i c_i\widehat{\hat{x}}_i$ in the numerator of the fraction (8) indicates the minimum cost. The value of the fraction $\frac{\sum c_i\widehat{\hat{x}}_i}{\sum c_i x_{io}}$ is always less than or equal to 1. Since the denominator of the fraction is strictly positive and its numerator is always non-negative, the cost efficiency will be a real number between 0 and 1. If the unit under evaluation spends the minimum cost to produce the output, then the value of the fraction, or in other words, the cost efficiency of that unit will be equal to 1. Cost efficiency scores smaller than 1 indicate that the cost of the inputs can be reduced. The closer the efficiency score is to 1, the more efficient the unit will be.

Now, by taking into account the output price vector P, which is the same for all units, we intend to calculate the revenue efficiency for the network units. Therefore, it is necessary to obtain the maximum total revenue of DMU$_o$ through the following model:

$$\begin{aligned} \max \quad & \sum_r p_r y_r \\ & \sum_{k\in p_I(i)} \sum_j \lambda_j^k x_{ij}^k \le x_{io} \quad \forall i \\ & \sum_{k\in p_O(r)} \sum_j \lambda_j^k y_{rj}^k \ge y_r \quad \forall r \\ & \sum_{k\in R^{\text{out}}(k)} \sum_j \lambda_j^k z_{gj}^k - \sum_{k\in R^{\text{in}}(k)} \sum_j \lambda_j^k z_{gj}^k \ge 0 \quad \forall g \\ & \lambda_j^k \ge 0, \quad y_r \ge 0 \end{aligned} \tag{9}$$

λ_j^k and y_r are the variables of Model (9). Using the optimal solution of Model (9), we will have the following network revenue efficient target operation point:

$$\begin{cases} x_i^{**} = \sum_{k\in p_I(i)} x_i^{k^{**}} = \sum_k \sum_j \lambda_j^{k^{***}} x_{ij}^k \quad \forall i \\ y_r^{**} = \sum_{k\in p_O(r)} y_r^{k^{**}} = \sum_k \sum_j \lambda_j^{k^{***}} y_{rj}^k \quad \forall r \end{cases} \tag{10}$$

$\lambda_j^{k^{***}}$ in (10) is the optimal solution of Model (9). The real total revenue of DMU$_o$ (unit with a network structure that is under evaluation) is calculated as follows:

$$E_o = \sum_{r=1}^{s} p_{ro} y_{ro} = \sum_{i=1}^{m} p_{ro}^k \sum_{k\in p_O(r)}^{K} y_{ro}^k \tag{11}$$

Using Eq. (10), the network revenue efficiency is defined as follows:

$$E_o^* = \frac{\sum_r p_r y_{ro}}{\sum_r p_r y_r^{**}} \tag{12}$$

In the fraction (12), $\sum p_r y_r^{**}$ (the denominator) indicates the maximum revenue. Furthermore, $\sum_r p_r y_{ro}$ is the observed real total revenue of DMU_0.

E_o^* ($0 < E_o^* \leq 1$) will never equal zero as $\sum p_r y_{ro}>0$ (the numerator will never equal zero). Moreover, the network unit under evaluation will be revenue efficient if and only if $E_o^* = 1$.

Here, it will be demonstrated though an example that the method proposed by Lozano [15] does not work properly when finding the network cost efficiency. We will use a numerical example adopted from Liu and Wang [17] to find the network cost efficiency C_o^*. The input prices are considered as (C_1, C_2, C_3), which are the same for all units. Table 1 includes 17 DMUs. Each unit consists of two components, the first of which has 3 exogenous inputs and the second one includes 1 final output. Also, each unit has two intermediate products as presented in Columns 4 and 5 of Table 1. The first component produces the intermediate products as outputs, and the second component consumes them as inputs.

Using Model (6) and taking into account the input price vector $C^1 = (C_1, C_2, C_3) = (500, 1, 1)$, we calculate the cost efficiency of the network DMUs, which is denoted by $C_{o_1}^*$ in Column 2 of Table 2.

Next, we obtain the cost efficiency of each network unit again, this time for the input price vector $C^2 = (1000, 2, 2)$, which is indicated by $C_{o_2}^*$.

By comparing Columns 2 and 3 of Table 2, it can be found that despite doubling the input price vector, the cost efficiency has remained unchanged as $C_{o_2}^*$ is equal to $C_{o_1}^*$

It can be deduced that the cost efficiency model proposed by Lozano [15] may fail to yield a proper estimation of the network cost efficiency in cases where the price information varies from one network DMU to another. Thus, the next section presents a method to resolve this ambiguity.

Table 1: Inputs, intermediate products, and output of the PCB-producing companies

DMU	X_1	X_2	X_3	Z_1	Z_2	Y_1
1	4183	756090	146092	8408000	15440539	512057
2	3000	401059	48629	4543000	9283600	263359
3	2715	465372	77507	2995980	8650485	688227
4	1893	858696	207128	11663363	9535196	855515
5	4578	1065000	331238	15318200	20817313	3072695
6	2134	781780	74154	8888590	11891722	805816
7	1059	261071	32324	5034254	3213303	83753
8	937	325130	78685	2346822	2857752	24067
9	701	190321	62251	2121270	2621901	163756
10	418	74445	13173	1728000	1190986	158142
11	582	92077	12805	4620185	1971958	105173
12	380	65696	7691	3472150	1342532	52973
13	2190	576821	68126	5588146	6812709	226023
14	523	79801	3673	1954550	1038792	8139
15	373	89923	6321	881038	1836709	200129
16	383	70581	5432	2134779	1040018	49248
17	736	97700	9356	2937134	1751369	147000

Table 2: Lozano's network technical efficiency and network cost efficiency with identical input prices for all units

DMU	$C^*_{o_1}$	$C^*_{o_2}$
1	0.1843	0.1843
2	0.1456	0.1456
3	0.3903	0.3903
4	0.4582	0.4582
5	0.8986	0.8986
6	0.4516	0.4516
7	0.1097	0.1097
8	0.0297	0.0297
9	0.2926	0.2926
10	0.5746	0.5746
11	0.2863	0.2863
12	0.2168	0.2168
13	0.1400	0.1400
14	0.0254	0.0254
15	0.7628	0.7628
16	0.1984	0.1984
17	0.3335	0.3335

3 Proposed Method for Network Cost Efficiency Estimation in DMUs with Varying Input Price Vectors

The previous section discussed the network cost efficiency proposed by Lozano [15] in a competitive space. In this section, we evaluate the network cost efficiency in cases where the DMUs have varying input prices (non-competitive space). Since the units have network structures, to evaluate them in a non-competitive space, we need to define new production possibility sets for each component and each system individually.

Based on Tone et al. [24], the following cost-based production possibility set is proposed for each component:

$$\overline{T}_k=\left\{\begin{array}{c}\left(\overline{x}_i^k,y_r^k,z_g^k\right):\ \exists\mu_j^k\in\Lambda\quad\forall j\quad \overline{x}_i^k\geq\sum_j\mu_j^k\overline{x}_{ij}^k\quad\ \forall i\in I\left(k\right)\\ ,y\ _r^k\leq\sum_j\mu_j^k y_{rj}^k\quad\forall r\in O\left(k\right)\\ z_g^k\geq\sum_j\mu_j^k z_{gj}^k\quad\forall g\in R^{\text{in}}\left(k\right)\quad z_g^k\leq\sum_j\mu_j^k z_{gj}^k\quad\forall g\in R^{\text{out}}\left(k\right)\end{array}\right\} \tag{13}$$

In the set $\overline{T}_k$ (13), $\overline{x}_j^k=\left(\overline{x}^k{}_{1j},\overline{x}^k{}_{2j},\ldots,\overline{x}^k{}_{m_kj}\right)=\left(c_{1j}^k\ \ \hat{x}_{1j}^k,c_{2j}^k\hat{x}_{2j}^k,\ldots,c_{m_kj}^k\hat{x}_{m_kj}^k\right)$, $\overline{x}^k{}_{io}=c_{io}^k\ \ \hat{x}_{io}^k$, $\overline{x}_j^k\in R^{m_k}$, and $\widehat{x}_j^k=\left(\ \hat{x}_{1j}^k,\hat{x}_{2j}^k,\ldots,\hat{x}_{m_kj}^k\right)$ (input of the projection point of each component in DMU_j, which is obtained from (3)).

