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A Family of Optimal Derivative Free Iterative Methods with Eighth-Order Convergence for Solving Nonlinear Equations

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Abstract. In this paper, modification of Steffensen's method with eight-order convergence is presented. We propose a family of optimal three-step methods with eight-order convergence for solving the simple roots of nonlinear equations by using the weight function and interpolation methods. Per iteration this method requires four evaluations of the function which implies that the efficiency index of the developed methods is 1.682. Some numerical examples illustrate that the algorithm is more efficient and performs better than other methods.

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1. Introduction

The numerical solution of a nonlinear equation $f(x) = 0$ is a fundamental task in scientific computation. The most famous approach is probably Newton's method $(NM) : x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} (n = 0, 1, 2, \dots)$ where x_0 is an initial guess of the root. It uses two evaluations of f and f' to achieve second-order convergence. However, Steffensen's method $(SM) : x_{n+1} =$

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$x_n - \frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)}$ is well known as a noticeable improvement of Newton's method, because it maintains quadratic convergence without any derivative. Ostrowski's method[8], given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \end{cases} \tag{1}$$

is an improvement of Newton's method. Chun and Ham developed a family of variants of Ostrowski's method with sixth-order methods by weight function methods in [3], which is written as:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - H(\mu_n) \frac{f(z_n)}{f'(x_n)}, \end{cases} \tag{2}$$

where $\mu_n = \frac{f(y_n)}{f(x_n)}$ and $H(t)$ represents a real-valued function with $H(0) = 1$, $H'(0) = 2$ and $|H''(0)| < \infty$. Kou et al, presented a family of variants of Ostrowski's method [5] with seventh-order convergence, which is given by:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \left[\left(\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)} \right)^2 + \frac{f(z_n)}{f(y_n)-\alpha f(z_n)} \right], \end{cases} \tag{3}$$

where α is constant. Bi et al, presented a new family of eighth-order iterative methods [2], which is given by:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - H(\mu_n) \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(x_n)+(\gamma+2)f(z_n)}{f(x_n)+\gamma f(z_n)} \frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}, \end{cases} \tag{4}$$

where $\gamma \in \mathbb{R}$ is constant, $\mu_n = \frac{f(y_n)}{f(x_n)}$ and $H(t)$ represents a real-valued function with $H(0) = 1$, $H'(0) = 2$, $H''(0) = 10$ and $|H'''(0)| < \infty$. Liu and Wang in [7] presented the following family of optimal order eight

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \left[\left(\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)} \right)^2 + \frac{f(z_n)}{f(y_n)-\mu f(z_n)} + \frac{4f(z_n)}{f(x_n)+\beta f(z_n)} \right], \end{cases} \tag{5}$$

with μ and β are in $\mathbb{R}$. Sharma in [9] to produce optimal eighth-order method in the following form

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \left[1 + \frac{f(z_n)}{f(x_n)} + \left(\frac{f(z_n)}{f(x_n)} \right)^2 \right] \frac{f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}. \end{cases} \tag{6}$$

Recently, there are several optimal eighth-order methods proposed in [1, 4, 6].

1.1 Preliminaries and Notation

Definition 1.1.1 *The efficiency index is defined as* $p^{\frac{1}{m}}$ *, where* p *is the order of the method and* m *is the number of functional evaluations per iteration required by the method [10].*

Definition 1.1.2 *The computational order of convergence* (COC) *is computed by using [10]*

$$COC = \frac{\log(\|X_{n+1} - X_n\|_\infty / \|X_n - X_{n-1}\|_\infty)}{\log(\|X_n - X_{n-1}\|_\infty / \|X_{n-1} - X_{n-2}\|_\infty)}, \tag{7}$$

where X_{n+1}*,* X_n*,* X_{n-1} *and* X_{n-2} *are iterations close to a zero of the nonlinear system.*

Definition 1.1.3 *Suppose* $\{X_n\}_{n=0}^{\infty}$ *is a sequence that converges to* x^**, with* $X_n \neq \alpha$ *for all* n*. If positive constants* c *and* r *exist with*

$$\lim_{n\to\infty} \frac{\|X_{n+1} - x^*\|}{\|X_n - x^*\|^r} = c \neq 0, \tag{8}$$

then $\{X_n\}_{n=0}^{\infty}$ converges to of order r, with asymptotic error constant c. Now after furnishing the outlines of the present work and a short study on the available high order developments of the classical Newton's method, we will provide our contribution in the next section. In the section 2 gives a general class of efficient three-step eight-order methods including four evaluations of the function per cycle. In the section 3,where the numerical comparisons are made to manifest the accuracy of the new methods from our class. Finally, the conclusion of the paper will be drawn in section 4.

