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Some Results on Topological BL-Algebras

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Abstract. In this paper, we generalize the concepts of para and quasi topological MV-algebras, which was first introduced by Najafi et al. in 2017, to BL-algebras as para and quasi topological BL-algebras and elaborate these concepts via some examples. We further derive and prove some theorems by employing pre-filters and a fundamental system of neighborhoods.

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1 Introduction

Algebra and topology in mathematics sometimes play complementary roles to each other. By studying some topological concepts through algebraic methods and also by applying some topological notions in algebra,

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the delicacy of these branches of mathematics becomes more evident. Therefore, the study of algebraic logics as algebraic structures, through topological concepts, has been considered by mathematical researchers. BL-algebras have been introduced by Hájek [7] in order to investigate many valued logic by algebraic way. He provided an algebraic counterpart of a propositional logic, called Basic Logic, which typifies a portion common to some of the most important many-valued logics, namely, Łukasiewicz logic, Gödel logic and Product logic. This Basic Logic (BL for short) is proposed as the most general many-valued logics with truth values in [0,1] and BL-algebras which are the corresponding to Lindenbaum-Tarski algebras. Also, Hájek presented an algebraic mean for the study of continuous t-norms (or triangular norms) on the unit real interval [0,1]. Apart from their logical interest, BL-algebras have important algebraic and topological properties and they have been intensively studied from an algebraic point of view.

In 1958, C. C. Chang defined MV-algebras [4] as the algebraic counterpart of $\aleph_0$-valued Łukasiewicz logic, which allowed him to give another completeness proof for this logic. In fact, MV-algebras are BL-algebras but, the converse is not true. Proved by Höhle [8], a BL-algebra A becomes an MV-algebra if, we adjoin to the axioms the double negation law, i.e., $x = x^{--}$, for every $x \in A$. Thus, a BL-algebra is in some intuitive way, a non-double negation MV-algebra. Hence the theory of MV-algebras, becomes one of the guiding to the development of the theory of BL-algebras.

Several authors have claimed that in BL-algebras, the notions of ideals are less studied than filters, because in these algebras, there is no appropriate algebraic addition. Therefore, the study of topological BL-algebras has been done mostly through its filters.

Najafi and Kohestani [10], introduced the notions of the quasi and para topological MV-algebras. We generalize these concepts through ideals to BL-algebras. This paper is organized as follows:

In Section 2, we recall some basic concepts on BL-algebra, topological spaces and topological BL-algebras. In Section 3, we define the notions of para and quasi-topological BL-algebra and derive some theorems and relations between them.

2 Preliminaries

In this section, we recall and summarize some definitions and propositions about topology and BL-algebras, which will be used in the following.

Definition 2.1. [7] A BL-algebra is a nonempty set A with four binary operations $\wedge$, $\vee$, $\odot$, $\rightarrow$ and two constants 0, 1, such that:
BL_1: $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;
BL_2: $(A, \odot, 1)$ is a commutative monoid;
BL_3: $x \odot y \leq z$ iff $x \leq y \rightarrow z$;
BL_4: $x \wedge y = x \odot (x \rightarrow y)$;
BL_5: $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

The main examples of BL-algebras are from the unit interval $[0, 1]$ endowed with the structure induced by continuous t-norms. Every BL-algebra has the negation operation defined by $x^- = x \rightarrow 0$.
A BL-algebra satisfying the double negation law is called an MV-algebra, that is $x^{--} = x$. Therefore, if A is a BL-algebra, then the set $MV(A) = \{x \in A \mid x^{--} = x\} = \{x^- \mid x \in A\}$ is an MV-algebra.

Definition 2.2. [3] A set A with a family $\mathcal{P}_A$ of its subsets is called a topological space, denoted by $(A, \mathcal{P}_A)$, if the following conditions hold:
(i) A, $\emptyset \in \mathcal{P}_A$;
(ii) The intersection of any finite number of the members of $\mathcal{P}_A$ is in $\mathcal{P}_A$;
(iii) The arbitrary union of members of $\mathcal{P}_A$ is in $\mathcal{P}_A$.

The members of $\mathcal{P}_A$ are called open sets of A, and the complement of an open set U, i.e., $A - U$ is a closed set. If B is a subset of A then the biggest open set contained in B, i.e., B° is called the interior of B [3].

Definition 2.3. [3] Let $(A, \mathcal{P}_A)$ be a topological space. Then
(i) A subfamily $\{U_i\}$ of $\mathcal{P}_A$ is called a basis of $\mathcal{P}_A$ if for each $x \in U \in \mathcal{P}_A$, there is an i in index set I such that $x \in U_i \subseteq U$ and if $x \in U_i \cap U_j$ for some $i,\ j \in I$, then there exists $k \in I$, with $x \in U_k \subseteq U_i \cap U_j$.
(ii) A sub-basis for $(A, \mathcal{P}_A)$ is a sub-collection $\mathcal{S}$ of $\mathcal{P}_A$ if any open set in $\mathcal{P}_A$ can be written as union of finite intersections of elements of $\mathcal{S}$.
(iii) Let $x \in A$, then $G \subseteq A$ is a neighborhood of x if $x \in W \subseteq G$, for some open set W.
(iv) Let $t \in A$ and $F = \{G \mid G \subseteq A,\ G$ is an open neighborhood of $t\}$.