The PPS of the system, which is denoted by $\overline{T}$, is as follows:

$$\overline{T}=\left\{\begin{array}{c}\left(\overline{x_i},y_r\right):\ \exists\left(\overline{x}_i^k,y_r^k,z_g^k\right)\in T_k\quad\forall k\ \forall j\quad \overline{x}_i\geq\sum_{k\in p_I(i)}\overline{x}_{ij}^k\quad\forall i\\ y_r\leq\sum_{k\in p_O(r)}y_{rj}^k\quad\forall r\\ \sum_{k\in R^{\text{out}}(k)}z_{gj}^k-\sum_{k\in R^{\text{in}}(k)}z_{gj}^k\geq0\quad\forall g\end{array}\right\}$$

Based on $\overline{T}$, the new input-oriented radial model is formulated as follows:

$$\begin{aligned}\min\ \ &\rho\\ &\sum_{k\in p_I(i)}\sum_j\mu_j^k\bar{x}_{ij}^k\ \leq\rho\bar{x}_{io}\quad\forall i\\ &\sum_{k\in p_O(r)}\sum_j\mu_j^k y_{rj}^k\geq y_{ro}\quad\forall r\\ &\sum_{k\in R^{\text{out}}(k)}\sum_j\mu_j^k z_{gj}^k-\sum_{k\in R^{\text{in}}(k)}\sum_j\mu_j^k z_{gj}^k\geq0\quad\forall g\\ &\mu_j^k\geq0,\quad\rho\text{ free}\end{aligned} \tag{14}$$

In this model, $\overline{x}^k{}_{io} = c^k_{io}\hat{x}^k_{io}$ is used instead of $\overline{x}^k{}_{io} = c^k_{io}x^k_{io}$, because by doing so, we can eliminate the maximum amount of technical inefficiency. ρ^* is the optimal solution of Model (14) and indicates the radial difference in the observed input price. The right side of the first constraint is $\rho^*\overline{x}_{io} = \rho^* \sum_k \overline{x}^k_{io} = \rho^* \sum_k c_i x^k_{io}$, where $\rho^* c_i$ represents the radial reduction in the input price vector x^*_i.

Model (14) is always feasible, because $= 1$, $\mu^k_j = 0$, $j \neq j_o$ $((j= 1, 2, \ldots, n)$ and $(k = 1, \ldots, K))$, and $\mu^k_o= 1$ is a feasible solution for this model. ρ is positive, because if we assume that $\rho= 0$ is a feasible solution, then it is concluded from the first constraint that $\mu^k= 0$, and hence, it will be concluded from the second constraint that $y_{ro} \leq 0$, which is a contradiction.

The target operation points for each section are obtained using the optimal solution of Model (14) as follows:

$$\begin{cases} \tilde{\overline{x}}^k_i = \sum_j \mu^{k^*}_j \overline{x}^k_{ij} & \forall i \in I(k) \\ \tilde{y}^k_i = \sum_j \mu^{k^*}_j y^k_{rj} & \forall r \in O(k) \\ \tilde{z}^k_i = \sum_j \mu^{k^*}_j z^k_{rj} & \forall g \in R^{in}(k) \cup R^{\text{out}}(k) \end{cases} \tag{15}$$

The technically efficient input and output targets for the system in $\overline{T}$, which are obtained from Eq. (15), can be defined as follows:

$$\begin{cases} \tilde{x}^*_i = \sum_{k\in p_I(i)} \tilde{\overline{x}}^k_o = \sum_{k\in p_I(i)} \sum_j \mu^{k^*}_j \overline{x}^k_{ij} = \sum_{k\in p_I(i)} \sum_j \mu^{k^*}_j c_{ij} x^{k\,*}_{ij} & \forall i \\ \tilde{y}^*_i = \sum_{k\in p_O(r)} \sum_j \mu^{k^*}_j y^k_{rj} & \forall r \end{cases} \tag{16}$$

The radial efficiency cost C^{**}_o (or price and technical efficiency costs) is defined in $\overline{T}$ as follows:

$$C^{**}_o = \sum_{i=1}^{m} \tilde{x}^*_i = \sum_{i=1}^{m} \sum_{k\in p_I(i)} \tilde{\overline{x}}^k_o = \sum_{i=1}^{m} \sum_{k\in p_I(i)} \sum_j \mu^{k^*}_j \overline{x}^k_{ij} \tag{17}$$

We define the loss cost L^{**}_0 resulting from price technical inefficiency as follows:

$$L^{**}_o = C^{CRS*}_o - C^{**}_o$$

Theorem 3.1. *The value of* $L^{**}_o = C^{CRS*}_o - C^{**}_o$ *is non-negative.*

Proof.

$$C_o^{**} = \sum_{i=1}^{m} \tilde{x}_i^* = \sum_{i=1}^{m} \sum_{k\in p_I(i)} \tilde{\overline{x}}_o^k = \sum_{i=1}^{m} \sum_{k\in p_I(i)} \sum_j \mu_j^{k^*} \overline{x}_{ij}^k$$
$$\leq \sum_{i=1}^{m} \sum_{k\in p_I(i)} \overline{x}_{io}^k = \sum_{i=1}^{m} \sum_{k\in p_I(i)} c_{io}^k \hat{x}_{io}^k = \sum_{i=1}^{m} c_{io}^k x_{io}^* = C_o^{CRS*}$$

The input price difference is known as the network price efficiency, which is denoted by β^* and defined as follows:

$$\beta^* = \frac{C_o^{**}}{C_o^{CRS*}}$$

Based on Theorem 3.1, it can be found that β^* is less than or equal to 1.

A model for finding the minimum network cost is proposed as follows:

$$\begin{aligned} C^{***} = \min \quad & e\bar{x}_i \\ & \sum_{k\in p_I(i)} \sum_j \mu_j^k \bar{x}_{ij}^k \leq \bar{x}_i \quad \forall i \\ & \sum_{k\in p_O(r)} \sum_j \mu_j^k y_{rj}^k \geq y_{ro} \quad \forall r \\ & \sum_{k\in R^{\text{out}}(k)} \sum_j \mu_j^k z_{gj}^k - \sum_{k\in R^{\text{out}}(k)} \sum_j \mu_j^k z_{gj}^k \geq 0 \quad \forall g \\ & \mu_j^k \geq 0, \quad \bar{x}_i \geq 0 \end{aligned} \tag{18}$$

$e \in R^m$ is a row vector in which all the elements are equal to 1. Using the optimal solution of Model (18), the allocative efficient target operation point is defined as follows:

$$\begin{cases} \tilde{\tilde{x}}_i = \sum_{k\in p_I(i)} \tilde{\tilde{\overline{x}}}_i^{\,k} = \sum_{k\in p_I(i)} \sum_j \mu_j^{k^{**}} \overline{x}_{ij}^k \\ \tilde{\tilde{y}}_r = \sum_{k\in p_O(r)} \tilde{\tilde{y}}_r^{\,k} = \sum_{k\in p_O(r)} \sum_j \mu_j^{k^{**}} y_{rj}^{\,k} \end{cases} \tag{19}$$

In (19), $\mu_j^{k^{**}}$ is obtained by solving Model (18). $\square$

Theorem 3.2. *Model (18) has a finite optimal value.*

Proof. The vector $(\tilde{\tilde{x}}_i,\ \mu_j^{k**})$ is a feasible value for Model (18) where $\overline{x}_i = x_{io} = \sum_{k\in P_I(i)}^{K} x_{io}^k \quad (x_j^k = (x_{1j}^k, x_{2j}^k, \ldots, x_{m_k j}^k), x_j^k \in R^{m_k}, (k = 1, \ldots,\ K))$, $\mu_j^k = 0$, $j \neq j_o$ $(j = 1, 2, \ldots, n)$, and $\mu_o^k = 1$. Therefore, the feasible region is nonempty, and based on the representation theorem, its optimal value is present as Model (18) is a linear programming model. The value of the objective function is non-negative, and therefore, the resulting optimal solution is finite.

Using the allocative efficient target operation point, the minimum cost in $\overline{T}$ is defined as follows:

$$C_o{}^{***} = \sum_i \tilde{\tilde{x}}_i = \sum_i \sum_{k\in p_I(i)} \tilde{\tilde{x}}_i^k = \sum_i \sum_{k\in p_I(i)} \sum_j \mu_j^{k**} \overline{x}_{ij}^k$$

The overall network allocative efficiency of DMU_0 is denoted by γ^*:

$$\gamma^* = \frac{C_o{}^{***}}{C_o{}^{**}} \tag{20}$$

Allocative efficiency indicates a unit's ability to use the inputs in optimal proportion to the production technology and the prices. Also, it represents the production of the best product combination using the lowest-cost combination of inputs. $\square$

Theorem 3.3. *In Eq. (20), the value of γ^* will not exceed 1 ($\gamma^* \leq 1$).*

Proof. Assuming that $(\mu_j^{k^*}, \rho^*)$ is the optimal solution of Model (14), then $(\tilde{x}_i^*,\ \mu_j^{k^*})$ will be a feasible solution for Model (18), and consequently, $\sum_{i=1}^m \tilde{x}_i^*$ (the objective function value for this feasible solution) will be greater than or equal to $\sum_i \tilde{\tilde{x}}_i$, and $\gamma^* \leq 1$, meaning that $C_o{}^{**} \geq C_o{}^{***}$.