2. Development of Method and Convergence Analysis

To develop the new method, let us consider the iteration scheme in the form

$$\begin{cases} y_n = x_n - \frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)}, \\ z_n = y_n - K(\mu_n)\frac{f(x_n)f(y_n)}{f(x_n+f(x_n))-f(x_n)}, \end{cases} \tag{9}$$

where $\mu_n = \frac{f(y_n)}{f(x_n)}$ and $K(t)$ represents a real-valued function, we have the following convergence result

Theorem 2.1. *Assume that $f \in C^4(D)$. Suppose $x^* \in D$, $f(x^*) = 0$ and $f^{'}(x^*) \neq 0$. If the initial point x_0 is sufficiently close to x^*, then the sequence $\{x_n\}$ generated of the iteration scheme (9) converges to x^*. If K is any function with $\{K(0) = 1, K^{'}(0) = \frac{3}{2}, |K^{''}(0)| < \infty\}$, then the convergence order of any method of the family (9) arrives to four.*

Proof. Since f is sufficiently differentiable, by expanding $f(x_n)$ and $f(x_n + f(x_n))$ about x^*, one obtains

$$f(x_n) = f^{'}(x^*)(e_n + \sum_{i=2}^{4} c_i e_n^i + O(e_n^5)), \tag{10}$$

and

$$f(x_n + f(x_n)) = f^{'}(x^*)(2e_n + 5c_2e_n^2 + (9c_3 + 4c_2^2)e_n^3 + O(e_n^4)), \tag{11}$$

where $c_k = \frac{1}{k!}\frac{f^{(k)}(x^*)}{f'(x^*)}$ for $k = 2, 3, \ldots$ Furthermore, with using the Maple software and by writing the Taylor's expansion for z_n about x^* we can get:

$$\begin{aligned} z_n &= x^* - 2c_2(-1+K(0))e_n^2 + 6(1-K(0))c_3 \\ &+ (-5+11K(0)-4K^{'}(0))c_2^2e_n^3 \\ &+ O(e_n^4). \end{aligned} \tag{12}$$

Solving system of the equations $\{1-K(0) = 0, -5+11K(0)-4K^{'}(0) = 0\}$ we find that $\{K(0) = 1, K^{'}(0) = \frac{3}{2}\}$, thereby we obtain $K(t) = \frac{1}{2}\alpha t^2 + \frac{3}{2}t + 1$, where $t = \frac{f(y_n)}{f(x_n)}, \alpha \in \mathbb{R}$.
Now, we consider an iteration scheme of the form,

$$\begin{cases} y_n = x_n - \frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)}, \\ z_n = y_n - \left[\frac{1}{2}\alpha\left(\frac{f(y_n)}{f(x_n)}\right)^2 + \frac{3}{2}\frac{f(y_n)}{f(x_n)} + 1\right]\frac{f(x_n)f(y_n)}{f(x_n+f(x_n))-f(x_n)}, \end{cases} \tag{13}$$

and satisfies the following error equation :

$$z_n - x^* = (19c_2^3 - 4c_2c_3 - 4K^{''}(0)c_2^3)e_n^4 + O(e_n^5). \quad \square \tag{14}$$

Remark 2.2. *The order of convergence of the iterative method* (13) *is* 4*. This method requires three evaluations of the function, namely,* $f(x_n)$*,* $f(x_n + f(x_n))$ *and* $f(y_n)$*. If we suppose that all the evaluations have the same cost, we have that the efficiency index of the method* (13) *is* $\sqrt[3]{4} \approx 1.5874$.
Now we construct a three-step iterative method:

$$\begin{cases} y_n = x_n - \frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)}, \\ z_n = y_n - \left[\frac{1}{2}\alpha\left(\frac{f(y_n)}{f(x_n)}\right)^2 + \frac{3}{2}\frac{f(y_n)}{f(x_n)} + 1\right]\frac{f(x_n)f(y_n)}{f(x_n+f(x_n))-f(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)^2}{f(z_n+f(z_n))-f(z_n)}. \end{cases} \tag{15}$$

