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Original Research Paper

Domination of Vague Graphs by Using of Strong Arcs

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Abstract. The present paper aims to introduce the concepts of adding a strong arc, cobondage set, cobondage number, t-cobondage set, and t-cobondage number in the vague graphs, as well as expressing some of the new segmentation of the additions of arc and reduce the effect of adding a strong arc on domination parameters in vague graphs. Finally, some of their applications are pinpointed.

AMS Subject Classification: 05C99.
Keywords and Phrases: Fuzzy graph, vague graph, additional arc, cobondage numbers.

1 Introduction

Euler first introduced the concept of graph theory in 1736. In the history of mathematics, the solution given by Euler regarding the well-known Konigsberg bridge problem is considered to be the first theorem of graph

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theory,which has now become a subject considered as a branch of combinatorics. The theory of graph is regarded as an extremely useful tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, operations research, optimization, and computer science. Cockayne and Hedetniemi [7] introduced the domination number and the independence number. In addition Zadeh [19] first proposed the theory of fuzzy sets. Then, Rosenfeld [16] introduced the concept of fuzzy graph theory as a generalization of Euler's graph. Gau and Buehrer [8] proposed the concept of vague set by replacing the value of an element in a set with a subinterval of $[0, 1]$. Namely, a true-membership function $t_v(x)$ and a false membership function $f_v(x)$ are used to describe the bounderies of the membership degree. Further, Janakiram and Kulli [9] suggested the concept of the cobondage number in graphs. Accordingly, Ramakrishna [11] introduced the concept of vague graphs, along with some of their properties. In another study, Somasundram [18] proposed the concept of domination in fuzzy graphs. Parvathi and Thamizhendhi [10] introduced domination in intuitionistic fuzzy graphs. In addition, Nagoor Gani and Prasanna Devi [10] suggested the reduction in the domination number of fuzzy graph. Borzooei and Rashmanlou studied different types of dominating set in vague graphs [3, 4, 5, 6, 13, 14, 15]. By considering the above-mentioned studies, the present paper seek to introduce the concepts of editions of an arc, cobondage sets, and cobondage numbers in vague graphs. Further, the new segmentation of the editions of arc is discussed in the vague graphs.

2 Preliminaries

In this section, we review some definitions and results from [3, 4, 9, 12], which we need in what follows.

A *graph* is an ordered pair $G^* = (V, E)$, where V is the set of vertices and E is the set of edges. A *subgraph* of a graph $G^* = (V, E)$ is a graph $H^* = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A *fuzzy graph* $G = (\sigma, \mu)$ on simple graph $G^* = (V, E)$ is a pair of functions $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$ where, for any $uv \in E$, $\mu(uv) \leq \min\{\sigma(u), \sigma(v)\}$. A vague set A on non-empty set X, is a pair (t_A, f_A), where $t_A : X \to [0,1]$ and $f_A : X \to [0,1]$ are true and false membership functions, respectively, such that for all $x \in X$, $0 \leq t_A(x) + f_A(x) \leq 1$. Note that $t_A(x)$ is considered as the lower bound for degree of membership of x in A and $f_A(x)$ is the lower bound for degree of non-membership of x in A. So, the degree of membership of x in the vague set A, is characterized by the interval $[t_A(x), 1 - f_A(x)]$. Hence, a vague set is a special case of interval-valued sets studied by many mathematicians and applied in many branches of mathematics.

It is worth to mention here that interval-valued fuzzy sets are not vague sets. In interval-valued fuzzy sets, an interval-valued membership value is assigned to each element of the universe considering the evidence for x only, without considering evidence against x. In vague sets both are independently proposed by the decision making. This makes a major difference in the judgment about the grade of membership.

A *vague graph* on simple graph $G^* = (V, E)$, is defined to be a pair $G = (A, B)$, where $A = (t_A, f_A)$ is a vague set on V and $B = (t_B, f_B)$ is a vague set on E such that for any edge $xy \in E$,

$$t_B(xy) \leq \min\{t_A(x), t_A(y)\} \quad , \quad f_B(xy) \geq \max\{f_A(x), f_A(y)\}.$$

Let $G = (A, B)$ be a vague graph on simple graph $G^* = (V, E)$. Then,
(i) the *vertex cardinality* of G is defined by,

$$|V| = \sum_{v_i \in V} \frac{t_A(v_i) + (1 - f_A(v_i))}{2},$$

(ii) the *edge cardinality* of G is defined by,

$$|E| = \sum_{v_i v_j \in E} \left(\frac{t_B(v_i v_j) + (1 - f_B(v_i v_j))}{2}\right),$$

(iii) the *cardinality* of G is defined by,

$$|G| = |V| + |E|$$

(iv) for any $U \subset V$, the *vertex cardinality* of U is denoted by $O(U)$ and defined by,

$$O(U) = \sum_{v_i \in U} \left(\frac{1 + t_A(v_i) - f_A(v_i)}{2}\right),$$

(v) for any $F \subset E$, the *edge cardinality* of F is denoted by $S(F)$ and defined by,

$$S(F) = \sum_{v_iv_j \in F} \left(\frac{1 + t_B(v_iv_j) - f_B(v_iv_j)}{2}\right).$$