The cost loss $L_o{}^{***}$, which is due to allocative inefficiency, is calculated as follows:

$$L_o{}^{***} = C_o{}^{**} - C_o{}^{***}$$

Using the proof of the previous theorem, it can be concluded that $L_o{}^{***}$ is non-negative. $\square$

3.1 Decomposation of the Observed Cost

Now, we intend to decompose the real cost C_o.

$$\begin{aligned}
L_o^* &= C_o - C_o^{CRS*} && \text{loss resulting from allocative inefficiency}\\
L_o^{**} &= C_o^{CRS*} - C_o^{**} && \text{loss resulting from price inefficiency}\\
{L_o}^{***} &= C_o^{**} - {C_o}^{****} && \text{loss resulting from allocative inefficiency}\\
C_o &= L_o^* + C_o^{CRS*} = L_o^* + L_o^{**} + {L_o}^{***} + {C_o}^{***}
\end{aligned}$$

The real cost C_o was decomposed into C_o^{***}, L_o^{***}, L_o^{**}, and L_o^*.

The overall network cost efficiency, which is indicated by α^*, is defined as follows:

$$\alpha^* = \frac{{C_o}^{***}}{C_o} = \frac{C_o^{CRS*}}{C_o} \times \frac{C_o^{**}}{C_o^{CRS*}} \times \frac{{C_o}^{***}}{{C_o}^{**}}$$

The C_o^{CRS*} to C_0 cost ratio, $\frac{C_0^{**}}{C_0^{CRS*}}$, and $\frac{C_0^{***}}{C_0^{**}}$ represent the technical efficiency, price efficiency, and allocative efficiency, respectively.

To decompose the cost efficiency, we will use the following algorithm:
Step 1: Determine C_o using Eq. (1)
Step 2: Evaluate the network DMUs using Model (2)
Step 3: Determine $\hat{x}_i^k$ $(\forall i \in I\,(k))$ for each component
Step 4: Determine C_o^{CRS*} using Eq. (5)
Step 5: Determine the network technical efficiency by dividing C_o^{CRS*} by C_o
Step 6: Determine C_o^* if the input price vector is identical for all units; otherwise, proceed to Step 7
Step 7: Determine ${\overline{x}^k}_i$ in $\overline{T}_k$
Step 8: Evaluate the network DMUs in a non-competitive space using Model (14)
Step 9: Determine $\tilde{x}_i^*$ using Eq. (16)
Step 10: Determine C_o^{**} using Eq. (17)
Step 11: Determine β^* (network price efficiency) by dividing C_o^{**} by C_o^{CRS*}
Step 12: Determine C_o^{***} (minimum network cost)
Step 13: Determine γ^* (network allocative efficiency) by dividing C_o^{***} by C_o^{**}

Step 14: Multiply the tehnical, price, and allocative efficiencies by each other, determine the network cost efficiency, and end the algorithm.

3.2 Extention

It is possible to extend the proposed network cost efficiency model to another situation, such as the estimation of network revenue efficiency.

3.2.1 Network Revenue Efficiency

In Section 2, the observed real total revenue of DMU_o is defined as Eq. (11).

We can evaluate the DMU using the radial network technical efficiency model (2) or the non-radial network technical efficiency model NSBM (introduced by Tone and Tsutsui [23]), and obtain the projection point $(x_i^*,\ y_r^*)$ from Eq. (4). The revenue corresponding to the projection point (x_i^*, y_r^*) (technically efficient revenue for DMU_o) is defined as follows:

$$
\begin{aligned}
E_o^{CRS*} &= \sum_{r=1}^{s} p_{ro} {y^*}_{ro} = \sum_{r=1}^{s} p_{ro}^k \sum_{k\in p_O(r)}^{K} \hat{y}_{ro}^k \\
&= \sum_{i=1}^{m} p_{ro}^k \sum_{k\in p_O(r)}^{K} \sum_j \lambda_j^{k*} y_{rj}^k \ \geq \sum_{r=1}^{s} p_{ro}^k \sum_{k\in O(r)}^{K} y_{ro}^k = \ E_o
\end{aligned}
$$

The loss due to network technical inefficiency $(\tilde{L}_o^*)$ is evaluated as follows:

$$
\tilde{L}_o^* = E_o^{CRS*} - E_o
$$

New cost-based production possibility sets are defined for each component as follows:

$$
\overline{P}_k = \left\{ \begin{array}{c} \left(x_i^k, \overline{y}_r^k, z_g^k\right) : \ \exists \mu_j^k \in \Lambda \quad \forall j \quad x_i^k \geq \sum_j \mu_j^k x_{ij}^k \quad \forall i \in I\left(k\right) \\ , \overline{y}\,_r^k \leq \sum_j \mu_j^k \overline{y}_{rj}^k \quad \forall r \in O\left(k\right) \\ z_g^k \geq \sum_j \mu_j^k z_{gj}^k \quad \forall g \in R^{\text{in}}\left(k\right) \quad z_g^k \leq \sum_j \mu_j^k z_{gj}^k \quad \forall g \in R^{\text{out}}\left(k\right) \end{array} \right\}
$$

In $\overline{P}_k$ (9), $\overline{y}_j^k = \left(\overline{y}^k{}_{1j},\ \overline{y}^k{}_{2j}, \ldots, \overline{y}^k{}_{s_k j}\right) = \left(p_{1j}^k \hat{y}_{1j}^k, p_{2j}^k \hat{y}_{2j}^k, \ldots, p_{m_k j}^k \hat{y}_{s_k j}^k\right)$ and $\hat{y}_j^k = \left(\hat{y}_{1j}^k, \hat{y}_{2j}^k, \ldots, \hat{y}_{s_k j}^k\right)$ (output of the projection point of each component in DMU_j, which is obtained from (3)) so that $\overline{y}^k{}_{ro} = p^k_{io}\ \ \hat{y}^k_{ro}$ and $\overline{y}^k_j \in R^{s_k}$.

We set up the system's PPS as follows:

$$\overline{P} = \left\{ \begin{array}{c} (x_i, \overline{y_r}):\ \exists \left(x^k_i, \overline{y}^k_r, z^k_g\right) \in T_k \quad \forall k\ \forall j \qquad x_i \geq \sum_{k \in p_I(i)} x^k_{ij} \quad \forall i \\ \overline{y}_r \leq \sum_{k \in p_O(r)} \overline{y}^k_{rj} \quad \forall r \\ \sum_{k \in R^{\mathrm{out}}(k)} z^k_{gj} - \sum_{k \in R^{\mathrm{in}}(k)} z^k_{gj} \geq 0 \quad \forall g \end{array} \right\}$$

Based on $\overline{P}$, similar to the case of Model (14), we formulate a network technical efficiency model as follows:

$$\begin{array}{ll} \min & \rho \\ & \sum\limits_{k \in p_I(i)} \sum\limits_j \mu^k_j x^k_{ij} \leq \rho x_{io} \quad \forall i \\ & \sum\limits_{k \in p_O(r)} \sum\limits_j \mu^k_j \overline{y}^k_{rj} \geq \bar{y}_{ro} \quad \forall r \\ & \sum\limits_{k \in R^{\mathrm{out}}(k)} \sum\limits_j \mu^k_j z^k_{gj} - \sum\limits_{k \in R^{\mathrm{out}}(k)} \sum\limits_j \mu^k_j z^k_{gj} \geq 0 \quad \forall g \\ & \mu^k_j \geq 0 \quad \rho \text{ free} \end{array} \tag{21}$$

Note that instead of the radial model (21), the non-radial model *NSBM* (Tone and Tsutsui [23]) can also be used. Technically efficient input and output targets can be defined for the system in $\overline{P}$ using the optimal solution of Model (21) as follows:

$$\begin{cases} \tilde{x}^{**}_i = \sum_{k \in p_I(i)} \sum_j \mu^{k^{**}}_j x^k_{ij} & \forall i \\ \tilde{y}^{**}_r = \sum_{k \in p_O(r)} \tilde{\overline{y}}^k_o = \sum_{k \in p_O(r)} \sum_j \mu^{k^{**}}_j \overline{y}^k_{rj} = \sum_{k \in p_O(r)} \sum_j \mu^{k^{**}}_j p_{rj} y^{k^{\ *}}_{rj} & \forall r \end{cases} \tag{22}$$

In (22), $\mu^{k^{**}}_j$ is obtained by solving Model (21).