Then a fundamental system of neighborhoods of t is a sub-collection E of F, if for every $W \in F$, $N \subseteq W$, for some $N \in E$.
(v) A function g from a topological space $(X, \mathcal{P}_X)$ to a topological space $(Y, \mathcal{P}_Y)$ is continuous, if the inverse image of each open set in Y is an open set in X.
(vi) A topological space $(X, \mathcal{P}_X)$ is compact if every open cover of X has a finite subcover.
(vii) A topological space $(X, \mathcal{P}_X)$ is locally compact if for every $x \in X$ x has a compact neighborhood, i.e., $x \in G \subseteq F$, for some compact set F and some open set G.

We summarize the main properties of BL-algebras that will be needed throughout the paper.

Proposition 2.4. [7,11] Let A be any BL-algebra. Then the following properties hold for every $x, y, z \in A$:
(i) $x \leq y$ iff $x \to y = 1$;
(ii) $1 \to x = x$, $x \to x = 1$, $x \to 1 = 1$;
(iii) $x \to (y \to z) = y \to (x \to z) = (x \odot y) \to z$;
(iv) $x \odot y \leq x \wedge y$ and $x \odot (x \to y) \leq y$;
(v) $x \leq y \to x$, $x \leq x^{--}$ and $x^{---} = x^-$;
(vi) $0^- = 1$, $1^- = 0$ and $x \leq y$ implies $x \odot z \leq y \odot z$;
(vii) $x \odot x^- = 0$ and $x \odot y = 0$ iff $x \leq y^-$.
(viii) $(x \odot y)^- = (x \to y^-) = (y \to x^-)$;
(ix) $(x \odot y)^{--} = x^{--} \odot y^{--}$, $(x \to y)^{--} = x^{--} \to y^{--}$;

From [5,6] we recall the following operations which are defined for every $x,\ y$ in BL-algebra A:
(i) $x \oplus y = (x^- \odot y^-)^-$;
(ii) $x \ominus y = x \odot y^-$;
(iii) $(x \ominus y)^- = (x \odot y^-)^- = x \to y^{--} = y^- \to x^-$;
(iv) $x \to y = (x \odot y^-)^- = (x \ominus y)^-$, for every $x \in A$ and y$\in MV(A)$.

Definition 2.5. [7] A subset F of a BL-algebra A is called a filter if it satisfies the following conditions:
(i) for every $x, y \in F$, $x \odot y \in F$;

(ii) for every $x, y \in A$ if $x \leq y$ and $x \in F$ then $y \in F$.

E. Turunen [11] defined a deductive system of a BL-algebra A to be a nonempty subset D of A such that (i) $1 \in D$ and (ii) $x \in D$ and $x \to y \in D$ imply $y \in D$. He proved that a subset F of a BL-algebra A is a deductive system of A if and only if F is a filter of A. From Proposition 2.4, it is clear that $x \in F$ implies $x^{--} \in F$.

C. Lele et.al. [9] proved that the operation "$\oslash$" on BL-algebra A by $x \oslash y = x^- \to y$, for every $x, y \in A$ is associative and monotone, i.e., for every $a, b, c, d \in A$, $a \leq b$ and $c \leq d$ imply $a \oslash c \leq b \oslash d$.

Definition 2.6. [9] Let $(A, \wedge, \vee, \odot, \to, 0, 1)$ be a BL-algebra and I be a nonempty subset of A. I is called an ideal of A if it satisfies:
(i) for every $x, y \in I$, $x \oslash y \in I$;
(ii) for every $x, y \in A$, if $x \leq y$ and $y \in I$ then $x \in I$.

It is easy to see that $0 \in I$ for every ideal I, and for every $x \in A$, $x \in I$ if and only if $x^{--} \in I$. The intersection of any family of ideals of a BL-algebra A is again an ideal of A [7].

Proposition 2.7 [9] A nonempty subset $\{0\} \subseteq I$ of a BL-algebra A is an ideal if and only if for every $s, t \in A$, if $s^- \odot t \in I$ and $s \in I$ then $t \in I$.

Definition 2.8. [2,14] Let A be a BL-algebra with topology $\mathcal{P}$. Then $(A, \mathcal{P})$ is called a topological BL-algebra, if all binary operations of A and its lattice structure are continuous.

For example, if we consider the BL-algebra $A = (I, \wedge, \vee, \odot, \to, 0, 1)$ (Product structure) with a topology $\mathcal{P}$, on the real unit interval $I = [0, 1]$, where for every $x, y \in I$, $x \odot y = x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$

and

$$x \to y = \begin{cases} 1 & x \leq y \\ \dfrac{y}{x} & o.w \end{cases}$$

Then $(A, \{\wedge, \vee\}, \mathcal{P})$ with the basis $S = \{[x, y) \cap I : x, y \in \mathbb{R}\}$ is a topological BL-algebra [2,9].

If A is a BL-algebra we denote for every $x, y \in A$ and $U, V \subseteq A$: $U \oplus V = \{u \oplus v | u \in U,\ v \in V\}$, $U \ominus V = \{u \ominus v | u \in U,\ v \in V\}$, $x \oplus V = \{x \oplus v | v \in V\}$, $U \oplus y = \{u \oplus y | u \in U\}$, $x \ominus V = \{x \ominus v | v \in V\}$, $U \ominus y = \{u \ominus y | u \in U\}$, $U \odot V = \{u \odot v | u \in U,\ v \in V\}$, $x \odot V = \{x \odot v | v \in V\}$, $U \odot y = \{u \odot y | u \in U\}$ and $U^- = \{u^- \mid u \in U\}$.