Let $G = (A, B)$ be a vague graph on simple graph $G^* = (V, E)$ and $u, v \in V$. Then,
(i) *t-strength of connectedness* between u and v is

$$t_B^{\infty}(uv) = \sup\{t_B^k(uv) \big| k = 1, 2, \cdots, n\},$$

and

$$t_B^k(uv) = \min\{ \quad t_B(ux_1), t_B(x_1x_2), \cdots, t_B(x_{k-1}v) \big| \\ u, x_1, \cdots, x_{k-1}, v \in V, k = 1, 2, \cdots, n\}.$$

(ii) *f-strength of connectedness* between u and v is

$$f_B^{\infty}(uv) = \inf\{f_B^k(uv) \big| k = 1, 2, \cdots, n\},$$

and

$$f_B^k(uv) = \max\{ \quad f_B(ux_1), f_B(x_1x_2), \cdots, f_B(x_{k-1}v) \big| \\ u, x_1, \cdots, x_{k-1}, v \in V, k = 1, 2, \cdots, n\}.$$

Let $G = (A, B)$ be a vague graph on simple graph $G^* = (V, E)$. An edge $uv \in E$ is said to be *strong edge* if

$$t_B(uv) \geq t_B^{\infty}(uv) \;\; and \;\; f_B(uv) \leq f_B^{\infty}(uv),$$

A *path* (or $u - v$ path) P in G is a sequence of distinct vertices $u = v_1, v_2, \cdots, v_n \in V = v$ such that either one of the following conditions is satisfied:
(i) $t_B(v_iv_j) > 0$ and $f_B(v_iv_j) = 0$, for some $1 \leq i, j \leq n$,
(ii) $t_B(v_iv_j) = 0$ and $f_B(v_iv_j) > 0$, for some $1 \leq i, j \leq n$,
(iii) $t_B(v_iv_j) > 0$ and $f_B(v_iv_j) > 0$, for some $1 \leq i, j \leq n$.

Let P be a $u-v$ path in a vague graph G. Then P is called a *strongest* $u - v$ *path* in G, if t-strength of P is a maximum and f-strength of P is a minimum among all paths between u and v.

An edge e in a vague graph G is called
(i) *t-bridge*, if deleting e reduce the t-strength of connectedness between some pair of vertices.
(ii) *f-bridge*, if deleting e increases the f-strength of connectedness between some pair of vertices.
(i) *vague bridge*, if it is a t-bridge and f-bridge.
Let $G = (A, B)$ be a vague graph on simple graph $G^* = (V, E)$ and $u, v \in V$. Then,
(i) we say that u dominate v in G, if there exists a strong edge between u and v.
(ii) $S \subset V$ is called a *dominating set* in G, if for any $v \in V - S$, there exists $u \in S$ such that u dominates v.
(iii) a dominating set S in G is called *minimal dominating* set if no proper subset of S is a dominating set.
(iv) minimum vertex cardinality among all minimal dominating sets of G is called *lower domination number* of G and is denoted by $d_V(G)$.
(v) maximum vertex cardinality among all minimal dominating sets of G is called *upper domination number* of G and is denoted by $D_V(G)$.

Let $G = (A, B)$ be a vague graph on simple graph $G^* = (V, E)$ without isolated vertices. Then,
(i) $S \subset V$ is called a *total dominating set* in G, if for any $v \in V$, there exists $u \in S$ such that $u \neq v$ and u dominates v.
(ii) a total dominating set S of G is called a *minimal total dominating set* if no proper subset of S is a total dominating set of G.
(iii) minimum vertex cardinality among all minimal total dominating sets of G is called *lower total domination number* of G and is denoted by $t_V(G)$.
(iv) maximum vertex cardinality among all minimal total dominating sets of G is called *upper total domination number* of G and is denoted by $T_V(G)$.

Let $G = (A, B)$ be a vague graph on simple graph $G^* = (V, E)$. Then,
(i) two vertices $u,\ v \in V$ are called *independent* if there is no any strong edge between them.
(ii) $S \subset V$ is called a *independent set* in G, if for any $u, v \in S$,

$$t_B(uv) < (t_B)^\infty(uv) \text{ and } f_B(uv) > (f_B)^\infty(uv).$$

(iii) an independent set S in G is said called *maximal independent set* if for any vertex $v \in V - S$, the set $S \cup \{v\}$ is not independent.
(iv) minimum vertex cardinality among all maximal independent set is called *lower independent number* of G and is denoted by $i_V(G)$.
(v) maximum vertex cardinality among all maximal independent set is called upper independent number of G and is denoted by $I_V(G)$.

A connected vague graph G on simple graph G^* is said to be *firm* if

$$\max\{t_B(uv) \mid uv \in E\} \leq \min\{t_A(u) \mid u \in V\}$$

and

$$\min\{f_B(uv) \mid uv \in E\} \geq \max\{f_A(u) \mid u \in V\}.$$

The cobondage set of a fuzzy graph G is the set C of additional strong arcs to G, that reduce the domination number of G and the cobondage number of fuzzy graph G is the smallest number of arcs in any cobondage set of G.

Notation. From now one, in this paper we let $G = (A, B)$ be a vague graph on simple graph $G^* = (V, E)$.

3 Study of domination by addition of strong arcs

In this section we discuss about domination of vague graph by adding a strong arc to this vague graph.