The revenue of price and technical efficiencies E^{**}_o is defeined as follows:

$$\begin{aligned} E^{**}_o &= \sum_{r=1}^{s} \tilde{y}^{**}_r = \sum_{r=1}^{s} \sum_{k \in p_O(r)} \tilde{\overline{y}}^k_o \\ &= \sum_{r=1}^{s} \sum_{k \in p_O(r)} \sum_j \mu^{k^{**}}_j \overline{y}^k_{rj} \geq \sum_{r=1}^{s} \sum_{k \in p_O(r)} \overline{y}^k_{ro} \\ &= \sum_{r=1}^{s} \sum_{k \in p_O(r)} p^k_{ro} \hat{y}^k_{ro} = \sum_{i=1}^{m} p^k_{ro} y^*_{ro} = \ E^{CRS*}_o \end{aligned}$$

The loss due to price inefficiency ($\tilde{L}_o^{**}$) is defined as follows:

$$\tilde{L}_o^{**} = E_o^{**} - E_o^{CRS*} \ \geq 0$$

Finally, using the following model, we determine the maximum network revenue in $\overline{P}$.

$$\begin{array}{ll} \max & e\bar{y}_r \\ & \sum\limits_{k\in p_I(i)} \sum\limits_j \mu_j^k x_{ij}^k \ \leq x_{io} \quad \forall i \\ & \sum\limits_{k\in p_O(r)} \sum\limits_j \mu_j^k \bar{y}_{rj}^k \geq \bar{y}_r \quad \forall r \\ & \sum\limits_{k\in R^{out}(k)} \sum\limits_j \mu_j^k z_{gj}^k - \sum\limits_{k\in R^{\text{out}}(k)} \sum\limits_j \mu_j^k z_{gj}^k \geq 0 \quad \forall g \\ & \mu_j^k \geq 0, \quad \bar{y}_r \ \geq 0 \end{array} \tag{23}$$

$e \in R^S$ is a row vector in which all the elements are equal to 1. The allocatively efficient target operation point will be defined as follows:

$$\begin{cases} \tilde{\tilde{x}}_i^{\,*} = \sum_{k\in p_I(i)} \tilde{\tilde{x}}_i^{\,k} = \sum_{k\in p_I(i)} \sum_j \mu_j^{k***} x_{ij}^k \\ \tilde{\tilde{y}}_r^{\,*} = \sum_{k\in p_O(r)} \tilde{\tilde{\overline{y}}}_r^{\,k} = \sum_{k\in p_O(r)} \sum_j \mu_j^{k***} \overline{y}_{rj}^k \end{cases} \tag{24}$$

In (24), μ_j^{k***} is obtained by solving Model (23).

Using the allocatively efficient target operation point, the maximum revenue $E_o{}^{***}$ in $\overline{P}$ is calculated as follows:

$$E_o{}^{***} = \sum_r \tilde{\tilde{y}}_r = \sum_r \sum_{k\in p_O(r)} \tilde{\tilde{\overline{y}}}_r^{\,k} = \sum_r \sum_{k\in p_O(r)} \sum_j \mu_j^{k***} \overline{y}_{rj}^k$$

The loss due to allocative inefficiency ($\tilde{L}_o^{***}$) is obtained through the following equation:

$$\tilde{L}_o^{***} = E_o^{***} - E_o^{**} \ \geq 0$$

Finally, the real revenue can be decomposed into maximum revenue and losses due to technical, price, and allocative inefficiencies.

The network revenue efficiency, which is denoted by γ^*, can be obtained as follows:

$$\gamma^* \ = \frac{E_o}{E_o{}^{***}}$$

4 Numerical Example

In Section 2, it was demonstrated through an example with the two input price vectors $C^1 = (500, 1, 1)$ and $C^2 = (1000, 2, 2)$ that despite doubling the input price vector, the network cost efficiency score proposed by Lozano [15] remained unchanged. In the current example, based on Table 1, the network cost efficiency score is calculated again for C^1 and C^2 using the method proposed in this paper. This would allow a better comparison between the two methods. Moreover, the network cost efficiency is also calculated through both Lozano's method and our proposed method in a case where the input price vector varies from one DMU to another. Finally, the network cost efficiency obtained through the proposed method is decomposed into technical, price, and allocative efficiencies.

Columns 2 and 3 of Table 3 provide the cost efficiency scores of the network DMUs for the input price vectors C^1 and C^2, which are obtained using the method proposed in this paper and are denoted by α_1^* and α_2^*, respectively.

Based on the α_1^* and α_2^* values in Table 3, it can be found that the proposed model yields the same network cost efficiency scores for both input price vectors C^1 and C^2. A comparison between Tables 3 and 4 shows that when the input price vector is the same for all units, our proposed model produces a lower network cost efficiency score than Lozano's model.

Table 4 includes the input prices for the network DMUs introduced in Section 2, Table 1. Columns 3, 5, and 7 in Table 4 show the input price for each unit, which varies from one unit to another and has been considered with presumptive values.

With the help of Table 5, the network cost efficiency can be decomposed.

The last column in Table 5 includes the minimum cost for each unit in the PPS $\overline{T}_k$, which is indicated by C_o^{***}. For all units, the values in this column are smaller than the values in the other columns of Table 4. Also, for all units, the total values of the real cost C_o in the first column of Table 5 are larger than the values in the other columns. It can be observed that in all units, the total cost corresponding to the technical efficiency (C_o^{CRS}) is less than the total input cost corresponding

Table 3: Network cost efficiency calculated through the proposed method with the input price vectors C^1 and C^2, which were identical for all units

DMU	α_1^*	α_2^*
1	0.010656011	0.010656012
2	0.008415213	0.008415213
3	0.022561788	0.022561788
4	0.026485719	0.026486788
5	0.051945085	0.051943728
6	0.026106794	0.02610594
7	0.006340712	0.006340711
8	0.001718821	0.001718821
9	0.016916484	0.016916483
10	0.033214808	0.03321481
11	0.01655504	0.016555042
12	0.012529745	0.012529747
13	0.008092797	0.008092797
14	0.001469829	0.001469828
15	0.044095946	0.044095945
16	0.011469005	0.011469005
17	0.019277671	0.019277671

Table 4: Inputs of the DMUs and the price of each input

DMU	X_1	C_1	X_2	C_2	X_3	C_3
1	4183	0.71	756090	0.11	146092	0.18
2	3000	0.49	401059	0.24	48629	0.27
3	2715	0.54	465372	0.15	77507	0.21
4	1893	0.8	858696	0.05	207128	0.15
5	4578	0.69	1065000	0.08	331238	0.23
6	2134	0.987	781780	0.005	74154	0.008
7	1059	0.58	261071	0.09	32324	0.19
8	937	0.7	325130	0.1	78685	0.2
9	701	0.75	190321	0.07	62251	0.18
10	418	0.61	74445	0.22	13173	0.17
11	582	0.6	92077	0.012	12805	0.28
12	380	0.68	65696	0.08	7691	0.24
13	2190	0.85	576821	0.05	68126	0.1
14	523	0.54	79801	0.18	3673	0.28
15	373	0.75	89923	0.05	6321	0.2
16	383	0.65	70581	0.1	5432	0.25
17	736	0.72	97700	0.06	9356	0.23

Table 5: Real cost, cost corresponding to technical efficiency, input cost corresponding to pure technical efficiency, radial efficiency cost that consists of technical and price efficiency costs, and minimum cost in PPS $\overline{T}_k$

DMU	C_o	${C_o}^{VRS*}$	${C_o}^{CRS*}$	${C_o}^{**}$	${C_o}^{***}$
1	112436.39	24379.66569	21240.60566	1554.8946	884.88602
2	110853.99	24560.94862	21584.8012	774.5818	455.110637
3	87548.36	42305.2173	37458.17898	2281.3862	1189.324945
4	75518.4	24705.26894	24535.8991	2410.8595	1478.415305
5	164543.56	164543.56	1032274.609	8647.3673	5309.923633
6	6608.39	3040.254306	3033.25101	1392.2622	1392.2622
7	44826.07	44826.07	3159.39438	253.1608	144.733543
8	48905.9	10517.6	1241.317	69.7551	41.590178
9	25053.4	7712.14	5343.91788	464.9335	282.986712
10	15808.69	14618.58696	9794.09672	542.8455	273.285159
11	14983.84	10265	4961.58724	302.0506	181.749441
12	7359.92	5514.32	1811.38876	148.6779	91.542631
13	22254.3	5103.823	5064.2262	634.82	390.590302
14	15675.04	15675.04	610.6864	32.3664	14.065004
15	6040.1	6040.1	4573.5957	541.5347	345.842886
16	8665.05	8665.05	1991.353582	195.7005	85.105459
17	8453.8	5869.21897	3648.321769	397.3421	254.030671