Definition 2.9. [1] Let A be a BL-algebra. Then, the operation $\odot$ is semi-continuous, if for each open set U with $x \odot y \in U$, there exist two open sets V, W such that $x \in V$, $y \in W$, $V \odot y \subseteq U$ and $x \odot W \subseteq U$.

In a similar way semi-continuity is defined for any binary operation on a BL-algebra.

3 On (para, quasi) topological BL-algebras

In this section, we define the para and quasi-topological BL-algebras and derive some results.

Definition 3.1. Let A be a BL-algebra with a topology $\mathcal{P}$, then
(i) $(A, \mathcal{P})$ is called a para-topological BL-algebra if the operation " $\oplus$ " is continuous, or equivalently, if for any $x, y \in A$ and any open neighborhood W of $x \oplus y$, there exist two open neighborhoods U and V of x and y, respectively, such that $U \oplus V \subseteq W$.
(ii) $(A, \mathcal{P})$ is called a quasi-topological BL-algebra if the operation " $\odot$ " be semi-continuous and " $-$ " is continuous.

Example 3.2. We consider the BL-algebra $A = (I, \wedge, \vee, \odot, \rightarrow, 0, 1)$ (Gödel structure) on the real unit interval $I = [0, 1]$, where for every $x, y \in I$, $x \odot y = x \wedge y =$min$\{x, y\}$, $x \vee y =$max$\{x, y\}$ and

$$x \rightarrow y = \begin{cases} 1 & x \leq y \\ y & o.w \end{cases}$$

Let $\mathcal{P}$ be a topology on A induced by the base $\mathcal{B} = \{[x, y] \cap I : x, y \in \mathbb{R}\}$.

Then $(A, \mathcal{P})$ is a quasi-topological BL-algebra, since the operation " $\odot$ "

is semi-continuous and the operation " $-$ " is continuous [2,11].

Example 3.3. Consider the BL-algebra $S_3 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ with the following operations and topology $\mathcal{P} = \{\emptyset, \{0\}, \{0, \frac{1}{3}\}, S_3\}$. Then S_3 with the topology $\mathcal{P}$ is a para-topological BL-algebra.

Table 1: Product operation

$\odot$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
0	0	0	0	0
$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
1	0	$\frac{1}{3}$	$\frac{2}{3}$	1

Table 2: Implication operation

$\rightarrow$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
0	1	1	1	1
$\frac{1}{3}$	0	1	1	1
$\frac{2}{3}$	0	$\frac{2}{3}$	1	1
1	0	$\frac{1}{3}$	$\frac{2}{3}$	1

Proof. It is easy to see that S_3 is a BL-algebra. Let $a, b \in A$, then for an open neighborhood D of $a \oplus b$, if $a \oplus b \geqslant \frac{1}{3}$, the only open neighborhood of $a \oplus b$ is S_3. Thus, S_3 is the open neighborhood of both a and b with $S_3 \oplus S_3 \subseteq S_3$.
For $a = b = 0$, since $a \oplus b = 0$, so D is $\{0, \frac{1}{3}\}$, $\{0\}$ or S_3. We choose, $G_1 = \{0\}$ and $G_2 = \{0\}$, then $G_1 \oplus G_2 = \{0\} \subseteq D$.
Finally, for $a = 0$ and $b = \frac{1}{3}$, we deduce $a \oplus b = \frac{1}{3}$, hence D will be $\{0, \frac{1}{3}\}$ or S_3. In this case, we set $G_1 = \{0\}$ and $G_2 = \{0, \frac{1}{3}\}$, so $G_1 \oplus G_2 = \{0, \frac{1}{3}\} \subseteq D$. Therefore, the map $\oplus$ is continuous, and S_3 is a para-topological BL-algebra.

Theorem 3.4. Let $(A, \mathcal{P})$ be a topological BL-algebra, then the following hold:
(i) $(A, \mathcal{P})$ is a quasi-topological BL-algebra;
(ii) $(A, \mathcal{P})$ is a para-topological BL-algebra.
Proof. (i) Let $(A, \mathcal{P})$ be a topological BL-algebra, then the operations $\vee, \wedge, \odot, \rightarrow$ are continuous. We consider $- : A \longrightarrow A$ by $-(x) = x^-$ and $\rightarrow: A \times A \longrightarrow A$ by $\rightarrow (x, y) = x \rightarrow y$. From the continuity of the $\rightarrow$, we set the restriction of the g on the $A \times \{0\} \subseteq A \times A$ to A with $g(x, 0) = x^- = -(x)$, then the continuity of g is concluded. By the continuity of the $\odot$, this binary operation is semi-continuous and the operation $-$ as the restriction g on the $A \times \{0\} \subseteq A \times A$ to A is continuous. Therefore A is a quasi-topological BL-algebra.
(ii) If A is a topological BL-algebra, then A is a para-topological BL-algebra. Consider $x \oplus y = (x^- \odot y^-)^-$. From hypothesis, "$\odot$" and "$-$" are continuous, thus $\oplus$ is continuous.

Proposition 3.5. Let A be a BL-algebra endowed by a topology $\mathcal{P}$. Consider the following assertions on the operations $(\odot,\ -,\ \ominus)$:
(i) $(A, \mathcal{P})$ is a topological BL-algebra;
(ii) $\odot$ and $-$ are continuous operations;
(iii) $\ominus$ and $-$ are continuous operations;
Then (i) implies (ii) and (ii) implies (iii).
Proof. (i) $\Rightarrow$ (ii) Since the binary operation "$\rightarrow$" $: A \times A \longrightarrow A$ is continuous, so by the continuity of the restriction $h : A \times \{0\} \subseteq A \times A \longrightarrow A$ by $h(x, 0) = x \rightarrow 0 = x^-$, we conclude that the operation $-$ is continuous.
(ii) $\Rightarrow$ (iii) We consider the continuous function $h : A \times A \longrightarrow A \times A$ by $h(x, y) = (x, y^-)$ (both components are continuous), therefore, the function $\ominus = \odot \circ h$ is also continuous.