We can easily prove that scheme (15) *is eight-order convergent and it requires five evaluations of the function, which satisfies the error equation* $e_{n+1} = (2(4c_3 - 19c_2^2 + 4\alpha c_2^2)^2c_2^3)e_n^8$*. Scheme* (15) *has an efficiency index* *of* $8^{\frac{1}{5}} \approx 1.5157$. *We construct an optimal efficiency index with an eight-order convergence for solving the simple roots of nonlinear equations by interpolation methods. We can express* $f(z_n + f(z_n))$ *as follows: by using the interpolation on five points* $(P_z, f(P_z))$, $(z_n, f(z_n))$, $(y_n, f(y_n))$, $(P_x, f(P_x))$ *and* $(x_n, f(x_n))$, *where* $P_z = z_n + f(z_n)$ *and* $P_x = x_n + f(x_n)$. *We can approximate* $f(z_n + f(z_n))$ *if we solve the equations* (16).

$$\begin{cases} a_0 + a_1 P_z + a_2 P_z^2 + a_3 P_z^3 = a_4 f(P_z) \\ a_0 + a_1 z_n + a_2 z_n^2 + a_3 z_n^3 = a_4 f(z_n) \\ a_0 + a_1 y_n + a_2 y_n^2 + a_3 y_n^3 = a_4 f(y_n) \\ a_0 + a_1 P_x + a_2 P_x^2 + a_3 P_x^3 = a_4 f(P_x) \\ a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 = a_4 f(x_n) \end{cases} \tag{16}$$

for the coefficients a_i, $0 \leqslant i \leqslant 4$ *and* $a_4 = 1$. *If*

$$M_z = \begin{bmatrix} 1 & P_z & P_z^2 & P_z^3 & -f(P_z) \\ 1 & z_n & z_n^2 & z_n^3 & -f(z_n) \\ 1 & y_n & y_n^2 & y_n^3 & -f(y_n) \\ 1 & P_x & P_x^2 & P_x^3 & -f(P_x) \\ 1 & x_n & x_n^2 & x_n^3 & -f(x_n) \end{bmatrix}$$

then the system (16) *has a unique solution if and only if the* $Deter-minant(M_z) = 0$. *By expanding* $Determinant(M_z)$ *about fifth column we obtain*

$$(A)f(k_z) - (B)f(z_n) + (C)f(y_n) - (D)f(k_x) + (E)f(x_n) = 0, \tag{17}$$

As, we have

$$f(k_z) = \frac{(B)}{(A)} f(z_n) - \frac{(C)}{(A)} f(y_n) + \frac{(D)}{(A)} f(k_x) - \frac{(E)}{(A)} f(x_n), \tag{18}$$

where

$$\begin{cases} A=-f(x_n)(x_n-y_n)(x_n-z_n)(y_n-z_n)(f(x_n)+x_n-y_n)(f(x_n)+x_n-z_n), \\ B=-f(x_n)(x_n-y_n)(-y_n+z_n+f(z_n))(f(z_n)+z_n-x_n)(f(x_n)+x_n-y_n)(f(x_n)-f(z_n)-z_n+x_n), \\ C=-f(x_n)f(z_n)(x_n-z_n)(f(x_n)+x_n-z_n)(f(z_n)-x_n+z_n)(f(x_n)-f(z_n)+x_n-z_n), \\ D=-f(z_n)(x_n-y_n)(x_n-z_n)(y_n-z_n)(f(z_n)-x_n+z_n)(f(z_n)-y_n+z_n), \\ E=f(z_n)(y_n-z_n)(f(z_n)-y_n+z_n)(-z_n+f(x_n)+x_n)(x_n-y_n+f(x_n))(f(x_n)-f(z_n)-z_n+x_n). \end{cases} \tag{19}$$

Substituting $\{A,B,C,D,E\}$ *of* (19) *in* (18) *and simplifying* $\{\frac{B}{A},\frac{C}{A},\frac{D}{A},\frac{E}{A}\}$ *we have* $f(k_z)\approx\Psi_f(x_n,y_n,z_n)$*:*

