Journal of Mathematical Extension
Vol. 16, No. 3, (2022) (6)1-10
URL: https://doi.org/10.30495/JME.2022.1465
ISSN: 1735-8299
Original Research Paper

Notes on Lifting of Mannheim Partner Curve on Tangent Space TR^3

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Abstract. In this paper, firstly, we define the Mannheim partner curve of Mannheim curve with respect to the vertical, complete and horizontal lifts on space R^3 to its tangent space $TR^3 = R^6$. Secondly, we examine the Frenet-Serret aparatus $\{T^*(s), N^*(s), B^*(s), \kappa^*(s), \tau^*(s)\}$ of the Mannheim partner curve α^* according to the vertical, complete and horizontal lifts on TR^3 by depend on the lifting of Frenet-Serret aparatus $\{T(s), N(s), B(s), \kappa(s), \tau(s)\}$ of the first curve α on space R^3. In addition, we include all special cases the curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ of the Frenet-Serret aparatus of the Mannheim partner curve α^* with respect to the vertical, complete and horizontal lifts on space R^3 to its tangent space TR^3. As a result of this transformation on space R^3 to its tangent space TR^3, we can speak about the features of Mannheim partner curve of any curve on space TR^3 by looking at the characteristics of the first curve α.

AMS Subject Classification: 28A51; 53A04; 57R25
Keywords and Phrases: Vector fields, Mannheim partner curve, vertical lift, complete lift, horizontal lift, tangent space

Received: November 2019; Accepted: September 2020
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1 Introduction

In differentiable gometry, lift method has an important role. Because, it is possible to generalize to differentiable structures on any space (resp. manifold) to extended spaces (resp. extended manifolds) using lift function [4, 5, 6, 7, 8, 10]. Thus, it may be extended the following theorem given on space R^3 to its tangent space TR^3. Mannheim partner curve of a given curve is a well-known concept in R^3. Let α and α^* be the curves in Euclidean $3-$space. It is well-known that, if a curve is diferentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve α, is called Frenet-Serret aparatus of the curves. Let Frenet vector fields be $T(s), N(s), B(s)$ of α and let the curvature and torsion of the curve $\alpha(s)$ be $\kappa(s)$ and $\tau(s)$, respectively. The quantities $\{T(s), N(s), B(s), D, \kappa(s), \tau(s)\}$ are collectively Frenet-Serret aparatus of the curves. Let a rigid object move along a regular curve described parametrically by $\alpha(s)$. For any unit speed curve α, in terms of the Frenet-Serret aparatus, the Darboux vector can be expressed as [1]

$$D(s) = \tau(s)T(s) + \kappa(s)B(s)$$

where curvature and torsion function are defined by $\kappa(s) = \| T^{'}(s) \|$ and $\tau(s) = -\left\langle B^{'}(s), N(s) \right\rangle$.

Definition 1.1. Let $\alpha : I \to R^3$ and $\alpha^* : I \to R^3$ be the C^2-class differentiable unit speed curves. In addition, $\{T(s), N(s), B(s), \kappa, \tau\}$ and $\{T^*(s), N^*(s), B^*(s), \kappa^*(s), \tau^*(s)\}$ are collectively Frenet-Serret aparatus of the curves α and α^*, respectively. If the principal normal vector N of the curve α is the linearly dependent on the binormal vector B of the curve α^*, then (α) is called a Mannheim curve and (α^*) a Mannheim partner curve of (α). The pair (α, α^*) is said to be Mannheim pair [2, 3]. *It is just known that a space curve in R^3 is a Mannheim curve if and only if its curvature $\kappa(s)$ and torsion $\tau(s)$ satisfy the formula $\kappa(s)$ $= \lambda(\kappa^2(s) + \tau^2(s))$, where λ is a nonzero constant.* The relations between the Frenet frames $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$

are as follows:

$$\alpha^*(s) = \alpha(s) - \lambda N(s) \tag{1}$$

$$\begin{aligned} T^* &= \cos\theta T - \sin\theta B \\ N^* &= \sin\theta T + \cos\theta B \\ B^* &= N, \end{aligned} \tag{2}$$

$$\begin{aligned} \cos\theta &= \frac{ds^*}{ds} \\ \sin\theta &= \lambda\tau^*\frac{ds^*}{ds} \end{aligned}$$

where $\measuredangle(T, T^*) = \theta, \forall s \in I, \lambda =$nonzero constant [3].