Corollary 3.6. Let $(A, \mathcal{P})$ be a topological BL-algebra, then the function $g : A \times A \longrightarrow A$ with $g(s, t) = s \odot t^-$ is continuous.
proof. Since $(A, \mathcal{P})$ is a topological BL-algebra, so " $\odot$ " and " $-$ " are continuous, therefore $g(s, t) = s \odot t^- = s \ominus t$ is continuous by Proposition 3.5 (iii).

The converse of Corollary 3.6, holds for $MV(A) = \{x \in A \mid \; x^{--} = x\}$. Indeed, If we consider the function p as the restriction of g to $\{1\} \times A$ by $p(1, s) = 1 \odot s^- = s^-$ is a continuous function. It is trivial that the map $q : A \longrightarrow A \times A$ by $q(s) = (1, s)$ is continuous therefore, for every $s \in A$, $(p \circ q)(s) = p(1, s) = 1 \odot s^- = s^-$ is continuous. This means that $-$ is continuous. Since each component of the map $l : A \times A \longrightarrow A \times A$ by $l(s, t) = (s, t^-)$ are continuous, so l is a continuous function. Hence, for every $(s, t) \in A^2$, we have

$$\begin{aligned}(g \circ l)(s,t) &= g(s, t^-) \\ &= s \odot t^{--} \\ &= s \odot t \qquad (in \;\; MV(A))\end{aligned}$$

This means that $\odot$ is continuous.

Proposition 3.7. If $(A, \mathcal{P})$ is a topological BL-algebra. Then the following statements hold:
(i) There exists a continuous function $h : A \hookrightarrow A$ by $h(x) = y$ for every $x, y \in A$.
(ii) If L_{a^-} or R_a is an open map, for $a \in MV(A)$, then T_a, is an open map, where $T_a : A \hookrightarrow A$, $T_a(x) = a \oplus x = (a^- \odot x^-)^-$, $L_a : A \hookrightarrow A$, $L_a(x) = a \ominus x = a \odot x^-$ and $R_a : A \hookrightarrow A$, $R_a(x) = x \ominus a = x \odot a^-$.
proof.(i) Let $(A, \mathcal{P})$ be a topological BL-algebra. Then by Theorem 3.4, $(A, \mathcal{P})$ is quasi and para-topological BL-algebra. So we conclude the continuity of the $-, \odot, \oplus$ and functions R_a and L_a. Thus, $h = L_y \circ R_x$ is a continuous map. We now have

$$\begin{aligned}h(x) &= L_y \circ R_x(x) \\ &= L_y(x \ominus x) \\ &= L_y(x \odot x^-) \\ &= L_y(0) \\ &= y \ominus 0 \\ &= y\end{aligned}$$

(ii) We know that if $(A, \mathcal{P})$ is a topological BL-algebra, then we conclude $(MV(A), MV(A) \cap \mathcal{P})$ is a topological BL-algebra. Since $x^{--} = x$, for every $x \in MV(A)$, so the map $- : MV(A) \hookrightarrow MV(A)$ is invertible and its inverse is equal to itself. By the continuity of " $-$ ", we conclude the map " $-$ " is homeomorphism. Suppose that $a \in MV(A)$ and $U \in MV(A) \cap \mathcal{P}$. First, let us assume that R_a is an open map, $\mathcal{P}$ is a topology on $MV(A)$ such that $(MV(A), \mathcal{P})$ is a topological BL-algebra. Since " $-$ " is homeomorphism, so $U^- \in MV(A) \cap \mathcal{P}$. Therefore, $U^- \odot a^- = R_a(U^-) \in MV(A) \cap \mathcal{P}$, $T_a(U) = a \oplus U = (a^- \odot U^-)^- \in MV(A) \cap \mathcal{P}$, where T_a and $R_a : MV(A) \hookrightarrow MV(A)$. Now let $a \in MV(A)$ and $L_{a^-} : MV(A) \hookrightarrow MV(A)$ be an open map, then

$$\begin{aligned} L_{a^-}(x) &= a^- \ominus x \\ &= a^- \odot x^-. \end{aligned}$$

Let $U \in MV(A) \cap \mathcal{P}$, then $U^- \in MV(A) \cap \mathcal{P}$ and $a^- \odot U^- = L_{a^-}(U^-) \in MV(A) \cap \mathcal{P}$.