If in graph $G^* = (V, E)$, we add an arc e to E, then we denote it by $E_e = E \cup \{e\}$ and $G^*_e = (V, E_e)$ and we say it is an additional arc. Moreover, if vague graph $G = (A, B)$ on G^* extend to G^*_e, then we denoted it by $G_e = (A_e, B_e)$.

Example 3.1. Consider a vague graph $G = (A, B)$ on simple graph $G^* = (V, E)$, where $V = \{a, b, c, d, k\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$, as in Figure 1. If we add an arc e_6 to G^* and define $t_B(e_6) = 0.4$ and $f_B(e_6) = 0.5$, then $G_{e_6} = (A_{e_6}, B_{e_6})$ is a vague graph on simple graph $G^*_{e_6} = (V_{e_6}, E_{e_6})$ and e_6 is an strong arc.

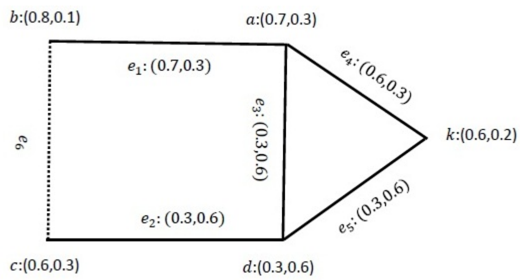

Figure 1: Vague graph G on G^*

Note 3.2. Arc e in vague graph $G_e = (A_e, B_e)$ is the strong arc if and only if there exists $u, v \in V$ such that $u - v$ path of G_e that includes $e = xy$ is a strongest path between two nodes u and v.

Note 3.3. If arc e in vague graph G_e is a vague bridge, then e is a strong arc.

Notation. If arc e in vague graph G_e is an strong arc, then we denote $G_e^s = (A_e^s, B_e^s)$ instead of $G_e = (A_e, B_e)$.

Sometimes, by adding an edge to the vague graph G, only the upper domination number is changed and the lower domination number still remains constant and sometimes vic versa. Now, since the main purpose of this paper is to investigate the effect of adding edge on the domination of the vague graph, we define the domination number of vague graph as follows.

Definition 3.4. Let $G = (A, B)$ be a vague graph. The domination number of G is denoted by $\Delta_V(G)$ and is defined as follows,

$$\Delta_V(G) = \frac{d_V(G) + D_V(G)}{2}.$$

Example 3.5. Consider a vague graph $G = (A, B)$ on simple graph $G^* = (V, E)$, as in Figure 2.

Then, it is clear that $d_V(G_2) = 0.75$ and $D_V(G_2) = 1.2$ and so,

$$\Delta_V(G_2) = \frac{0.75 + 1.2}{2} = 0.975.$$

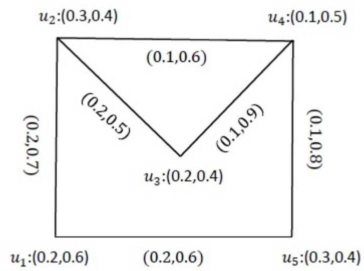

Figure 2: Vague graph G on G^*

Theorem 3.6. *Let $e = uv$ be an additional strong arc in G_e^s. Then*
(i) $\Delta_V(G_e^s) \leq \Delta_V(G)$.
(ii) $0 \leq \Delta_V(G) - \Delta_V(G_e^s) \leq \max\{|\{u\}|, |\{v\}|\}$.

Proof. (i) Assume that S is a minimal dominating set of G and $e = uv$ be an additional strong arc in G_e^s. If u or v is an isolated node, then $S - \{u\}$ or $S - \{v\}$ is a minimal dominating set in G_e^s. Otherwise, S is a minimal dominating set in G_e^s. Hence, $d_V(G_e^s) \leq d_V(G)$ and $D_V(G_e^s) \leq D_V(G)$. Therefore, $\Delta_V(G_e^s) \leq \Delta_V(G)$.
(ii) By the proof of (i), we have:

$$0 \leq d_V(G) - d_V(G_e) \leq \max\{|\{u\}|, |\{v\}|\}$$

and

$$0 \leq D_V(G) - D_V(G_e) \leq \max\{|\{u\}|, |\{v\}|\}.$$

Then

$$0 \leq \Delta_V(G) - \Delta_V(G_e^s) \leq \max\{|\{u\}|, |\{v\}|\}.$$

□

Definition 3.7. The independent number of G is denoted by $I(G)$ and is defined as follows,

$$I(G) = \frac{i_V(G) + I_V(G)}{2}.$$

Example 3.8. Consider the vague graph G as Figure 2. By routine calculations, it is easy to see that, $i_V(G) = 0.9$ and $I_V(G) = 1.15$. Then, $I(G) = \dfrac{0.9 + 1.15}{2} = 1.025$.

Proposition 3.9. *Let e be an additional strong arc in G_e^s. Then $I(G_e^s) \leq I(G)$.*
Proof. *Straightforward.* $\square$

Now, we shall discuss about the types of addition of arcs in vague graphs.

