Journal of Mathematical Extension
Vol. 15, No. 3, (2021) (7)1-18
URL: https://doi.org/10.30495/JME.2021.1453
ISSN: 1735-8299
Original Research Paper

Spatial Behavior of Solutions for a Class of Hyperbolic Equations with Nonlinear Dissipative Terms

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Abstract. This paper deals with the spatial behavior of solutions for a viscoelastic wave equations with nonlinear dissipative terms in a semi-infinite n-dimensional cylindrical domain. An alternative of Phragmén-Lindelöf type theorems is obtained in the result. In the case of decay, an upper bound will be derived for the total energy by means of the boundary data. The main point of the contribution is the use of energy method.

AMS Subject Classification: 35B44; 35L20; 35L70
Keywords and Phrases: Spatial estimates, hyperbolic equations, viscoelasticity

1 Introduction

Starting the twentieth century, the study of spatial estimates have been received a lot of attention of researchers in mechanics and applied mathematics. These studies are motivated by the Saint-Venant principle which has been widely investigated in the study of asymptotic behavior of end effects for partial differential equations and systems in semi-infinite domains. Saint-Venant principle is initiated by Toupin [25] and developed by Horgan and Knowles [15] and the updated articles by Horgan [13],

Received: December 2019; Accepted: April 2020

[14]. The same kind of results can be found in the studies by Knowles [16], [18], Oleinik [19], Flavin [7], Flavin *et al.* [9]-[11] and Horgan [12]. In recent years, many problems deal with spatial growth or decay end effects for the solutions of initial-boundary value problems containing hyperbolic equations. We may recall the studies by Flavin *et.al* [8] and Chirita *et.al* [2]-[4] and the references cited in these works. Quintanilla in [20] established spatial estimates for some classes of hyperbolic heat equation and proved same results in nonlinear viscoelasticity (See [21] and [22]). In linear viscoelasticity, Diaz and Quintanilla [5] proved simillar results. In another work by Quintanilla and Saccomandi [23], a dispersive wave equation in the form

$$u_{tt} - \epsilon\Delta u_{tt} + bu_t = a\Delta u,$$

has been considered. They proved a Phragmén-Lindelöf alternative of exponential type in the presence of dissipative term u_t. In the case $b = 0$, while they showed that an exponential decay is possible, they obtained a polynomial decay rate for solutions. In [26], Yilmaz obtained the spatial behavior of solutions for the quasiliniear equation modelling dynamic viscoelasticity

$$u_{tt} - \nabla.(W(|\nabla u|)\nabla u) + b|u_t|^{p-2}u_t = \alpha\Delta u_t,$$

where $b, \alpha > 0, p > 1$ and $W(s) = a+s^{p-2}, a > 0$. In this regard, we may also recall another work by Y. Liu *et al.* [17] in which authors studied equations in the form

$$|\dot{u}|^{\varepsilon}\ddot{u} - (|\nabla u|^p u_{,i})_{,i} - \gamma\dot{u}_{,jj} = 0,$$

where ε, p and γ are positive constants with $p \leq \varepsilon$. They showed that the solution grows at least exponentially or decays at least algebraically with distance along the cylinder from the base. In this work, motivated by the above studies, we investigate the equation:

$$\begin{aligned} u_{tt} - \Delta u + (g*\Delta u)(t) + au_t + u_t|u_t|^m \\ = \operatorname{div}(\nabla u|\nabla u|^p) + \operatorname{div}(\nabla u_t|\nabla u_t|^p), \end{aligned} \tag{1}$$

in $\Omega \times (0,T)$ under the initial-boundary conditions

$$u(x',0,t) = h(x',t), \ (x',t) \in D_0 \times (0,T), \tag{2}$$
$$u(x,t) = 0, \quad (x,t) \in S_0 \times (0,T), \tag{3}$$
$$u(x,0) = u_t(x,0) = 0, \quad x \in \Omega, \tag{4}$$

where $a > 0$, $m \geq 0$, $p > 0$, ν is the outward normal to the boundary and

$$(g * v)(t) := \int_0^t g(t-\tau)v(\tau)d\tau.$$

Ω is the cylinder

$$\Omega = \left\{x \in \mathbb{R}^n : x_n \in R^+, x' \in D_{x_n}, n \geq 2\right\},$$

where the cross-section D_z is a bounded simply-connected region in R^{n-1} and

$$S_z = \{x \in \mathbb{R}^n : x' \in \partial D_{x_n}, z \leq x_n < \infty\}.$$

Equations of the form (1) have their origin in the mathematical description of motion of viscoelastic materials. Viscoelastic materials exhibit effects of both elasticity and viscosity. Such materials have memory. The stress is a functional of the past history of the strain, instead of being a function of the present strain value (elastic) or of the present value of the time derivative of strain (viscous). The principal type of stress-strain relation has either an explicit elastic term or an explicit viscous term, plus an integral over the past history of some nonlinear function of the strain. Existence theorems can be obtained by an extension of methods used for pure partial differential equations, principally energy estimates for hyperbolic equations. Mathematical analysis on the motions of materials with memory can be found in [6], [24] and references therein. In our work, we assume the existence of classical solutions of the problem (1)-(4).