Theorem 1.2. *The distance between corresponding points of the Mannheim partner curves in in R^3 is constant [2].*

Theorem 1.3. *Let (α, α^*) be a Mannheim pair curves in R^3. For the curvatures and torsions of the Mannheim curve pair (α, α^*) we have [2, 3]*

$$\begin{aligned} \kappa &= \tau^*\sin\theta\frac{ds^*}{ds} \\ \tau &= -\tau^*\cos\theta\frac{ds^*}{ds} \end{aligned}$$

$$\begin{aligned} \kappa^* &= \frac{d\theta}{ds^*} = \theta'\frac{\kappa}{\lambda\tau\sqrt{\kappa^2+\tau^2}} \\ \tau^* &= (\kappa\sin\theta - \tau\cos\theta)\frac{ds^*}{ds} = \frac{\kappa}{\lambda\tau} \end{aligned} \tag{3}$$

where T, N, B, κ, τ is respectively tangent vector, normal vector, binormal vector, curvature and torsion of the curve $\alpha(s)$.

The paper is structured as follows. In section 2, the vertical, complete and horizontal lifts of a vector field defined on any manifold M of dimension m and their lift properties will be extended to space TR^3. In section 3, vertical, complete and horizontal lifts of the Mannheim partner curve $\alpha^*(s)$ obtained, respectively. Later, we include some special cases the curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ of the Mannheim partner curve with respect to lifts on space R^3 to its tangent space TR^3.

In this study, all geometric objects will be assumed to be of class C^∞ and the sum is taken over repeated indices. Also, v, c and H denote the vertical, complete and horizontal lifts any differentiable geometric structures defined on R^3 to its tangent space TR^3, respectively.

2 Lift of Vector Field

The vertical lift of a vector field X on space R^3 to extended space $TR^3(= R^6)$ is the vector field $X^v \in \chi(TR^3)$ given as [4, 10]:

$$X^v(f^c) = (Xf)^v, \ \forall f \in F(R^3)$$

The vector field $X^c \in \chi(TR^3)$ defined by

$$X^c(f^c) = (Xf)^c, \ \forall f \in F(R^3)$$

is called the complete lift of a vector field X on R^3 to its tangent space TR^3.

The horizontal lift of a vector field X on space R^3 to TR^3 is the vector field $X^H \in \chi(TR^3)$ determined by

$$X^H(f^v) = (Xf)^v, \ \forall f \in F(R^3)$$

the general properties of vertical, complete and horizontal lifts of a vector field on R^3 as follows:

Proposition 2.1. *[8, 9, 10]Let be functions all $f, g \in F(R^3)$ and vector* *fields all $X,Y\in\chi(R^3)$. Then it is satisfied the following equalities.*

$$\begin{aligned}(X+Y)^v &= X^v+Y^v,\ \ (X+Y)^c=X^c+Y^c, \\ (X+Y)^H &= X^H+Y^H,(fX)^v=f^v+X^v, \\ (fX)^c &= f^cX^v+f^vX^c,\ X^v(f^v)=0, \\ (fg)^H &= 0, X^c(f^v)=X^v(f^c)=(Xf)^v, \\ X^c(f^c) &= (Xf)^c, X^H(f^v)=(Xf)^v, \\ \chi(U) &= Sp\left\{\frac{\partial}{\partial x^\alpha}\right\},\ \chi(TU)=Sp\left\{\frac{\partial}{\partial x^\alpha},\frac{\partial}{\partial y^\alpha}\right\}, \\ \left(\frac{\partial}{\partial x^\alpha}\right)^c &= \frac{\partial}{\partial x^\alpha},\left(\frac{\partial}{\partial x^\alpha}\right)^v=\frac{\partial}{\partial y^\alpha},\ \left(\frac{\partial}{\partial x^\alpha}\right)^H=\frac{\partial}{\partial x^\alpha}-\chi\Gamma^\alpha_\beta\frac{\partial}{\partial y^\alpha}.\end{aligned} \tag{4}$$