Definition 3.10. Let $e = uv$ be an additional strong arc in G_e^s. Then $e = uv$ is called
(i) an α-strong arc in G_e if

$$t_{B^e}(uv) > t^{\infty}_{B^e}(uv) \quad , \quad f_{B^e}(uv) < f^{\infty}_{B^e}(uv).$$

(ii) a β-strong arc in G_e if

$$t_{B^e}(uv) = t^{\infty}_{B^e}(uv) \quad , \quad f_{B^e}(uv) = f^{\infty}_{B^e}(uv).$$

(iii) a δ-arc in G_e if

$$t_{B^e}(uv) < t^{\infty}_{B^e}(uv) \quad , \quad f_{B^e}(uv) > f^{\infty}_{B^e}(uv).$$

Example 3.11. Consider a vague graph $G = (A, B)$ on simple graph $G^* = (V, E)$, as Figure 3.

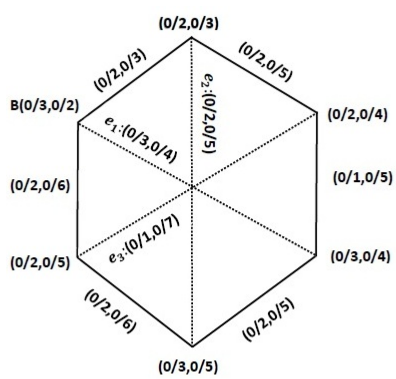

Figure 3: Vague graph G on G^*

Consider additional arcs e_1, e_2 and e_3 in Figure 3. Then e_1 is an α-strong arc, e_2 is a β-strong arc and e_3 is a δ-arc.

Theorem 3.12. *If G is a firm and e is an additional β-strong arc or δ-arc of G_e^*, then G_e is a firm, too.*

Proof. If G is a firm, then,

$$\max\{t_B(uv) \mid (u,v) \in V \times V\} \leq \min\{t_A(u) \mid u \in V\},$$

and

$$\min\{f_B(uv) \mid (u,v) \in V \times V\} \geq \max\{f_A(u) \mid u \in V\}.$$

Now, if $e = xy$ is a β-strong arc of G_e, then

$$\begin{aligned} t_B(e) &= t_B(xy) = t_B^{\infty}(xy) \\ &\leq \max\{t_B(uv) \mid (u,v) \in V \times V\} \leq \min\{t_A(u) \mid u \in V\}, \end{aligned}$$

and

$$\begin{aligned} f_B(e) &= f_B(xy) = f_B^{\infty}(xy) \\ &\geq \min\{f_B(uv) \mid (u,v) \in V \times V\} \geq \max\{f_A(u) \mid u \in V\}. \end{aligned}$$

Similarly, if $e = xy$ is a δ-arc of G_e, then

$$\begin{aligned} t_B(e) &= t_B(xy) < t_B^{\infty}(xy) \\ &\leq \max\{t_B(uv) \mid (u,v) \in V \times V\} \leq \min\{t_A(u) \mid u \in V\}, \end{aligned}$$

and

$$\begin{aligned} f_B(e) &= f_B(xy) > f_B^{\infty}(xy) \\ &\geq \min\{f_B(uv) \mid (u,v) \in V \times V\} \geq \max\{f_A(u) \mid u \in V\}. \end{aligned}$$

Hence, G_e is a firm. □

In the following example, we show that for any α-strong arc, Theorem 3.11 may not be true, in general.

Example 3.13. Vague graph G in Figure 4, is a firm and additional arc e_4 is an α-strong arc, but vague graph G_e in Figure 4 is not a firm, since $t_B(e_4) > 0.3$ and $f_B(e_4) < 0.4$.

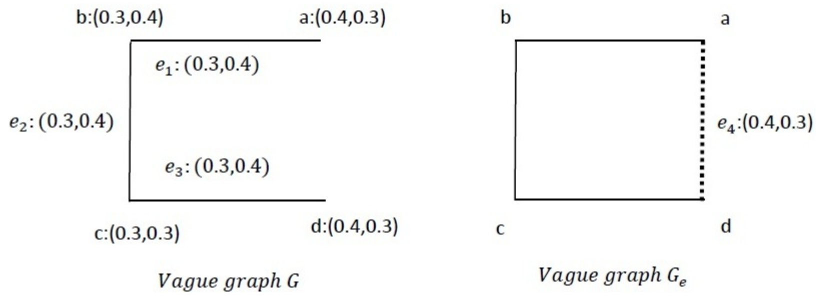

Figure 4: Vague graphs G, G_e

Theorem 3.14. *Let G be a vague graph and e be an additional arc in G_e^*. Then e is an α-strong arc if and only if there exists nodes u and v such that $u - v$ path of G_e that includes e is an unique strongest path between two nodes u and v.*

Proof. Let $e = xy$ be an α-strong arc in G_e. Then,

$$t_B(xy) > t_B^{\infty}(xy) \quad , \quad f_B(xy) < f_B^{\infty}(xy)$$

If we let $u = x$ and $v = y$, then the proof is clear.

Conversely, if there exists nodes u, v such that $u - v$ path P_e of G_e that includes $e = xy$ is an unique strongest path between two nodes u and v, then for any $x - y$ path P without arc $e = xy$ in G, we have:

$$t_B(xy) > t_P(xy) \quad , \quad f_B(xy) < f_P(xy)$$

Hence,

$$t_B(xy) > t_B^{\infty}(xy) \quad , \quad f_B(xy) < f_B^{\infty}(xy).$$

Therefore, $e = xy$ is an α-strong arc in G_e. □

Note 3.15. Additional arc $e = uv$ in G_e^* is an $\alpha - strong$ arc if and only if $e = uv$ is a vague bridge in vague graph G_e.