Proposition 3.8. Let A be a BL-algebra which is endowed by a topology $\mathcal{P}$. Consider the following assertions :
(i) The topological space $(A, \mathcal{P})$ is a quasi-topological BL-algebra;
(ii) $\odot$ and $-$ are semi-continuous and continuous, respectively;
(iii) $\ominus$ and $-$ are semi-continuous and continuous, respectively;
(iv) $\rightarrow$ and $-$ are semi-continuous and continuous, respectively.
Then (i) implies (ii), (ii) implies (iii) and for $MV(A)$, (iii) implies (iv).
proof. (i)$\Rightarrow$(ii) By Definition 3.1, it becomes clear.
(ii)$\Rightarrow$(iii) Let W_1 be an open set such that $x \ominus y \in W_1 \in \mathcal{P}$, for $x, y \in A$. Then $x \ominus y = x \odot y^- \in W_1$. By (ii), since $\odot$ is semi-continuous, so there exist open neighborhoods U_1 of x and U_2 of y^- with $x \odot U_2 \subseteq W_1$ and $U_1 \odot \mathrm{y}^- \subseteq W_1$. From (ii), "$-$" is continuous, thus $y \in W_2$ and ${W_2}^- \subseteq U_2$, for some $W_2 \in \mathcal{P}$. For any $k \in U_1$ and $l \in W_2$, we have $k \ominus y = k \odot \mathrm{y}^- \in U_1 \odot \mathrm{y}^- \subseteq W_1$ and $x \ominus l = x \odot l^- \in x \odot {W_2}^- \subseteq x \odot U_2 \subseteq W_1$. Hence, $U_1 \ominus y \subseteq W_1$ and $x \ominus W_2 \subseteq W_1$, this means that "$\ominus$" is semi-continuous.
(iii)$\Rightarrow$(iv) We consider $(MV(A), \mathcal{P})$, where $\mathcal{P}$ is a topology on $MV(A)$ and "$\ominus$" is a semi-continuous and "$-$" is a continuous mapping and suppose that $x \rightarrow y \in W_1 \in \mathcal{P}$. Then $x \rightarrow y = (x \ominus y)^- \in W_1$. Since "$-$" is continuous, so there exists $U \in \mathcal{P}$, $x \ominus y \in U$ and $U^- \subseteq W_1$. From (iii), "$\ominus$" is semi-continuous, then there exist open neighborhoods

U_1 and W_2 of x and y, with $U_1 \ominus y \subseteq U$ and $x \ominus W_2 \subseteq U$. For any $k \in U_1$ and $l \in W_2$, $k \to y = (k \ominus y)^- \in (U_1 \ominus y)^- \subseteq U^- \subseteq W_1$ and $x \to l = (x \ominus l)^- \in (x \ominus W_2)^- \subseteq U^- \subseteq W_1$, then $U_1 \to y \subseteq W_1$ and $x \to W_2 \subseteq W_1$. Therefore, "$\to$" is semi-continuous.

From [1], a family ξ of nonempty subsets of a set X is called a pre-filter on X if $X \in \xi$ and for each finite collection $\{A_i\}_{i=1}^k$ of elements of ξ, there exists $B \in \xi$ such that $B \subseteq \cap_{i=1}^k A_i$.

Definition 3.9. [10] Let A be a BL-algebra, $a \in A$ and $\emptyset \neq U \subseteq A$. We define $U(a) = \{x \in A \mid R_a(x) \in U \text{ and } L_a(x) \in U\}$, ($L_a(x)$ and $R_a(x)$ are defined in Proposition 3.7).
It is trivial that if $U \subseteq W \subseteq A$, then $U(a) \subseteq W(a)$.

Remark 3.10. Let A be a BL-algebra. From [13], we recall that, for every $x, y, z \in A$, $z \leq x \oslash y$ iff $z \ominus x \leq y$; since, $z \leq x \oslash y$ iff $z \leq x^- \to y$ iff $z \odot x^- \leq y$ iff $z \ominus x \leq y$. We also have $x \ominus z \leq (y \ominus z) \oslash (x \ominus y)$; because, $(x \ominus z) \ominus (y \ominus z) = (x \odot z^-) \ominus (y \odot z^-) = (x \odot z^-) \odot (y \odot z^-)^- = (x \odot z^-) \odot (y \to z^{--}) \leq (x \odot z^-) \odot (z^{---} \to y^-) = x \odot z^- \odot (z^- \to y^-) \leq x \odot y^- = x \ominus y$. This means that $(x \ominus z) \ominus (y \ominus z) \leq (x \ominus y)$, i.e., $x \ominus z \leq (y \ominus z) \oslash (x \ominus y)$.

Theorem 3.11. [9] Let A be a BL-algebra and U be an ideal of A. Then relation "$\sim_U$" on A defined by:
for every $x, y \in A$, "$x \sim_U y$ iff $x \ominus y \in U$ and $y \ominus x \in U$" is a congruence on A.

Corollary 3.12. Let A be a BL-algebra and U be an ideal of A. If for every $x, y, z, t \in A$; $x \ominus y,\ y \ominus x,\ z \ominus t,\ t \ominus z \in U$, then $((x \odot z) \ominus (y \odot t)) \in U$ and $((y \odot t) \ominus (x \odot z)) \in U$.

Theorem 3.13. Let A be a BL-algebra and ξ a pre-filter on A such that for every $U \in \xi$ and $t, s \in U$:
(i) $0 \in \cap \xi$;
(ii) $R_s \circ R_t(x) = 0$ implies that $x \in U$.
Then, the following statements hold:

(1) The set $\mathcal{P} = \{W \subseteq A \mid$ for every $a \in W,\ U(a) \subseteq W$ for some $U \in \xi\}$ is a topology on A;
(2) ξ is a fundamental system of 0;
(3) $U \in \xi$ is an ideal of A;
(4) $U(s)$ is an open set, for $s \in A$ and $U \in \xi$;
(5) "$\odot$" is a continuous operation on $(A, \mathcal{P})$, where $\mathcal{P}$ is defined in (1).
proof. (1): We set $\mathcal{P} = \{W \subseteq A \mid$ for every $a \in W,\ U(a) \subseteq W$ for some $U \in \xi\}$. Obviously, $\mathcal{P}$ contains A and $\emptyset$. We suppose that $\{W_\alpha\}$ is a sub-collection of $\mathcal{P}$, and a be an element of $\cup W_\alpha$. Then $a \in W_\alpha$ for some α and there exists $U \in \xi$ such that $U(a) \subseteq W_\alpha \subseteq \cup W_\alpha$. So, $\cup W_\alpha \in \mathcal{P}$. Let $W_\alpha,\ W_\beta \in \mathcal{P}$, $W = W_\alpha \cap W_\beta$ and $a \in W_\alpha \cap W_\beta$. Then, there exist $U_1 \in \xi$ and $U_2 \in \xi$ such that $U_1(a) \subseteq W_\alpha$ and $U_2(a) \subseteq W_\beta$. Since ξ is a pre-filter, there exists $U \in \xi$, $U \subseteq U_1 \cap U_2$. Now, $U(a) \subseteq (U_1 \cap U_2)(a) \subseteq U_1(a) \cap U_2(a) \subseteq W_\alpha \cap W_\beta$, this means that $W_\alpha \cap W_\beta \in \mathcal{P}$.
(2): Let $t \in U \in \xi$. Since $0 \in \cap\xi$, so $0 \in U$. We assume that z is an element of $U(t)$, then, $z \ominus t$ and $t \ominus z \in U$. It is easy to see that $R_{z\ominus t} \circ R_t(z) = 0$, therefore by (ii), $z \in U$, i.e., $U(t) \subseteq U$ and $U \in \mathcal{P}$. Hence U is an open neighborhood of 0. If W is an open neighborhood of 0, then there is $V \in \mathcal{P}$ such that $0 \in V \subseteq W$. From definition of $\mathcal{P}$ we get that $U(0) \subseteq V$ for some $U \in \xi$. It is evident that for every $t \in U$, $t \ominus 0$ and $0 \ominus t \in U$, then $0 \in U \subseteq U(0) \subseteq V \subseteq W$, i.e., (2) holds.
(3): We show that U is an ideal of A, for an arbitrary element $U \in \xi$. Let $z \in U$ and $t \leq z$. Since $0, z \in U$, we have $R_0 \circ R_z(t) = R_0(t \ominus z) = R_0(t \odot z^-) = (t \odot z^-) \odot 0^- = (t \odot z^-) \odot 1 = (t \odot z^-)$. From the fact that $t \leq z$, we conclude $t \odot z^- \leq z \odot z^- = 0$, i.e., $t \odot z^- = 0$. Therefore,

by (ii), $t \in U$. Let $x, y \in U$, then

$$\begin{aligned} R_x \circ R_y(x \oslash y) &= R_x \circ R_y(x^- \to y) \\ &= R_x((x^- \to y) \ominus y) \\ &= R_x((x^- \to y) \odot y^-) \\ &= ((x^- \to y) \odot y^-) \ominus x \\ &= ((x^- \to y) \odot y^-) \odot x^- \\ &= (x^- \odot (x^- \to y) \odot y^-) \\ &= (x^- \wedge y) \odot y^- \\ &\leq y \odot y^- \\ &= 0. \end{aligned}$$

This means that, $(x \oslash y) \in U$ and U is an ideal of A.
(4): We claim that $U(s)$ is an open set, for $s \in A$ and $U \in \xi$. Let $t \in U(s)$. It is enough to show that $U(t) \subseteq U(s)$. Let $z \in U(t)$, then $R_s(t), L_s(t), R_t(z)$ and $L_t(z) \in U$, i.e., $t \ominus s, s \ominus t, z \ominus t$ and $t \ominus z \in U$. Since U is an ideal, so it is closed under "$\oslash$", therefore, $(t \ominus z) \oslash (s \ominus t)$ and $(t \ominus s) \oslash (z \ominus t) \in U$. By Remark 3.10, we have $(s \ominus z) \leq (t \ominus z) \oslash (s \ominus t)$ and $(z \ominus s) \leq (t \ominus s) \oslash (z \ominus t)$. Since U is an ideal, we conclude $(s \ominus z)$ and $(z \ominus s) \in U$. This means that $z \in U(s)$, i.e., $U(t) \subseteq U(s)$.
(5): We prove the continuity of "$\odot$". Suppose that K is an open neighborhood of $s \odot t$. Then, there exists $U \in \xi$ such that $U(s \odot t) \subseteq K$. Clearly, $L_s(s) = R_s(s) = L_t(t) = R_t(t) = 0 \in U$, i.e., $s \in U(s)$ and $t \in U(t)$. Now, we show that $U(s) \odot U(t) \subseteq U(s \odot t)$. Let $i \odot j \in U(s) \odot U(t)$, with $i \in U(s)$ and $j \in U(t)$, i.e., $L_s(i) = s \ominus i$, $R_s(i) = i \ominus s$, $R_t(j) = j \ominus t$ and $L_t(j) = t \ominus j \in U$. By Corollary 3.12, since U is an ideal, we conclude $((s \odot t) \ominus (i \odot j)) \in U$ and $((i \odot j) \ominus (s \odot t)) \in U$. This means that $L_{(s \odot t)}(i \odot j) \in U$ and $R_{(s \odot t)}(i \odot j) \in U$. Therefore, $(i \odot j) \in U(s \odot t)$.