where Γ^α_β are Christoper symbols, U and TU are respectively topological opens of R^3 and TR^3, $f^v, f^c\in F(TR^3)$, $X^v,Y^v,X^c,Y^c,X^H,Y^H\in\chi(TR^3), 1\le\alpha,\beta\le 3$.

3 Lifting of the Mannheim Partner Curve $\alpha^*(s)$

In this section, we compute the vertical, complete and horizontal lifts of the Mannheim partner curve $\alpha^*(s)$ with the curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ on space R^3 to TR^3.

3.1 The Vertical, Complete and Horizontal Lifting of the Mannheim Partner Curve $\alpha^*(s)$

Theorem 3.1. *Let $\alpha^*(s)$ be the Mannheim partner curve of the curve α on space R^3. Then, we get the following equalities with respect to vertical, complete and horizontal lifts on TR^3.*

i) $\alpha_1^*(s)=(\alpha^*(s))^v=\alpha(s)-\lambda(N(s))^v$,
ii) $\alpha_2^*(s)=(\alpha^*(s))^c=(\alpha(s))^c$,
iii) $\alpha_3^*(s)=(\alpha^*(s))^H=0$,
where λ =nonzero constant, $\forall s\in I$.

Proof. *i)* For a unit speed curve $\alpha_1^*(s)=(\alpha^*(s))^v$ on TR^3 with curvature $(\kappa^*(s))^v$ and torsion $(\tau^*(s))^v$ on TR^3. Let f be a function and λ =nonzero constant. Then we write $f^{\ v}=f$ and $\lambda^v=\lambda$ with respect to vertical lifts on tangent bunde TR^3 [10]. If we apply vertical lift to both side the equation (1), we get

$$\begin{aligned}(\alpha^*(s))^v &= (\alpha(s))^v - (\lambda)^v(N(s))^v \\ &= \alpha(s) - \lambda(N(s))^v\end{aligned}$$

ii) For a unit speed curve $\alpha_2^*(s) = (\alpha^*(s))^c$ on TR^3 with curvature $(\kappa^*(s))^c$ and torsion $(\tau^*(s))^c$ on TR^3. Let f be a function and $\lambda =$nonzero constant. Then we write $\lambda^c = 0$ with respect to complete lifts on tangent bunde TR^3[10]. If we apply complete lift to both side the equation (1), we get

$$\begin{aligned}(\alpha^*(s))^c &= (\alpha(s))^c - (\lambda)^c(N(s))^c \\ &= (\alpha(s))^c\end{aligned}$$

iii) For a unit speed curve $\alpha_3^*(s) = (\alpha^*(s))^H$ on TR^3 with curvature $(\kappa^*(s))^H$ and torsion $(\tau^*(s))^H$ on TR^3. Let f be a function and $\lambda =$nonzero constant. Then we write $f^{\ H} = 0$ and $\lambda^H = 0$ with respect to horizontal lifts on tangent bunde TR^3 [10]. If we apply horizontal lift to both side the equation (1), we get

$$\begin{aligned}(\alpha^*(s))^H &= (\alpha(s))^H - (\lambda)^H(N(s))^H \\ &= (\alpha(s))^H \\ &= 0\end{aligned}$$