Remark 3.16. Let $e = uv$ be an additional $\alpha - strong$ arc in G_e. Then, e is belong to the maximum spaning tree of G_e.

4 Cobondage numbers of a vague graph.

In this section, we discuss about cobondage set and cobondage number of a vague graph.

Definition 4.1. (i) The cobondage set of a vague graph G is the set C of additional strong arcs to G, that reduces the domination number, i.e,

$$\Delta_V(G_C) < \Delta_V(G).$$

(ii) A cobondage set C of G is said to be minimal cobondage set if no proper subset of C is a cobondage set.
(iii) Minimum edge cardinality among all minimal cobondage sets of G is called lower cobondage number of G and denoted by $b_E(G)$.
(iv) Maximum edge cardinality among all minimal cobondage sets of G is called upper cobondage number of G and denoted by $B_E(G)$.

Example 4.2. Consider the vague graph G in Figure 5. It is clear

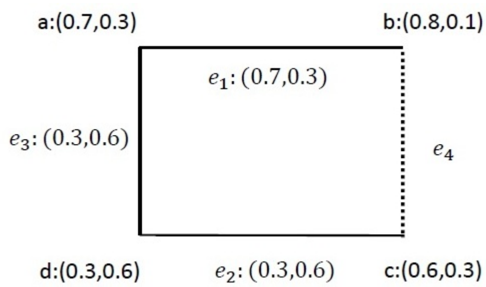

Figure 5: Vague graph G

that $D_1^* = \{a, d\}$ and $D_2^* = \{b, c\}$ are the minimal dominaiting sets of vague graph G ($d_V(G) = 1.05$, $D_V(G) = 1.50$ and $\Delta_V(G) = 1.275$). In this case, by adding $e_4 = (0.5, 0.5)$, the set $D_1 = \{c, d\}$ is a minimal dominaiting set with the cardinality of 1. Then, by adding e_5 as $bd = (0.3, 0.6)$, the set $D_2 = \{d\}$ is a minimal dominaiting set with the cardinality of 0.35, so $x_2 = \{e_5\}$ is a minimal cobondage set, and by adding e_6 as $ac = (0.6, 0.3)$, the set $D_3 = \{a\}$ is a minimal dominaiting set with the cardinality of 0/70. Thus, $x_3 = \{e_6\}$ is a minimal cobondage set and so $b_E(G)$ and $B_E(G)$ are 0.35 and 0.65, respectively.

Theorem 4.3. *If a vague graph G has an isolated node v, then*

$$b_E(G) \leq |\{v\}|.$$

Proof. Let v be an isolated node of G. Then v belongs to every minimal dominating set D of G. If $u \in D - \{v\}$ and e is an additional strong arc between v and u, then, $D - \{v\}$ is a minimal dominating set of G_e and $d_V(G_e) < d_V(G)$. Thus, $\Delta_V(G_e) < \Delta_V(G)$. Hence, by definition of additional arc, we have, $t_B(e) \leq t_A(v)$ and $f_B(e) \geq f_A(v)$. Hence, $S(e) \leq \dfrac{1 + t_A(v) - f_A(v)}{2}$ and so

$$b_E(G) \leq \frac{1 + t_A(v) - f_A(v)}{2} = |\{v\}|.$$

□

Theorem 4.4. *If G has not isolated node and $e = uv$ is an isolated edge, then*

$$b_E(G) \leq |\{u\}| + |\{v\}|.$$

Proof. If $e = uv$ is an isolated edge in G, then one of u or v belongs to every minimal dominating set D of G. Let $u \in D$ and $w \in D - \{u\}$. By adding the strong arcs $e_1 = (uw)$ and $e_2 = (vw)$, the set $D - \{u\}$ is a minimal dominating set of G_e. Moreover, if $C = \{e_1, e_2\}$, then $\Delta_V(G_C) < \Delta_V(G)$. Thus,

$$b_E(G) = S(C) \leq \frac{1 + t_A(u) - f_A(u)}{2} + \frac{1 + t_A(v) - f_A(v)}{2} = |\{u\}| + |\{v\}|.$$

□

Definition 4.5. The t-domination number of G is denoted by $\Delta_V^t(G)$ and is defined as follows,

$$\Delta_V^t(G) = \frac{t_V(G) + T_V(G)}{2}.$$

Definition 4.6. (i) The t-cobondage set of a vague graph G is the set C_t of additional strong arcs to G that reduces the t-domination number, i.e, $\Delta_V^t(G_{C_t}) < \Delta_V^t(G)$.

(ii) A t-cobondage set C_t of G is called a minimal t-cobondage set if no proper subset of C_t is a t-cobondage set.

(iii) Minimum edge cardinality among all minimal t-cobondage sets of G is called a lower t-cobondage number of G and denoted by $b^t_E(G)$.

(iv) Maximum edge cardinality among all minimal t-cobondage sets of G is called an upper t-cobondage number of G and denoted by $B^t_E(G)$.