In Theorem 3.13, if we consider the $MV(A)$ (as an MV-algebra which is a BL-algebra), then $(MV(A), \mathcal{P})$ is a quasi-topological BL-algebra. Indeed, we only need to prove the continuity of the unary operation "$-$". Suppose that $t \in MV(A)$ and N is an open neighborhood of t^-, then $Q(t^-) \subseteq N$, for some $Q \in \xi$. We show that $(Q(t))^- \subseteq Q(t^-)$. Let $i \in (Q(t))^-$, then $i = j^-$, for some $j \in Q(t)$. This means that $j \ominus t$

and $t \ominus j \in Q$. Since $j^{--} = j \in Q(t) \subseteq MV(A)$, so

$$\begin{aligned} t^- \ominus j^- &= t^- \odot j^{--} \\ &= t^- \odot j \\ &= j \odot t^- \\ &= j \ominus t \in Q. \end{aligned}$$

We also have

$$\begin{aligned} j^- \ominus t^- &= j^- \odot t^{--} \\ &= t^{--} \odot j^- \\ &= t \odot j^- \\ &= t \ominus j \in Q. \end{aligned}$$

Therefore, $i = j^- \in Q(t^-)$, i.e., $(Q(t))^- \subseteq Q(t^-)$. This means that $(Q(t))^- \subseteq N$, hence the continuity of "$-$" holds.

Definition 3.14. [10] Let A be a BL-algebra and $\mathcal{M}$ be a collection of subsets of A. Then $\mathcal{M}$ is a system of 1 if $\cap\mathcal{M}$ contains 1 and the following conditions hold:
(i) for every $x \in U \in \mathcal{M}$, there exists $V \in \mathcal{M}$ such that $x \odot V \subseteq U$;
(ii) for every $U \in \mathcal{M}$, there exists $V \in \mathcal{M}$ such that $V \odot V \subseteq U$;
(iii) for $U, V \in \mathcal{M}$, there exists $W \in \mathcal{M}$ such that $W \subseteq U \cap V$.

Proposition 3.15. If $\mathcal{M}$ is a fundamental system of open neighborhoods of 1 in a topological BL-algebra $(A, \mathcal{P})$. Then $\mathcal{M}$ is a system of 1 in A.
proof. Since $\mathcal{M}$ is a fundamental system of open neighborhoods of 1, so $1 \in \cap\mathcal{M}$. Let $x \in W \in \mathcal{M}$. Since $\odot$ is continuous and $x \odot 1 = x \in W$, there is an open neighborhood K of 1 such that $x \odot K \subseteq W$. There exists $U \in \mathcal{M}$ such that $1 \in U \subseteq K$, since $\mathcal{M}$ is a fundamental system of open neighborhoods of 1, thus $x \odot U \subseteq x \odot K \subseteq W$. Now, let $W \in \mathcal{M}$, then $1 \odot 1 = 1 \in W$ and the mapping "$\odot$" is continuous, so there exist open neighborhoods K_0 and K_1 of 1 such that $K_0 \odot K_1 \subseteq W$. We set $K = K_0 \cap K_1$. Since $\mathcal{M}$ is a fundamental system of open neighborhoods of 1, there is $U \in \mathcal{M}$ such that $1 \in U \subseteq K$. Therefore

$U \odot U \subseteq K_0 \odot K_1 \subseteq W$, (ii) holds. Since $\mathcal{M}$ is closed under the finite intersection, so (iii) holds.

Theorem 3.16. If A be a BL-algebra and $\mathcal{M}$ is a system of 1 in A, then the following statements hold:
(i) The set $\mathcal{P} = \{K \subseteq A \mid \text{for all } t \in K, \text{there exists } Q \in \mathcal{M};\ t \odot Q \subseteq K\}$, is a topology on A;
(ii) $\mathcal{M}$ is a fundamental system of open neighborhoods of 1 with respect to $\mathcal{P}$;
(iii) $(A, \odot, \mathcal{P})$ is a quasi-topological BL-algebra.
proof. We set $\mathcal{P} = \{K \subseteq A \mid \text{for all } t \in K, \text{there exists } Q \in \mathcal{M};\ t \odot Q \subseteq K\}$ and show that for every $t \in A$ and $Q \in \mathcal{M}$, $t \odot Q \in \mathcal{P}$. Obviously, $\emptyset, A \in \mathcal{P}$. Assume that $s \in t \odot Q$, then there exists $q \in Q$; $s = t \odot q$. By the properties of $\mathcal{M}$, $q \odot D \subseteq Q$, for some $D \in \mathcal{M}$. Therefore, $s \odot D = (t \odot q) \odot D \subseteq t \odot Q$. This means that $t \odot Q \in \mathcal{P}$.
We suppose that $\{K_j \mid j \in I\}$ is a sub-collection of $\mathcal{P}$. From the fact that $t \in \cup K_j$, there exists K_i; $t \in K_i$. This means that $t \odot Q \subseteq K_i$, for some $Q \in \mathcal{M}$ therefore, $t \odot Q \subseteq \cup K_j$, i.e., $\cup K_j \in \mathcal{P}$.
Let $K_1, K_2 \in \mathcal{P}$, and $K = K_1 \cap K_2$. We show that $K \in \mathcal{P}$. Let $t \in K$, then there exist Q_1 and $Q_2 \in \mathcal{M}$ such that $t \odot Q_1 \subseteq K_1$ and $t \odot Q_2 \subseteq K_2$. Since $\mathcal{M}$ is a system of 1, so there is $Q \in \mathcal{M}$, $Q \subseteq Q_1 \cap Q_2$. From the fact that $t \odot Q \subseteq t \odot (Q_1 \cap Q_2) \subseteq (t \odot Q_1) \cap (t \odot Q_2) \subseteq K_1 \cap K_2 = K$, we have $K \in \mathcal{P}$. Thus being $\mathcal{P}$ on A as a topology is now deduced. From Definitions 2.3 and 3.14, it is trivial that $\mathcal{M}$ is a fundamental system of open neighborhoods of 1.
Since every continuous operation is semi-continuous, so for the proof of (iii), it is enough to show that the continuity of the binary operation "$\odot$". We assume that $t = s_1 \odot s_2$ and K is an open neighborhood of t. Then $t \odot D \subseteq K$, for some $D \in \mathcal{M}$. Since $D \in \mathcal{M}$, so $P \odot P \subseteq D$, for some $P \in \mathcal{M}$. Hence there exist open neighborhoods $s_1 \odot P$ of s_1 and $s_2 \odot P$ of s_2, with $(s_1 \odot P) \odot (s_2 \odot P) = (s_1 \odot s_2) \odot P \odot P \subseteq (s_1 \odot s_2) \odot D = t \odot D \subseteq K$, which indicate the continuity of the "$\odot$".