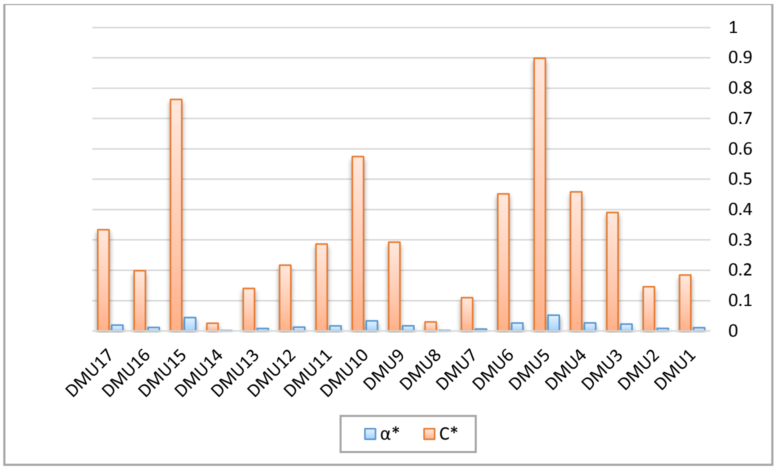

Figure 1: Comparison between cost efficiency scores produced by Lozano's method and the method proposed in this paper (comparison between Tables 2 and 3)

to the pure technical efficiency (C_o^{VRS}). On the whole, based on Table 4 and the explanations given, it can be concluded that for all units, the following relation holds:

$$C_o \geq C_o{}^{VRS*} \geq C_o{}^{CRS*} \geq C_o{}^{**} \geq C_o{}^{***}$$

Table 6 shows the causes behind the cost loss in each unit. The losses in each unit occur due to various reasons, such as technical, scale, price, and allocative inefficiency.

According to the second column in Table 6, for all units except Units 5 and 15, the biggest loss has been due to technical inefficiency, while the biggest loss in Units 5 and 15 was resulted from price inefficiency. For all units, the values in the last column of Table 5 are smaller than the values in the other columns. Therefore, among the causes of loss, allocative inefficiency has been the factor with the lowest contribution to the losses. In this regard, allocative inefficiency has had no effect on the losses in Unit 6, while technical inefficiency has had the highest contribution.

Table 6: Losses due to technical, price, and allocative inefficiency

DMU	L_o^*	L_o^{**}	${L_o}^{***}$
1	91195.78	19685.71	670.009
2	89269.19	20810.22	319.4712
3	50090.18	35176.79	1092.061
4	50982.5	22125.04	932.4442
5	61268.95	94627.24	3337.444
6	3575.139	1640.989	0
7	41666.68	2906.234	108.4273
8	47664.58	1171.562	28.16492
9	19709.48	4878.984	181.9468
10	6014.593	9251.251	269.5603
11	10022.25	4659.537	120.3012
12	5548.531	1662.711	57.13527
13	17190.07	4429.406	244.2297
14	15064.35	578.32	18.3014
15	1466.504	4032.061	195.6918
16	6673.696	1795.653	110.595
17	4805.478	3250.98	143.3114

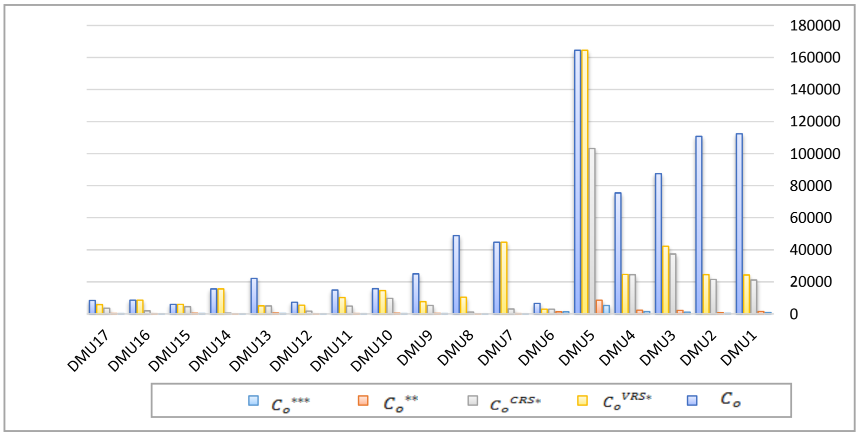

Figure 2: Relation between C_o, ${C_o}^{VRS*}$, ${C_o}^{CRS*}$, ${C_o}^{**}$, and ${C_o}^{***}$

By comparing the last two columns of Table 6, it can be concluded that in all units, the loss caused by price inefficiency is higher than the loss due to allocative inefficiency.

Based on Table 6, it can be found that for all of units except Units 5 and 15, ${L_o}^{***} \leq {L_o}^{**} \leq {L_o}^{**}$.

Table 6 shows that the cost efficiency can be decomposed into technical, price, and allocative efficiencies. The values provided in Table 8 indicate the technical efficiency (C_o^*/C_o), price efficiency (C_o^{**}/C_o^*), allocative efficiency (C_o^{***}/C_o^{**}), and cost efficiency ($\frac{{C_o}^{***}}{C_o}$) of each DMU. When $\frac{C_o^{***}}{C_o}$ equals 1, it means that C_o^{***} is equal to C_o, and thus the losses due to price, technical, and allocative inefficiencies will be equal to zero. If C_o is larger than C_o^{***}, it indicates that the unit is inefficient and the inefficiency is due to losses resulting from technical, price, and allocative inefficiencies.

According to Table 7, none of the units has a cost efficiency equal to 1, and all the scores are less than 1. This indicates that none of the units is cost-efficient, meaning that the input cost of all DMUs can be reduced considerably as all of them are too distant from the value of 1. The cost efficiency of the 6^{th} unit is higher than the others. This unit has an allocative efficiency of 1 and its loss is not due to allocative

Table 7: Technical, price, allocative, and network cost efficiencies obtained by dividing the costs by each other

DMU	C_o^*/C_o	C_o^{**}/C_o^*	C_o^{***}/C_o^{**}	$\frac{C_o{}^{***}}{C_o}$
1	0.188912199	0.073204	0.569096839	0.00787
2	0.194713796	0.035886	0.587556585	0.004105
3	0.427857004	0.060905	0.521316796	0.013585
4	0.324899615	0.098258	0.613231632	0.019577
5	0.627642974	0.083732	0.614050895	0.032271
6	0.459	0.459	1	0.210681
7	0.070481182	0.08013	0.571705979	0.003229
8	0.025381743	0.056194	0.596231358	0.00085
9	0.213301104	0.087002	0.608660619	0.011295
10	0.619538793	0.055426	0.503430827	0.017287
11	0.331129219	0.060878	0.601718523	0.01213
12	0.246115278	0.08208	0.615711084	0.012438
13	0.227561694	0.125354	0.615277247	0.017551
14	0.038959161	0.053	0.434555712	0.000897
15	0.757205295	0.118405	0.638634765	0.057258
16	0.229814436	0.098275	0.434876043	0.009822
17	0.431559981	0.108911	0.639324831	0.030049

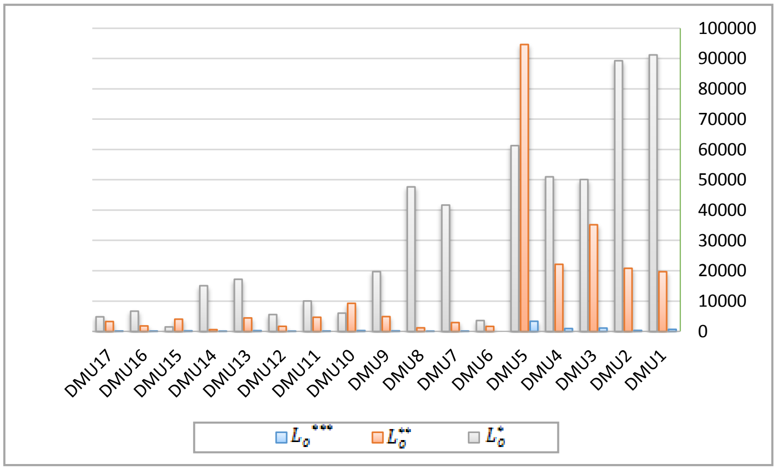

Figure 3: Relation between L_o^*, L_o^{**}, and $L_o{}^{***}$

inefficiency, because the respective L_o^{***} value in Table 6 is equal to zero. Therefore, it can be concluded that the loss in this unit is due to either price inefficiency or technical inefficiency. Based on Table 5, it can be found that the value of L_o^* is larger than L_o^{**}. Unit 15 has a cost efficiency of approximately 0.057. Among the values presented in Table 7, price efficiency has the lowest value, while in Table 6, the main cause of loss is price inefficiency since it has a larger value than the other columns in row 15 of Table 6.

According to Table 6, the allocative efficiency of almost all units is higher than their technical and price efficiencies.