$$\begin{cases} \Psi_f(x_n,y_n,z_n)=\frac{(f(z_n)-y_n+z_n)(f(z_n)+z_n-x_n)(f(x_n)-f(z_n)-z_n+x_n)f(z_n)}{(x_n-z_n)(y_n-z_n)(-z_n+f(x_n)+x_n)} \\ -\frac{f(z_n)(f(z_n)+z_n-x_n)(f(x_n)-f(z_n)-z_n+x_n)f(y_n)}{(x_n-y_n)(y_n-z_n)(x_n-y_n+f(x_n))}+\frac{f(z_n)(f(z_n)+z_n-x_n)(f(z_n)-y_n+z_n)f(k_x)}{f(x_n)(x_n-y_n+f(x_n))(-z_n+f(x_n)+x_n)} \\ +\frac{f(z_n)(f(z_n)-y_n+z_n)(f(x_n)-f(z_n)-z_n+x_n)}{(x_n-y_n)(x_n-z_n)}. \end{cases} \tag{20}$$

Now we construct a three-step iterative method:

$$\begin{cases} y_n=x_n-\frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)}, \\ z_n=y_n-\left[\frac{1}{2}\alpha\left(\frac{f(y_n)}{f(x_n)}\right)^2+\frac{3}{2}\frac{f(y_n)}{f(x_n)}+1\right]\frac{f(x_n)f(y_n)}{f(x_n+f(x_n))-f(x_n)}, \\ x_{n+1}=z_n-\frac{f(z_n)^2}{\Psi_f(x_n,y_n,z_n)-f(z_n)}. \end{cases} \tag{21}$$

We prove the following convergence theorem for the method (21)

Theorem 2.3. *Let* $x^*\in D$ *be a simple zero of a sufficiently differentiable function* $f:D\to R$ *for an open interval* D*. If* x_0 *is sufficiently close to* x^**, then the three-step iterative method* (21) *has eight-order convergence and satisfies the following error equation :*

Proof. Since f is sufficiently differentiable, by expanding $f(x_n)$ about x^*, one obtains

$$f(x_n) = f^{'}(x^*)(e_n + \sum_{i=2}^{8} c_i e_n^i + O(e_n^9)), \tag{22}$$

by expanding y_n about x_n , we obtain

$$y_n \;=\; x^* + 2c_2 e_n^2 + (6c_3 - 5c_2^2)e_n^3 + \cdots + O(e_n^9), \tag{23}$$

By expanding $f(y_n)$ about x_n, we have

$$\begin{aligned} f(y_n) \;=\; & f^{'}(x^*)(2c_2 e_n^2 + (6c_3 - 5c_2^2)e_n^3 + (-26c_2c_3 + 17c_2^3 + 14c_4)e_n^4 \\ + \; & \cdots + O(e_n^9)). \end{aligned} \tag{24}$$

By Substituting (22), (23) and (24) into the second formula of (21), using Taylor's expansion, and simplifying, we have

$$\begin{aligned} z_n \;=\; & x^* + (-4c_2c_3 + 19c_2^3 - 4\alpha c_2^3)e_n^4 + \cdots + (-34178c_2^2c_3c_4 \\ - \; & 13664\alpha c_2^7 + 2937c_2c_4^2 + 29428c_4c_2^4 + \frac{51093}{2}c_2^7 + 10272\alpha c_2^2c_3c_4 \\ - \; & 11187\alpha c_2^4c_4 + 37353\alpha c_2^5c_3 - 26902\alpha c_2^3c_3^2 - 756\alpha c_3^2c_4 + 4248\alpha c_2c_3^3 \\ - \; & 588\alpha c_2c_4^2 + 69690c_2^3c_3^2 - 14133c_2c_3^3 - 80957c_2^5c_3 \\ + \; & 3961c_4c_3^2)e_n^8 + O(x^9) \end{aligned} \tag{25}$$

By expanding $f(z_n)$ about x_n, we have

$$\begin{aligned} f(z_n) \;=\; & f^{'}(x^*)(-4c_2c_3 + 19c_2^3 - 4\alpha c_2^3)e_n^4 + \cdots \\ + \; & (-34178c_2^2c_3c_4 - 13816\alpha c_2^7 + 2937c_2c_4^2 + 29428c_4c_2^4 + \frac{51815}{2}c_2^7 \\ + \; & 10272\alpha c_2^2c_3c_4 - 11187\alpha c_2^4c_4 + 37385\alpha c_2^5c_3 \\ - \; & 26902\alpha c_2^3c_3^2 - 756\alpha c_3^2c_4 + 4248\alpha c_2c_3^3 - 588\alpha c_2c_4^2 + 16\alpha^2c_2^7 \\ + \; & 69706c_2^3c_3^2 - 14133c_2c_3^3 - 81109c_2^5c_3 + 3961c4c_3^2)e_n^8 \\ + \; & O(e_n^9)). \end{aligned} \tag{26}$$