□

Theorem 3.2. *Let $\alpha^*(s)$ be the Mannheim partner curve of the curve α on R^3 and a unit speed curve $\alpha_1^*(s) = (\alpha^*(s))^v$ on TR^3 with curvature $(\kappa^*(s))^v$ and torsion $(\tau^*(s))^v$ on TR^3. Then, we get the following equalities with respect to vertical lifts on tangent bunde TR^3.*

$$\begin{aligned}(T^*(s))^v &= \cos\theta(T(s))^v - \sin\theta(B(s))^v \\ \\ (N^*(s))^v &= \sin\theta(T(s))^v + \cos\theta(B(s))^v \\ \\ (B^*(s))^v &= (N(s))^v\end{aligned}$$

where T, N, B, κ, τ is respectively tangent vector, normal vector, binormal vector, curvature, torsion of the curve $\alpha(s)$, $\measuredangle(T, T^*) = \theta, \forall s \in I$.
Proof. If we apply the vertical lift to both side the equation (3) and using the Proposition 2.1, we easily get the following results.

$$\begin{aligned} (T^*(s))^v &= (\cos\theta T(s) - \sin\theta B(s))^v \\ &= (\cos\theta)^v (T(s))^v - (\sin\theta)^v (B(s))^v \\ &= \cos\theta (T(s))^v - \sin\theta (B(s))^v \end{aligned}$$

Similarly, from (3) and using the lifting properties, we get

$$\begin{aligned} (N^*(s))^v &= \sin\theta (T(s))^v + \cos\theta (B(s))^v \\ (B^*(s))^v &= (N(s))^v \end{aligned}$$

where $\measuredangle(T, T^*) = \theta, \forall s \in I$. □

Corollary 3.3. *Let the curvature κ and torsion τ of the curve $\alpha(s)$ on R^3 are constant or non-constant functions and $\alpha^*(s)$ be the Mannheim partner curve of the curve $\alpha(s)$ on R^3. For a unit speed curve $\alpha_1^*(s) = (\alpha^*(s))^v$ on with curvature $(\kappa^*(s))^v$ and torsion $(\tau^*(s))^v$ on TR^3, we say the curve $(\alpha^*(s))^v$ on TR^3 is similar structure and apperance to R^3 with respect to vertical lifts.*

Example 3.4. $\alpha^*(s)$ be the Mannheim partner curve of the a circular helis curve $\alpha(s)$ on R^3. Then $\alpha^*(s)$ has similar appearance with the curve $\alpha_1^*(s) = (\alpha^*(s))^v$ on TR^3. Because of the curvature κ and torsion τ of a circular helis curve is constant, we write $\kappa^v = \kappa$ and $(\tau)^v = \tau$. From (3) and (4), we get $(\kappa^*(s))^v = \kappa^*(s)$ and $(\tau^*(s))^v = \tau^*(s)$ on TR^3. Then, we can say the curve $\alpha_1^*(s) = (\alpha^*(s))^v$ has the same $\kappa^*(s)$ and $\tau^*(s)$ on TR^3.

3.2 The Complete and Horizontal Lifting of the Mannheim Partner Curve $\alpha^*(s)$

Theorem 3.5. *Let $\alpha^*(s)$ be the Mannheim partner curve of the curve α on R^3 and a unit speed curve $\alpha_2^*(s) = (\alpha^*(s))^c$ on TR^3 with curvature $(\kappa^*(s))^c$ and torsion $(\tau^*(s))^c$ on TR^3. Then, we get the following* *equalities with respect to complete lifts on tangent bunde TR^3.*

$$\begin{aligned}(T^*(s))^c &= (\cos\theta)^c(T(s))^v - (\sin\theta)^c(B(s))^v \\ (N^*(s))^c &= (\sin\theta)^c(T(s))^v + (\cos\theta)^c(B(s))^v \\ (B^*(s))^c &= (N(s))^c\end{aligned}$$

where T, N, B, κ, τ is respectively tangent vector, normal vector, binormal vector, curvature, torsion of the curve $\alpha(s)$, $\measuredangle(T, T^*) = \theta, \forall s \in I$.
Proof. Similarly to vertical lifts, the theorem easily proved with respect to complete lift. □