Based on Table 7, the allocative efficiency is higher than the technical and price efficiencies in the majority of the units, except for Units 5, 10, and 15, in which the allocative efficiency is lower than the technical efficiency. Therefore, the loss due to allocative inefficiency is smaller than the loss caused by technical and price inefficiencies.

Except for Units 7, 8, and 14 in which the price efficiency is higher than the technical efficiency, the price efficiency of all other units is lower than their technical and allocative efficiencies. Therefore, it can be concluded that the following relation holds for the majority of the

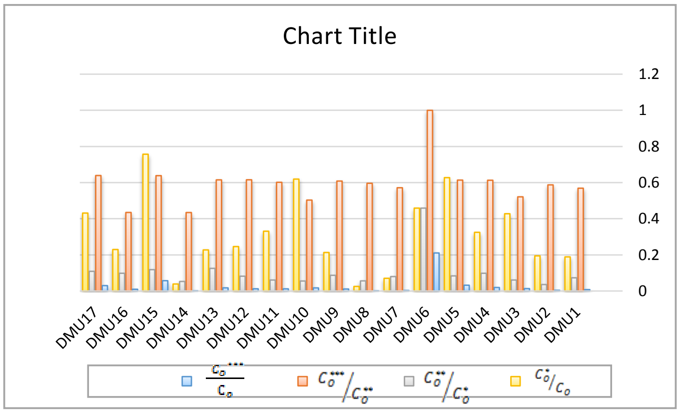

Figure 4: Relation between $\frac{C_o^*}{C_o}$, $\frac{C_o^{**}}{C_o^*}$, $\frac{{C_o}^{***}}{{C_o}^{**}}$, and $\frac{{C_o}^{***}}{C_o}$

units:

$$\frac{C_o^*}{C_o} < \frac{C_o^{**}}{C_o^*} < \frac{{C_o}^{***}}{{C_o}^{**}}$$

Based on Table 7, it can be concluded that the following equation is true for every DMU:

$$\frac{{C_o}^{***}}{C_o} = \frac{C_o^*}{C_o} \times \frac{C_o^{**}}{C_o^*} \times \frac{{C_o}^{***}}{{C_o}^{**}}$$

In Section 2, Lozano [15] obtained the cost efficiency of network DMUs in a case with identical input prices. Now, in this section, we intend to determine the cost efficiency as proposed by Lozano [15] using the input prices assumed in Table 2 for the network units in Table 1 (input prices vary from one DMU to another).

Table 8 compares the cost efficiency scores obtained through Lozano's method (C^*) with the scores produced by our proposed method (α^*) in a case with varying input prices.

As can be observed in Table 8, the cost efficiency values obtained through the two methods are different from each other. The values are greater than the values, meaning that Lozano's method has produced larger cost efficiency scores than our proposed method for every network unit.

Now, we will find the price, allocative, and cost efficiencies of the

Table 8: Cost efficiency using Lozano's method and the method proposed in this paper in a non-competitive space

DMU	C^*	α^*
1	0.185215	0.00787
2	0.194683	0.004105
3	0.41847	0.013585
4	0.243757	0.019577
5	0.622306	0.032271
6	0.400318	0.210681
7	0.065059	0.003229
8	0.018902	0.00085
9	0.18847	0.011295
10	0.726531	0.017287
11	0.32997	0.01213
12	0.241564	0.012438
13	0.206013	0.017551
4	0.033579	0.000897
15	0.752292	0.057258
6	0.227892	0.009822
17	0.464875	0.030049

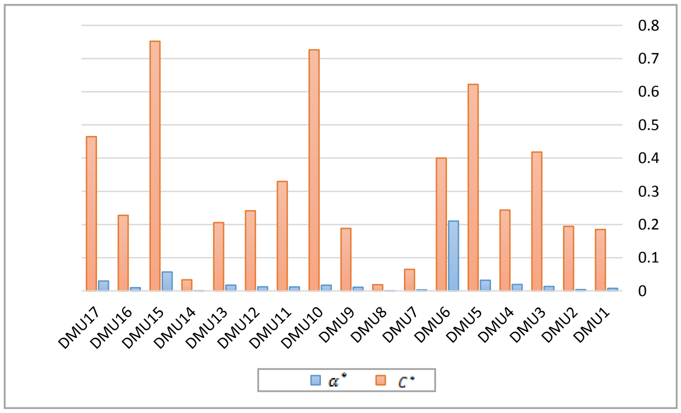

Figure 5: Comparison between the cost efficiency scores produced by Lozano's method and the method proposed in this paper in a non-competitive space

network units in Table 1 using our proposed method, the results of which are denoted in Table 10 by C_o^{**}/C_o^{*}, C_o^{***}/C_o^{**}, and $\frac{C_o{}^{***}}{C_o}$, respectively. All units have the same input price vector, $C^1 = (500, 1, 1)$.

We will use Table 9 to determine the price, allocative, and cost efficiencies.

Based on Table 9, it can be deduced that the relation $C_o \geq C_o{}^{CRS*} \geq C_o{}^{**} \geq C_o{}^{***}$ holds for all the network units.

4.1 Illustrative Application

This section provides a case study involving 16 airlines with two-stage network structures. Each airline consumes the two inputs x_1 and x_2 in the first stage to produce 4 intermediate products (these products are denoted by z), and these intermediate products are used as inputs in the second stage to produce 4 final outputs. Fig. 1 illustrates the inputs, intermediate products, and outputs of each airline. In this example, since an increased number of personnel and empty seats would impose great costs on the airlines, the number of personnel (x_1) and number of empty seats (x_2) are considered as the inputs of the first stage. We intend to reduce these inputs as much as possible, beause by reducing

Table 9: Real cost, cost corresponding to technical efficiency, radial efficiency cost that consists of technical and price efficiency costs, and minimum cost

DMU	C_o	${C_o}^{CRS*}$	${C_o}^{**}$	${C_o}^{***}$
1	2993682	624693.7	34342.62	31900.71
2	1949688	371941.5	17662.95	16407.04
3	1900379	863523.7	46157.98	42875.95
4	2012324	937243.6	53297.85	53297.85
5	3685238	3312679	203013.2	191000
6	1922934	882796.8	50201.64	50201.64
7	822895	90453.52	5487.197	5217.74
8	872315	26278.55	1499.353	1499.353
9	603072	177388.5	10578.22	10201.86
10	296618	194442.9	10606.26	9852.11
11	395782	137685.9	7053.739	6552.187
12	263387	66164.54	3552.791	3300.172
13	1739947	244606.2	14665.69	14081.04
14	344974	10283.44	545.8662	507.0527
15	282744	219792.4	13266.31	12467.86
16	267513	53478.77	3265.934	3068.108
17	475056	181486	9858.991	9157.973

Table 10: Technical efficiency, Price efficiency, Allocative efficiency, and Cost efficiency

DMU	C_o^*/C_o	C_o^{**}/C_o^*	C_o^{***}/C_o^{**}	$\frac{{C_o}^{***}}{C_o}$
1	0.208671	0.054975	0.928896	0.010656
2	0.19077	0.047489	0.928896	0.008415
3	0.454396	0.053453	0.928896	0.022562
4	0.465752	0.056867	1	0.026486
5	0.898905	0.061284	0.940826	0.051828
6	0.459088	0.056867	1	0.026107
7	0.109921	0.060663	0.950894	0.006341
8	0.030125	0.057056	1	0.001719
9	0.294141	0.059633	0.964421	0.016916
10	0.655533	0.054547	0.928896	0.033215
11	0.347883	0.051231	0.928896	0.016555
12	0.251207	0.053696	0.928896	0.01253
13	0.140583	0.059956	0.960135	0.008093
14	0.029809	0.053082	0.928896	0.00147
15	0.777355	0.060358	0.939814	0.044096
16	0.199911	0.06107	0.939427	0.011469
17	0.382031	0.054324	0.928896	0.019278

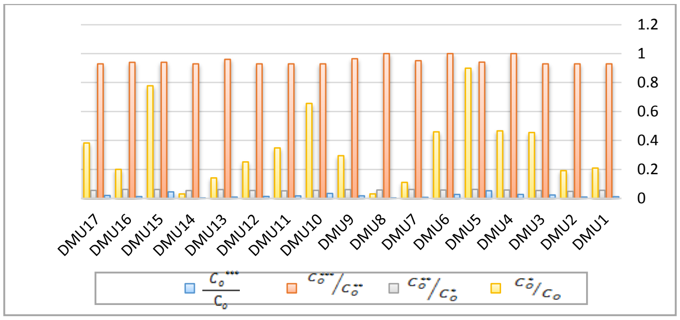

Figure 6: Relation between technical, price, allocative, and cost efficiency scores produced by our proposed method for the input price vector $C^1 = (500, 1, 1)$

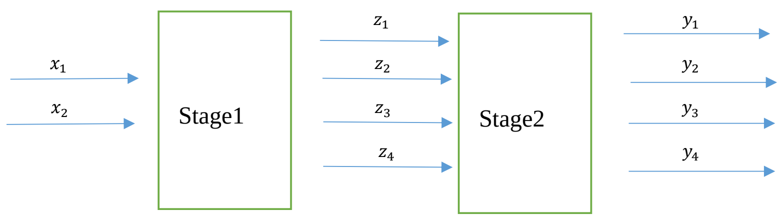

Figure 7: DMU as a two-stage system

x_1 and x_2 , the costs will be reduced as well. The intermediate products include the number of airplanes (z_1), average flight delay time (z_2), number of passengers (z_3), and number of flights (z_4), which are the inputs of the second stage and our aim is to reduce them. The number of flight lines (y_1), average weight capacity (y_2), number of pilots and flight attendants (y_3), and average passenger satisfaction (y_4) are all the final outputs of the system, which we aim to increase as much as possible, because an increase in these outputs would increase the satisfaction level of the customers and result in a larger clientele.