By expanding $\Psi_f(x_n, y_n, z_n) \approx f'(z_n)$ about x_n, we have

$$\begin{aligned}\Psi_f(x_n, y_n, z_n) &= f'(x^*)((-8c_2c_3 + 38c_2^3 - 8\alpha c_2^3)e_n^4 \cdots \\ &+ (5140c_2c_3c_4 - 168c_4^2 - 15644c_2^2c_3^2 - 8410c_2^3c_4 \\ &+ 27726c_2^4c_3 + 1150c_3^3 - 11263c_2^6 - 1008\alpha c_2c_3c_4 \\ &- 10294\alpha c_2^4c_3 + 2464\alpha c_2^3c_4 + 4584\alpha c_2^2c_3^2 \\ &+ 4983\alpha c_2^6 - 216\alpha c_3^3)e_n^7 + O(e_n^8) \qquad (27)\end{aligned}$$

By Substituting (25), (26) and (27) into the third formula of (21), using Taylor's expansion, and simplifying, we have we have

$$e_{n+1} = x_{n+1} - x^* = 2c_2^2(4c_3 - 19c_2^2 + 4\alpha c_2^2)(-19c_2^3 + 4\alpha c_2^3 + 4c_2c_3 - 2c_4)e_n^8 + O(e_n^9). \quad \square \qquad (28)$$

Remark 2.4. The order of convergence of the iterative method (21) is

8. This method requires four evaluations of the function, namely, $f(x_n)$, $f(x_n + f(x_n))$, $f(y_n)$ and $f(z_n)$. If we suppose that all the evaluations have the same cost, we have that the efficiency index of the method (21) is $\sqrt[4]{8} \approx 1.6821$.

Table 1: *Comparison of different methods in terms of orders and efficiencies.*

Methods	Order	Total number of evaluations	Efficiency index
Newton	2	2	$\sqrt[2]{2} \approx 1.414$
Steffensen	2	2	$\sqrt[2]{2} \approx 1.414$
Ostrowski	4	3	$\sqrt[3]{4} \approx 1.587$
(13)	4	3	$\sqrt[3]{4} \approx 1.587$
Chun	6	4	$\sqrt[4]{6} \approx 1.565$
Kou	7	4	$\sqrt[4]{7} \approx 1.626$
Bi	8	4	$\sqrt[4]{8} \approx 1.682$
Liu	8	4	$\sqrt[4]{8} \approx 1.682$
Sharma	8	4	$\sqrt[4]{8} \approx 1.682$
(21)	8	4	$\sqrt[4]{8} \approx 1.682$

3. Numerical Examples

This section deals with comparison of some numerical examples and obtaining the simple roots of the test problems. All the instances were done with Matlab 7.13 using 1000 digits, floating point (digits := 1000), with VPA Command. Unlike Section 2, we here use Matlab to show the readers that all of the iterative schemes can be implemented really well in all of the available Software. In examples considered in this article, the stopping criterion are the $| x_{n+1} - x_n |< \epsilon$, $| f(x_{n+1}) |< \epsilon$, where $\epsilon = 10^{-1000}$, The absolute value of the given test functions after some full iterations are listed in Table 2. As Table 2 illustrates, the new methods from the class gives reliable results in all cases, in contrast by the well-known methods with the same Total Number of Evaluations per cycle, i.e. Bi's optimal eight-order method (BM), Liu's optimal eight-order method (LM) and Sharma's optimal eight-order method (SM). We present some numerical test results with the following functions:

$$\begin{cases} f_1(x) = x^3 + 4x^2 - 15, & x^* \approx 1.93198055660636, \\ f_2(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5, & x^* \approx -1.20764782713091, \\ f_3(x) = 10xe^{-x^2} - 1, & x^* \approx 1.67963061042845, \\ f_4(x) = \sin^2(x) - x^2 + 1, & x^* \approx 1.40449164821534, \\ f_5(x) = x^5 + x^4 + 4x^2 - 15, & x^* \approx 1.34742809896830, \\ f_6(x) = \ln(x) + \sqrt{x} - 5, & x^* \approx 8.30943269423157, \\ f_7(x) = e^{x^2+7x-30} - 1, & x^* = 3.00000000000000, \\ f_8(x) = \sin(x)e^{x} - 2x - 5, & x^* \approx -2.52324523073255, \\ f_9(x) = \sqrt{x} - \frac{1}{x} - 3, & x^* \approx 9.63359556283269, \\ f_{10}(x) = \sqrt{x^2 + 2x + 5} - 2\sin(x) - x^2 + 3, & x^* \approx 2.33196765588396. \end{cases} \tag{29}$$

The computational results presented in Table 2 shows that in almost all of cases, the presented method converge more rapidly than other methods. As shown in Tables 2, the proposed method (21) is preferable to Bi's method, Liu's methods and sharma's method with optimal eight-order convergence.