Corollary 3.6. *Let the curvature κ and torsion τ of the curve $\alpha(s)$ on R^3 are constant or non-constant functions and $\alpha^*(s)$ be the Mannheim partner curve of the curve $\alpha(s)$ on R^3. For a unit speed curve $\alpha_2^*(s) = (\alpha^*(s))^c$ on with curvature $(\kappa^*(s))^c$ and torsion $(\tau^*(s))^c$ on TR^3, we say the Frenet-Serret aparatus of the curve $(\alpha^*(s))^c$ on TR^3 is similar structure and apperance to R^3 with respect to complete lifts.*

Theorem 3.7. *Let $\alpha^*(s)$ be the Mannheim partner curve of the curve α on R^3 and a unit speed curve $\alpha_3^*(s) = (\alpha^*(s))^H$ on TR^3 with curvature $(\kappa^*(s))^H$ and torsion $(\tau^*(s))^H$ on TR^3.Then, we say the horizontal lifts of binormal vector of the Mannheim partner curve $\alpha^*(s)$ is the same direction with the horizontal lifts basic normal vector of the principal curve $\alpha(s)$ on TR^3.*

Proof. For all functions on R^3, we write $f^H = 0$ with respect to proporties of the horizontal lifts on TR^3 [10]. So, we can write $(\cos\theta)^H = 0$, $(\sin\theta)^H = 0$. Consequently, we get $(B^*(s))^H = (N(s))^H$. □

Theorem 3.8. *Let $\alpha^*(s)$ be the Mannheim partner curve with curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ on R^3. Then, the curvature $(\kappa^*(s))^v$ and torsion $(\tau^*(s))^v$ of a unit speed curve $(\alpha^*(s))^v$ on TR^3 is the same as the curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ of Mannheim partner curve $\alpha^*(s)$, respectively, with respect to vertical lifts on TR^3.*

Proof. Let the curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ be constant or nonconstant functions. If we apply a vertical lift to both sides of the equality

(3) and using the properties of vertical lifts, we easily get the following results.

$$
\begin{aligned}
(\kappa^*)^v &= \frac{(\theta^{'})^v(\kappa)^v}{(\lambda)^v(\tau)^v\sqrt{(\kappa^2)^v+(\tau^2)^v}} = \frac{\theta^{'}\kappa}{\lambda\tau\sqrt{\kappa^2+\tau^2}} = \kappa^* \\
(\tau^*)^v &= \frac{(\kappa)^v}{(\lambda)^v(\tau)^v} = \frac{\kappa}{\lambda\tau} = \tau^*
\end{aligned}
$$

where κ, τ is respectively curvature and torsion of the curve $\alpha(s)$, $\lambda =$nonzero constant, $\measuredangle(T, T^*) = \theta$.

Similarly to vertical lifts, using the lifting properties, we get the following results. $\square$

Corollary 3.9. *Let $\alpha^*(s)$ be the Mannheim partner curve with curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ on R^3. Then, the curvature $(\kappa^*(s))^c$ and torsion $(\tau^*(s))^c$ of the curve $(\alpha^*(s))^c$ are already zero on TR^3.*

Corollary 3.10. *Let $\alpha^*(s)$ be the Mannheim partner curve with curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ on R^3. Then, the curvature $(\kappa^*(s))^H$ and torsion $(\tau^*(s))^H$ of the curve $(\alpha^*(s))^H$ are already zero on TR^3.*

Proof. For all functions on R^3, we write $f^H = 0$ with respect to proporties of the horizontal lifts on TR^3 [10]. So, we can write $(\kappa^*(s))^H = (\tau^*(s))^H = 0$. $\square$

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