Tables 11, 12, and 13 present the inputs, intermediate products, and final outputs of the airlines, respectively.

First, we will use the model proposed by Lozano [15] and calculate the network cost efficiency of the airlins in a competitive space with

Table 11: Inputs of the 16 airlines under study

DMU	x_1	x_2
1	309	185
2	174	196
3	674	109
4	52	174
5	32	176
6	58	165
7	122	141
8	31	169
9	23	132
10	205	124
11	37	127
12	30	199
13	36	132
14	7	69
15	27	148
16	30	46

Table 12: Intermediate products of the airlines under study

DMU	z_1	z_2	z_3	z_4
1	63	37	3129799	13122
2	35	57	2975231	2178
3	49	62	2044068	9823
4	15	52	2295410	2267
5	21	54	1917329	2548
6	11	51	1837346	3196
7	14	43	1959569	2007
8	7	62	1242314	244
9	10	43	1998118	976
10	22	41	1047006	5384
11	9	58	763924	310
12	7	50	556006	1306
13	4	49	226198	307
14	10	24	10318	302
15	5	40	10538	350
16	2	32	3836	342

Table 13: Final outputs of the 16 airlines under study

DMU	y_1	y_2	y_3	y_4
1	48	44.6	1182	38.99
2	34	84	532	4.48
3	27	72	1037	29.38
4	20	92.9	290	6.51
5	24	63.1	233	2.16
6	28	91.3	195	0.71
7	22	80.4	176	3.68
8	22	88.2	217	1.84
9	22	71.4	150	1.03
10	22	77.2	267	5.29
11	29	68.6	104	0.77
12	9	89.7	117	4.87
13	13	87	43	0.02
14	41	36.5	46	0.06
15	6	53.4	60	0.01
16	3	64.6	40	0.2

Table 14: Input prices, which vary from one unit to another

DMU	x_1	Price of input x_1 (c_1)	x_2	Price of input x_2 (c_2)
1	309	0.2	185	0.8
2	174	0.58	196	0.42
3	674	0.15	109	0.85
4	52	0.68	174	0.32
5	32	0.75	176	0.25
6	58	0.82	165	0.18
7	122	0.6	141	0.4
8	31	0.75	169	0.25
9	23	0.7	132	0.3
10	205	0.35	124	0.65
11	37	0.62	127	0.38
12	30	0.82	199	0.18
13	36	0.7	132	0.3
14	7	0.92	69	0.08
15	27	0.8	148	0.2
16	30	0.6	46	0.4

the input price vector $C = (5, 1)$ (the input prices are hypothetical and are the same for all the airlines), and then we will compare the results with the network cost efficiency scores calculated through our proposed method in Section 3. Next, we will calculate the network cost efficiency of the airlines in a non-competitive space using the proposed method, and compare the results with Lozano's network cost efficiency. In this case, the input prices vary from one airline to another and are selected hypothetically, as presented in Table 14. Finally, we will decompose the network cost efficiency and determine the causes of cost loss in each airline.

According to Table 15, except for Airline 14 that shows the same

Table 15: Comparison between the network cost efficiency scores produced by Lozano's method and our proposed method in a competitive space with the input price vector $C = (5, 1)$

DMU	c^*lozano	α^* proposed
1	0.88660347	0.602175145
2	0.39393058	0.272363977
3	0.3338123	0.226723196
4	0.72732719	0.497165899
5	0.64458333	0.471547619
6	0.54846154	0.424527473
7	0.31724368	0.227976032
8	0.7695679	0.582839506
9	0.78663968	0.609716599
10	0.23474326	0.160765883
11	0.56660256	0.46349359
12	0.72667622	0.534641834
13	0.61730769	0.476826923
14	1	1
15	0.43024735	0.333568905
16	0.7075	0.546479592

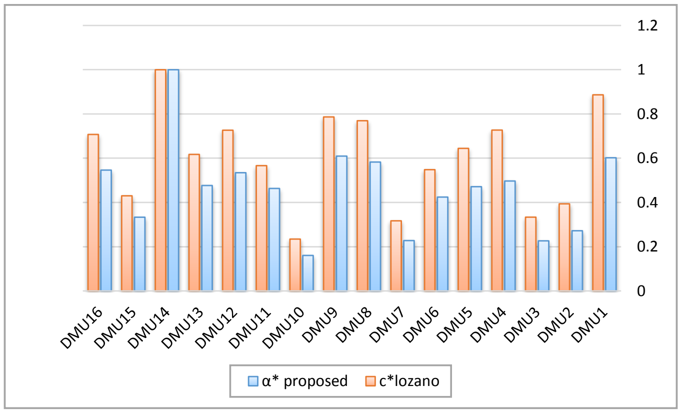

Figure 8: Comparison between the network cost efficiency scores produced by Lozano's method and our proposed method in a competitive space

network cost efficiency score in both methods (this airline is network cost efficient based on both methods), in the other airlines, the network cost efficiency scores produced by our method (α^*) are smaller than the ones resulting from Lozano's method (C^*).

Based on the input prices specified in Table 14, which vary from one airline to another, we find the network cost efficiency scores of the airlines in Table 16. In Column 2 of Table 16, the network cost efficiency of the airlines are determined using Lozano's method [15], which is indicated by $c*$, and in Column 3, we find the network cost efficiency scores produced by our proposed method (α^*). A comparison between Columns 2 and 3 of Table 16 reveals that the network cost efficiency scores produced by our method are less than or equal to their corresponding scores based on Lozano's method. Airlines 1 and 14 were network cost efficient in both methods. According to Lozano's method, Airlines 1, 3, 14, and 16 were network cost efficient.

We will use Tables (14) and (15) to decompose the network cost efficiency into technical, price, and allocative efficiencies.

In Table 17, the relation $C_o \geq {C_o}^{CRS*} \geq {C_o}^{**} \geq {C_o}^{***}$ holds for all the airlines. In Airlines 1 and 14, the equation

Table 16: Comparison between the network cost efficiency scores produced by Lozano's method and our proposed method in a non-competitive space

DMU	c^*Lozano	α^* proposed
1	1	1
2	0.639817	0.310194
3	1	0.821265
4	0.809205	0.45815
5	0.657206	0.4175
6	0.561222	0.405514
7	0.499537	0.232562
8	0.797863	0.486107
9	0.829982	0.437163
10	0.600656	0.233607
11	0.691854	0.297612
12	0.728401	0.527971
13	0.731636	0.341821
14	1	1
15	0.442188	0.279297
16	1	0.438187

Table 17: Real cost, cost corresponding to technical efficiency, radial efficiency cost that consists of technical and price efficiency costs, and minimum cost

DMU	C_o	${C_o}^{CRS*}$	${C_o}^{**}$	${C_o}^{***}$
1	209.8	209.8	209.8	209.8
2	183.24	139.4489	66.9919	56.84
3	193.75	193.75	161.3285	159.12
4	91.04	78.10371	44.2128	41.71
5	68	46.18515	32.0649	28.39
6	77.26	56.24246	56.2425	31.33
7	129.6	77.65848	34.2943	30.14
8	65.5	53.50085	38.8706	31.84
9	55.7	46.60009	38.0601	24.35
10	152.35	101.7332	42.1424	35.59
11	71.2	49.98878	32.779	21.19
12	60.42	44.8631	36.3166	31.9
13	64.8	47.84762	26.9149	22.15
14	11.96	11.96	11.96	11.96
15	51.2	24.22292	17.4995	14.3
16	36.4	36.4	19.3783	15.95

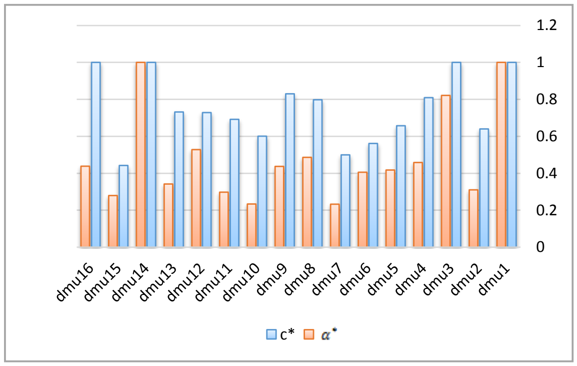

Figure 9: Comparison between the network cost efficiency scores produced by Lozano's method and our proposed method in a non-competitive space

${C_o} = {C_o}^{CRS*} = {C_o}^{**} = {C_o}^{***}$ holds true.