Example 4.7. In Figure 2, the set $\{u_1, u_2\}$ is a total dominating set and $\Delta^t_V(G) = 0.75$. By adding the strong arc $u_1u_3 = (0.2, 0.6)$, the set $\{u_1, u_3\}$ is a total dominating set and $\Delta^t_V(G) = 0.7$. Thus $\{u_1u_3\}$ is a t-cobondage set.

Remark 4.8. Consider the next example show that t-cobondage set and cobondage set are not necessarily equivalent.

Example 4.9. Consider the vague graph G in Figure 6. In Figure 6,

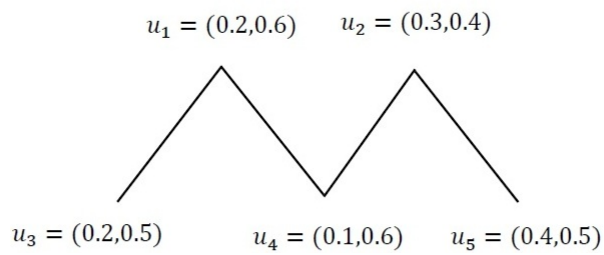

Figure 6: Vague graph G

$D_1 = \{u_1, u_2\}$, $D_2 = \{u_1, u_5\}$ and $D_3 = \{u_2, u_3\}$ are minimal dominating sets, then $C_1 = \{u_1u_2, u_2u_3\}$ and $C_2 = \{u_1u_2, u_1u_5\}$ are cobondage sets. Also, $D_t = \{u_1, u_2, u_4\}$ is a total dominating set, then $C_t = \{u_2u_3\}$ is a t-cobondage set in G.

Theorem 4.10. *If G has an isolated edge $e = uv$, then*

$$b^t_E(G) \leq \max\big\{|\{u\}|, |\{v\}|\big\}.$$

Proof. Let $e = uv$ be an isolated edge in G. Then both vertices u and v belong to every minimal total dominating set D_t of G. Now by adding strong arc e' with end node u or v, $D_t - \{u\}$ or $D_t - \{v\}$ is a minimal total dominating set of $G_{e'}$. Thus, if $C_t = \{e'\}$, then

$$\Delta_V^t(G_{C_t}) < \Delta_V^t(G).$$

Hence, C_t is a t-cobondage set and we have

$$\begin{aligned} S(C_t) &= S(e') \le \max\Big\{\frac{1+t_A(u)-f_A(u)}{2}, \frac{1+t_A(v)-f_A(v)}{2}\Big\} \\ &= \max\big\{|\{u\}|, |\{v\}|\big\} \end{aligned}$$

and

$$b_E^t(G) \le \max\big\{|\{u\}|, |\{v\}|\big\}.$$

□

Theorem 4.11. *$b_E^t(G) \le b_E(G)$ and $B_E^t(G) \le B_E(G)$.*

Proof. Every total dominating set is a dominating set, then

$$\Delta(G) \le \Delta_t(G)$$

since, $C_t \subseteq C$, where C is a minimal cobondage set and C_t is a minimal t-cobondage set, we have $b_E^t(G) \le b_E(G)$ and $B_E^t(G) \le B_E(G)$. □

5 Application

Domination in graphs has many applications in different areas, especially in operations research, neural networks, electrical networks and monitoring communication. Therefore, it is important to minimize the dominating sets and reduce the associated parameters. By considering the application of the dominating sets as controlling or guiding sections in the above mentioned fields, examining the addition of strong arcs and reducing effects on domination parameters in vague graphs indicate that the velocity or accuracy can be increased by adding a string or a link in the above circuits, while some of domination parameters can decrease instead of being increased.

5.1 Fire stations

Consider the vague graph of Figure 6. Suppose the graph G represents the map of a city, in which the vertices representing the regions and edges represent their communication paths in the city. In this case, $D_1^* = \{A, E, F\}$ set represents the fire stations located all around the city. It is worth noting that D_1^* is the minimal dominating set of the

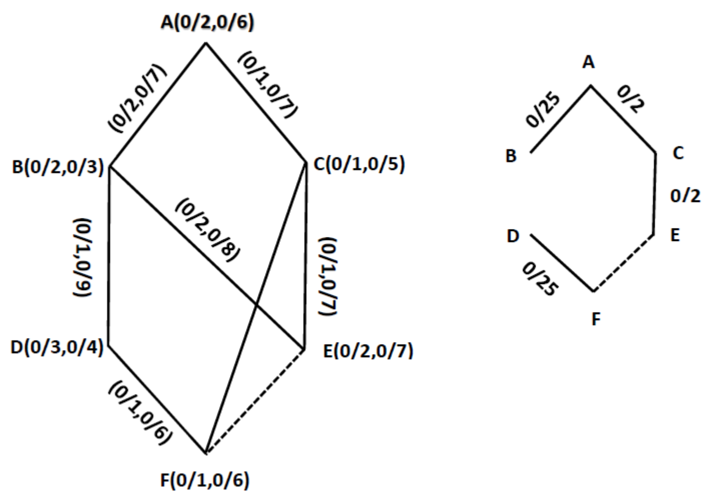

Figure 7: Vague graph G

graph G with vertex cardinality of 0.8. We define the f-strength and t-strength values in each vertex and edge(path) as follows. For each $v \in V$ and $e \in E$, we have:
$t_A(v)$: The minimum assurance of non-incidentalism in region v.
$f_A(v)$: The minimum assurance of incidentalism in region v.
$t_B(e)$: The minimum assurance of the timely presence at the incident scene through the e path.
$f_B(e)$: The minimum assurance of the timely absence at the incident scene through the e path.
Thus, the size of each vertex stands is $|v| = \dfrac{1 + t_A(v) - f_A(v)}{2}$, for each $v \in V$, which represents the optimal level of assurance of the non-incidentalism of that region. In addition, the optimum value in the