Table 2: *Comparison of iterative methods.*

Function	Methods	iteration	$\|x_k - x_{k-1}\|$	$\|f(x_k)\|$	COC
$f_1(x)$, $x_0 = 2$	(21)	4	0.000000e+00	0.000000e+00	8.000000
	BM	4	1.600405e-330	0.000000e+00	7.000000
	LM	4	1.880395e-387	0.000000e+00	8.000000
	SM	4	4.565243e-428	0.000000e+00	8.000000
$f_2(x)$, $x_0 = -1.5$	(21)	4	2.280003e-143	1.966920e-848	7.999974
	BM	4	4.425546e-66	2.443099e-455	7.001710
	LM	4	1.630467e-234	1.200000e-998	8.000010
	SM	4	2.342471e-280	1.100000e-998	8.000005
$f_3(x)$, $x_0 = 1.5$	(21)	4	1.074982e-90	6.620299e-539	7.999477
	BM	4	1.250541e-46	2.401965e-323	7.117036
	LM	4	3.716627e-45	7.005386e-355	7.890752
	SM	4	1.378981e-50	4.941091e-399	7.923618
$f_4(x)$, $x_0 = 2$	(21)	4	5.767842e-30	2.667055e-275	7.999999
	BM	4	4.384097e-29	1.459166e-200	6.999961
	LM	4	2.790447e-25	1.401650e-196	7.999899
	SM	4	4.310256e-29	9.953621e-228	7.999973
$f_5(x)$, $x_0 = 1.4$	(21)	4	3.827346e-112	2.819039e-660	7.999722
	BM	4	8.621099e-62	2.391218e-426	7.000000
	LM	4	8.575358e-79	3.508366e-623	8.000000
	SM	4	5.463271e-85	1.808119e-673	8.000000
$f_6(x)$, $x_0 = 8$	(21)	4	4.181664e-424	0.000000e+00	8.000000
	BM	4	1.023748e-82	2.311776e-582	7.000000
	LM	4	1.850555e-113	4.581926e-912	8.000000
	SM	4	1.417600e-124	0.000000e+00	8.003890
$f_7(x)$, $x_0 = 3.5$	(21)	5	1.299147e-55	7.597901e-320	8.000000
	BM	5	4.455908e-08	1.253740e-52	7.000000
	LM	5	2.233053e-24	4.987826e-183	7.728782
	SM	5	3.060757e-39	1.124354e-302	7.940283
$f_8(x)$, $x_0 = -2.4$	(21)	3	7.481338e-109	0.000000e+00	8.005681
	BM	3	2.982942e-107	1.506077e-753	6.976066
	LM	3	8.857841e-134	0.000000e+00	7.980231
	SM	3	4.669022e-134	0.000000e+00	7.982322
$f_9(x)$, $x_0 = 9$	(21)	3	1.699554e-110	3.183989e-668	8.000903
	BM	3	2.071706e-72	4.020242e-511	7.012641
	LM	3	7.375387e-99	3.856457e-796	8.010044
	SM	3	1.030470e-121	1.941528e-981	8.001482
$f_{10}(x)$, $x_0 = 2$	(21)	3	2.889024e-83	1.184087e-499	8.000000
	BM	3	2.073513e-57	1.521116e-400	6.882628
	LM	3	3.187186e-77	2.045299e-616	7.869719
	SM	3	1.116946e-85	6.055121e-685	7.876306

4. Conclusions

In this work we presented an approach which can be used to constructing of eight-order convergence iterative methods that do not require the computation of first and second or higher derivatives. Numerical examples also show that the numerical results of our new three-step methods, in equal iterations, improve the results of other existing three-step methods with eight-order convergence. Finally, it is hoped that this study makes a contribution to nonlinear equation solving.

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