According to Table 18, Airlines 1 and 14 have network technical, price, allocative, and cost efficiency values equal to one. Airlines 1, 3, 14, and 16 are all network-technically efficient. The technical and allocative efficiency scores of Airline 8 are almostequal to each other. Airlines 1,6, and 14 are price efficient. Except for units 1 and 14, the rest of the airlines have lower network cost efficiency scores in comparison with other types of efficiency.

To determine the causes of cost loss in the airlines, we need to make use of Table 19.

According to Table 19, Airlines 1 and 14 have no cost loss as the network tehnical, price, allocative, and cost efficiency scores equal 1 in these airlines. In Table 18, Airlines 3 and 16 have no cost loss due to technical inefficiency, because they are technically efficient. In Table 19, the causes of cost loss in Airline 6 are technical and allocative inefficiencies (since the value of C_o^{**}/C_o^* equals 1 in Table 18, there is no cost losss due to price inefficiency in this case). Airlines 2 and 10 have the greatest cost losses, and their losses are due to price inefficiency. In the majority of the airlines, the greatest cost losses are due to technical and price inefficiencies.

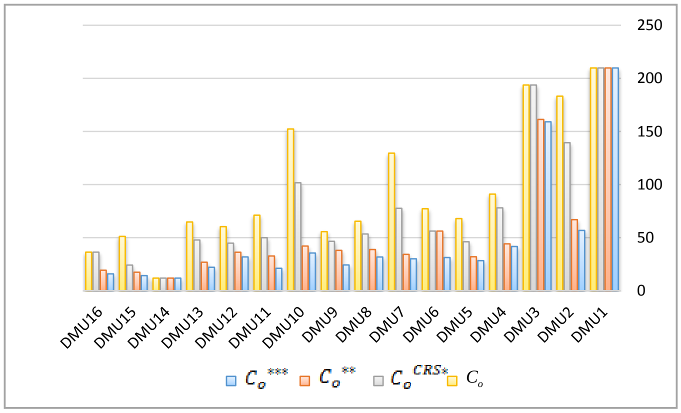

Figure 10: Relation between C_o, ${C_o}^{CRS*}$, ${C_o}^{**}$, and ${C_o}^{***}$in the airlines under study

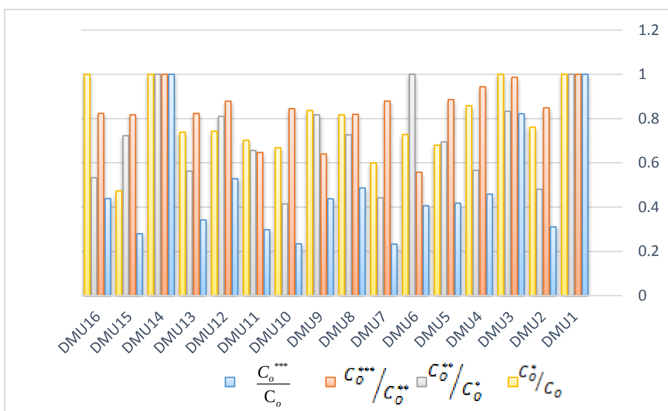

Figure 11: Relation between $\frac{C_o^*}{C_o}$, $\frac{C_o^{**}}{C_o^*}$, $\frac{{C_o}^{***}}{{C_o}^{**}}$, and $\frac{{C_o}^{***}}{C_o}$ in a non-competitive space

Table 18: Network technical, price, allocative, and cost efficiency scores of the airlines under study

DMU	C_o^*/C_o	C_o^{**}/C_o^*	C_o^{***}/C_o^{**}	$\frac{C_o{}^{***}}{C_o}$
1	1	1	1	1
2	0.761018	0.480405	0.848461	0.310194
3	1	0.832663	0.986311	0.821265
4	0.857905	0.566078	0.943392	0.45815
5	0.679193	0.694269	0.885392	0.4175
6	0.727963	1	0.557052	0.405514
7	0.599217	0.441604	0.878863	0.232562
8	0.816807	0.726542	0.819128	0.486107
9	0.836626	0.816739	0.639778	0.437163
10	0.66776	0.414244	0.844518	0.233607
11	0.70209	0.655727	0.64645	0.297612
12	0.742521	0.809498	0.878386	0.527971
13	0.738389	0.562513	0.822964	0.341821
14	1	1	1	1
15	0.473104	0.722436	0.817166	0.279297
16	1	0.532371	0.823086	0.438187

Table 19: Causes of cost loss in the airlines under study

DMU	L_o^*	L_o^{**}	${L_o}^{***}$
1	0	0	0
2	43.7911	72.457	10.1519
3	0	32.4215	2.2085
4	12.93629	33.89091	2.5028
5	21.81485	14.12025	3.6749
6	21.01754	0	24.9125
7	51.94152	43.36418	4.1543
8	11.99915	14.63025	7.0306
9	9.09991	8.53999	13.7101
10	50.6168	59.5908	6.5524
11	21.21122	17.20978	11.589
12	15.5569	8.5465	4.4166
13	16.95238	20.93272	4.7649
14	0	0	0
15	26.97708	6.72342	3.1995
16	0	17.0217	3.4283

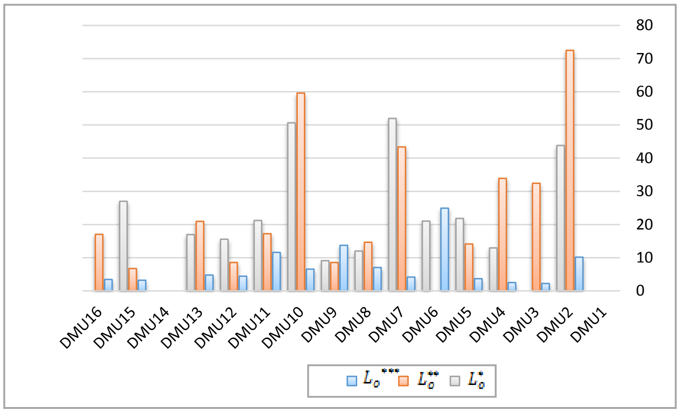

Figure 12: Relation between L_o^{*},L_o^{**} and ${L_o}^{***}$ in a non-competitive space

5 Conclusion

In cost efficiency measurement, the value and price of the inputs in the unit under evaluation are considered as the factors affecting efficiency. Given that some units have a network structure, in the present paper, we determined the technical and cost efficiencies for such units with explicit, identical, and precise input prices.

However, since the input prices might vary from one unit to another, it is necessary to introduce a new PPS and evaluate the DMUs in a cost space.

In this study, to evaluate the cost efficieny of DMUs with network structures, we used two numerical examples, one of which was an example used by Lozano in [15], and the other a case study relating a number of airlines. In this respect, first, by obtaining the network cost efficiency through Lozano's method [15] with identical input prices for all units, it was demonstrated that when the input price vector is doubled, the cost efficiency scores remain unchanged. Now, the same results are also achieved when the network efficiency scores are calculated for these two vectors using the method proposed in this paper, but it must be noted that when all DMUs have the same input prices, our proposed method yields lower cost efficiency scores than Lozano's method. In addition, we calculated the cost efficiency of each network unit again using both methods in a case where input prices varied from one DMU to another. A comparison between the results produced by the two methods in this case revealed that the methods did not yield identical network cost efficiency scores (although, note that in a non-competitive space, using Lozano's method to calculate the cost efficiency would be problematic, because if among our DMUs, we have the two units DMU_A and DMU_B with equal inputs and outputs (x_A=x_B and $y_A = y_B$), Lozano's model would yield the same network cost efficiency for both units, even when one DMU has higher input prices). Next, the causes of cost loss in each network decision-making unit were outlined, i.e. it was determined that the cost loss is due which type of inefficiency, technical, price, or allocative. Therefore, considering the fact that real-world data are usually imprecise, we suggest investigating the revenue efficiency, cost efficiency, and profit efficiency models with imprecise data such as fuzzy and interval data for future research. Also, for units with network structures, we can determine the cost efficiency with explicit and imprecise input prices, again, such as fuzzy and interval data.

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