vertices A, E, F has the lowest value which justifies the location of fire stations in these areas. It is worth noting that some factors such as urban texture, type of industry and presence of high-risk industries in one area can contribute to the estimation of the incidentalism or non-incidentalism in that area. Also, for any $e \in E$, the size of each edge in the graph G stands is $|e| = \dfrac{1 + t_B(e) - f_B(e)}{2}$ which indicated the optimal amount of assurance of timely presence in the incident scene through that edge(path). We see that the AB, AC, CE, FD edges have the most assurance of timely presence and the least assurance of timely absence on the incident scene relative to other interurban routes. In other words, the strong edges are graph G. It should be noted that some factors such as the volume of traffic, number of traffic lights, squares, overpasses and pedestrian underpasses, as well as maximum and minimum speed of vehicles per path, etc., are affected by the route in estimating the assurance of timely presence or absence. In addition, by considering the interpretation of t-strength and f-strength, it is also logical to have:

$$t_B(xy) \leq \min(t_A(x), t_A(y)) \quad , \quad f_B(xy) \geq \max(f_A(x), f_A(y)).$$

Purpose: How can we increase the optimum amount of assurance in timely presence in each of the incident scenes, but reduce the number of fire stations in the city simultaneously?
Answer: If a new path with coordinates $(0.1, 0.7)$ between E and F areas is re-opened (which is in fact an addition of strong arc), the D_1^* set of fire stations can be $D_2^* = \{A, F\}$ set of fire stations with vertex value of 0.65, while increasing the optimum level of assurance in the timely presence in the city by 0.2 simultaneously.

5.2 Mastering and evaluating performance in an organization

Controlling and ensuring compliance of the organization with its program, as well as comparing actual performance with predetermined standards, are considered one of the issues which plays an important role in enhancing the effectiveness and efficiency of the organization, and is regarded as one of the main activities of managers and leaders of an

organization. The main elements for measuring and evaluating system performance are related to goals or standards, indices and data collection systems. A set of indices of a goal can be considered as a vague graph. We define the f-strength and t-strength values in each vertex and edge(path) as follows. For each $x, y \in V$ and $xy \in E$, we have:
$t_A(x)$: The weight of the effectiveness of index x on goal achievement.
$f_A(x)$: The weight of ineffectiveness of index x on goal achievement.
$t_B(xy)$: The weight of the effective affiliation of xy on goal achievement.
$f_B(xy)$: The weight of the ineffective affiliation of xy on goal achievement.
In this case, the following relations seem logical:

$$t_B(xy) \leq \min(t_A(x), t_A(y))$$

$$f_B(xy) \geq \max(f_A(x), f_A(y))$$

The relationship between two indices is effective (strong edge) when the effective dependence weight between two indices is higher than that of all communication paths and the ineffective dependence weight between the two indices is less than of all communication pathways. Therefore, the dominating set in this graph consists of the indices that all other indices have relationships with at least one of the elements (indices) of this set. In fact, the dominating set provides an opportunity for managers and leaders of the organization to focus only on observing and controlling the enumeration of indices, rather than observing and controlling a large number of indices for realizing the goals. This issue helps managers in the crisis in the organization to make quick decisions in a short time. In addition, it is possible for managers and leaders to be relatively sure about the realization of other indicators and goal achievement by realizing the domination set indicators due to the effective relationship between the indicators of the domination set with other indicators. For example, Fig. 7, displays the graph of goal indices, in which the set of $\{A, D, E\}$ is an domination set. In other words, instead of controlling the six indices, only indices E, D, A can be controlled and observed and be relatively sure about goal achievement. It is worth noting that some factors such as common variables in calculating indices, dependent calculation formula, and relationship between the variables of calculating the indices of the factors play significant role in creating an effective relation between the indices. The size of any effective relationship (strong edge), i.e $|e| = \dfrac{1 + t_B(e) - f_B(e)}{2}$ for each $e \in E$, represents the optimal effective weight of that relationship (edge) on the goal achievement, and the total optimal effective weight of all effective relations (total of all the strong edges) represents the optimal effective weight of the index graph on goal achievement. For example, Figure 8, illustrates the optimal effective weight of the index graph is 1.05 on goal achievement. Now, if possible, the optimal effective weight of the indices graph on goal achievement can be increased by reinforcing the constructive factors of an effective relationship, which results in increasing the accuracy and confidence in performance measurement and evaluation and decreasing the vertex cardinality of the dominating set. For example, as shown n Figure 8, the domination set of indices decreases to the set $\{A, D\}$ when establishing an effective relationship is possible between e and d indices with coordinates $(0.1, 0.6)$, while the optimal effective weight of graph upgrades to 1.30.