Proposition 3.17. If A, J, G and $\mathcal{P}$ are a BL-algebra, an ideal of A, a filter of A and a topology on A respectively, such that $(A, \mathcal{P})$ be a topological BL-algebra. Then the following hold:

(i) If 0 is an interior point of J ($0 \in J^\circ$), then J is an open set;
(ii) If 1 is an interior point of G, then G is a closed set.
proof. (i) Suppose that J is an ideal of A, $t \in J$ and $0 \in J^\circ$. Then $0 \in K \subseteq J$, for some $K \in \mathcal{P}$. From the continuity of "$\ominus$" (Proposition 3.5) and the fact that $0 = t \odot t^- = t \ominus t \in J$, we conclude $Q \ominus Q \subseteq K \subseteq J$, for some open neighborhood Q of t. We claim that $Q \subseteq J$. Since for any such Q there is an open set $U_t \in \mathcal{P}$ such that $t \in U_t \subseteq Q \subseteq J$. We get that $J = \cup_{t\in J} U_t \in \mathcal{P}$. Therefore, J is an open set.
(ii) Let G be filter of A and $1 \in G^\circ$. It is enough to show that the complement of G ($G^c = A - G$) is an open set. We assume that $a \in A - G$. Since $1 \in G^\circ$, so $K \subseteq G$, for some open neighborhood K of 1. By the continuity of "$\rightarrow$" (Definition 2.8) and $a \rightarrow a = 1$, (Proposition 2.4), we have $E \rightarrow E \subseteq K \subseteq G$, for some open neighborhood E of a. We claim that $E \subseteq A - G$, otherwise, there exists $t \in E$; $t \notin A - G$, i.e., $t \in E$ and $t \in G$. Since for every $b \in E$, $t \rightarrow b \in E \rightarrow E \subseteq G$, and $t \in G$, so according to the paragraph after the Definition 2.5, we conclude $b \in G$, i.e., $E \subseteq G$, which is a contradiction with the fact that $a \in E$, but $a \notin G$ ($a \in A - G$). This means that, $a \in E \subseteq A - G$ and $A - G$ is an open set.

Proposition 3.18. Let $(A, \mathcal{P})$, $G \subseteq A$ and $H \subseteq A$ be a topological BL-algebra, compact subset of A and closed subset of A respectively. If G and H are disjoint, then there exists an open neighborhood U of 1 such that $(G \odot U) \cap H = \emptyset$.
proof. Since $G \cap H = \emptyset$ and H is closed, so $A - H$ is an open set containing t, for every $t \in G$ ($t \notin H$, i.e., $t \in A - H$). Thus, there exists an open neighborhood K_t of t such that $K_t \subseteq G \subseteq A - H$. Since $(A, \mathcal{P})$ is a topological BL-algebra and $t \odot 1 = t \in K_t$, there exists an open neighborhood Q_t of 1 such that $t \odot Q_t \subseteq K_t$ (▼).
We consider $S_t \odot S_t \subseteq Q_t$, (♦) for some open neighborhood S_t of 1 (It exists, from the fact that $1 = 1 \odot 1$ and $\odot$ is a continuous mapping). If $C = \{t \odot S_t | t \in G\}$, then $G \subseteq \bigcup_{c\in C} c$, i.e., the collection C of open neighborhoods is covers G, with $(t \odot S_t) \cap H = \emptyset$. Since G is compact, so $G \subseteq \bigcup_{c_\alpha \in C} c_\alpha$, for some finite sub-collection $C_\alpha \subseteq C$. This means that $G \subseteq \cup_{t\in B}(t \odot S_t)$, for some finite subset $B \subset G$. Let $Q = \cap_{t\in B} S_t$. Then Q is an open neighborhood of 1. We show that $(G \odot Q) \cap G = \emptyset$.

Suppose that r is an arbitrary element of G. Since $r \in G \subseteq \cup_{t\in B}(t \odot S_t)$, there exists $b \in B$, $r \in b \odot S_b$. Thus

$$\begin{array}{rcll} r \odot Q & \subseteq & b \odot S_b \odot Q & \\ & \subseteq & b \odot S_b \odot S_b & \textit{by } (Q = \cap_{t\in B} S_t) \\ & \subseteq & b \odot Q_b & \textit{by } (\blacklozenge) \\ & \subseteq & K_b & \textit{by } (\blacktriangledown) \\ & \subseteq & A - H. & \end{array}$$

Therefore for every $r \in G$, $(r \odot Q) \cap H = \emptyset$, i.e., $(G \odot Q) \cap H = \emptyset$.

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