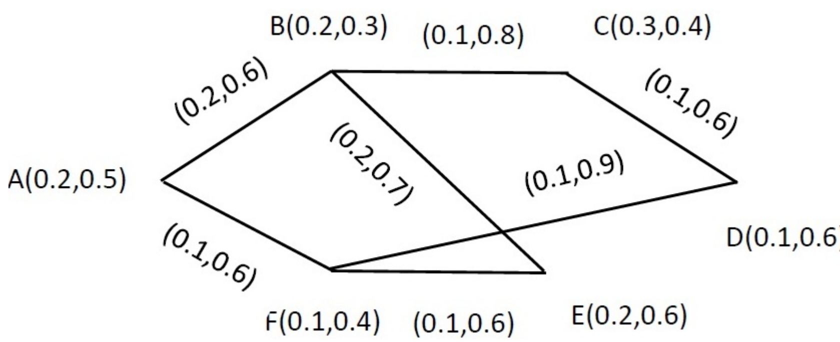

Figure 8: Vague graph G

6 Conclusion

The theory of vague graphs has many applications in new sciences and technologies. Now days, the concepts of domination and dominating set and numbers are considered as fundamental concepts in the theory of vague graphs. In this paper, we introduced of the concepts of editions of edge, cobondage set, cobondage number, as well as their effects on reducing the domination parameters in vague graphs. Specially, a new segmention of the the editions of arc and their results in vague graphs was pinpointed. Also, it is proven that in firm vague graph G by adding strong arc or arc e, is a firm and for any strong arc, it may not be true, in general. And then we expressed the relationship between the additional strong arc and the notion of strongest path between the two vertices in vague graph . Finally, by considering some special conditions, we obtained results for the upper bound of the cobondage numbers.

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References

[1] M. Akram, A. Farooq, A. Borumand Saeid and K. P. Shum, Certain types of vague cycles and vague trees, *Journal of Intelligent Fuzzy Systems*, 28 (2015), 621-631.

[2] M. Akram, W. Chen and K. P. Shum, Some properties of vague graphs, *Southeast Asian Bulletin of Mathematics*, 37 (2013), 307-324.

[3] R. A. Borzooei and H. Rashmanlou, Domination in vague graph and its application, *Journal of Intelligent and Fuzzy Systems*, 29 (2015), 1933-1940.

[4] R. A. Borzooei and H. Rashmanlou, Degree of vertices in vague graphs, *Journal of Applied Mathematics and Informatics*, 33 (2015), 545-557.

[5] R. A. Borzooei and H. Rashmanlou, Semi global domination sets in vague graphs with application, *Journal of Intelligent and Fuzzy Systems*, 30(6) (2016), 3645-3652.

[6] R. A. Borzooei, H. Rashmanlou, S. Samanta and M. Pal, Regularity of vague graphs, *Journal of Intelligent and Fuzzy Systems*, 30(6) (2016), 3681-3689.

[7] E. J. Cockayne and S. Hedetniem, Towards a theory of domination in graphs, *Networks*, 7 (1977), 247-261.

[8] W. L. Gau and D. J. Buehrer, Vague sets, *IEEE Transactions on Systems, Man and Cybernetics*, 23(2) (1993), 610-614.

[9] V. R. Kulli and B. Janakiram, The cobondage number of a graph, *Discussiones Mathematicae Graph Theory*, 16(2) (1996), 111-117.

[10] A. Nagoor Gani and K. Prasanna Devi, Reduction of domination parameter in fuzzy graphs, *Global Journal of Pure and Applied Mathematics*, 7 (2017), 3307-3315.

[11] R. Parvathi, G. Thamizhendhi, Domination in intuitionistic fuzzy graphs, *Fourteenth International Conference on Intuitionistic Fuzzy Sets, Sofia*, 6(2) (2010), 39-49.

[12] N. Ramakrishna, Vague graphs, *International Journal of Computational Cognition*, 7 (2009), 51-58.

[13] H. Rashmanlou, S. Samanta, M. Pal and R. A. Borzooei, A study on bipolar fuzzy graphs, *Journal of Intelligent and Fuzzy Systems*, 28 (2015), 571-580.

[14] H. Rashmanlou, S. Samanta, M. Pal and R. A. Borzooei, Bipolar fuzzy graphs with categorical properties, *International Journal of Computational Intelligent Systems*, 8(5) (2015), 808-818.

[15] H. Rashmanlou, S. Samanta, M. Pal and R. A. Borzooei, Product of bipolar fuzzy graphs and their degree, *International Journal of General Systems*, 45 (2016), 1-14.

[16] A. Rosenfeld, Fuzzy graphs, *Fuzzy Sets and their Applications* (L. A. Zadeh, K. S. Fu, and M. Shimura, Eds.), Academic Press, New York, (1975), 77-95.

[17] S. Samanta, M. Pal, H. Rashmanlou and R. A. Borzooei, Vague graphs and strengths, *Journal of Intelligent and Fuzzy Systems*, 30(6) (2016), 3675-3680.

[18] A. Somasundaram and S. Somasundaram, Domination in fuzzy graphs-I, *Pattern Recogni- tion Letters*, 19 (1998), 787-791.

[19] L. A. Zadeh, Fuzzy sets, *Information and Control*, 8 (1965), 338-353.

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