In this article we study the spatial behavior of solutions for the problem (1)-(4) in the case $m \leq p$. An essential ingredient of our proof is based on construction a cross sectional energy function in order to obtain linear differential inequalities to finally prove that the energy grows or decays exponentially. Our method of proof follows closely the arguments of [23] and [26] with modifications needed for our problem. It is worth noting that one can find similar results when $m > p$ by defining a suitable weighted energy function and employing similar arguments in [17]. Finally, in the case of decay, an upper bound will be derived for the total energy (the amplitude term) by means of the boundary datum.

The rest of the paper is organized as follows. In section 2 we state some preliminaries and assumptions about the problem (1)-(4). Our main result is obtained in Section 3.

2 Preliminaries

In this section, we present some material that will be needed throughout the article. We denote by $(u, v)_D$ the usual L^2 inner product on D and for $1 \leq s \leq \infty$ we denote the L^s norm on D by $\|u\|_{s,D}$. In the sequel we use

$$\Omega_z = \Omega \cap \{x \in \mathbb{R}^n : 0 < x_n < z\}, \quad R_z = \Omega \cap \{x \in \mathbb{R}^n : z < x_n < \infty\}.$$

Lemma 2.1. (Sobolev-Poincaré inequality [1]) *Let s be a number with $2 \leq s \leq \infty$ $(n = 1, 2, ..., r)$ or $2 \leq s \leq \frac{nr}{n-r}$ $(n \geq r+1)$, then there exists a constant $B = B(D, s)$ such that*

$$\|u\|_{s,D} \leq B\|\nabla u\|_{r,D}, \quad for \quad u \in W_0^{1,r}(D).$$

We will also use the embedding $L^r(D) \hookrightarrow L^s(D)$ for $s < r$ with the same embedding constant B in the above inequality. We also assume that ∂D_z is sufficiently smooth to apply the divergence theorem such that

$$0 < \gamma_0 \leq \inf_z |D_z| \leq \sup_z |D_z| \leq \gamma_1 < \infty,$$

and the prescribed function h vanishes on the lateral surface S_0. For the memory kernel, we assume that

$$g \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad 1 - \int_0^\infty g(s)ds = l > 0, \tag{5}$$

and

$$g(s) \geq 0, \quad g'(s) \leq 0, \qquad \forall s \in \mathbb{R}^+. \tag{6}$$

For nonlinear terms we assume that $p \geq m \geq 0$ and

$$0 \leq m \leq \frac{4}{n-2} \quad \text{for} \quad n \geq 3. \tag{7}$$

Remark 2.2. There are many functions satisfying the conditions (5)-(6), such as

$$\begin{aligned}
g_1(t) &= \alpha(1+t)^{\mu}, \qquad \mu < -1,\\
g_2(t) &= \alpha e^{-\beta(1+t)^p}, \qquad 0 < p \leq 1,\\
g_3(t) &= \frac{\alpha e^{-\beta t}}{(1+t)^n}, \qquad n = 0, 1, 2, ...,\\
g_4(t) &= \frac{\alpha}{(2+t)^{\mu}\left(\ln(2+t)\right)^{\beta}}, \qquad \mu > 1,
\end{aligned}$$

where α and β are positive constants which are chosen properly.

Finally, we state the following lemma that will be needed throughout our proofs.

Lemma 2.3. *For $\psi \in C^1([0,+\infty), \mathbb{R})$, we have*

$$\begin{aligned}
(g * \psi)(t)\psi_t(t) = &-\frac{1}{2}\frac{d}{dt}(g \diamond \psi)(t) + \frac{1}{2}\frac{d}{dt}\left\{\int_0^t g(\tau)d\tau |\psi(t)|^2\right\}\\
&+\frac{1}{2}(g' \diamond \psi)(t) - \frac{1}{2}g(t)|\psi(t)|^2,
\end{aligned}$$

where

$$(g \diamond \psi)(t) := \int_0^t g(t-\tau)|\psi(t) - \psi(\tau)|^2 d\tau.$$

Proof. We have

$$\begin{aligned}
&(g * \psi)(t)\psi_t(t)\\
&\quad = \int_0^t g(t-\tau)\big(\psi(\tau) - \psi(t)\big)\psi_t(t)d\tau + \frac{1}{2}\int_0^t g(t-\tau)\frac{d}{dt}|\psi(t)|^2 d\tau\\
&\quad = -\frac{1}{2}\frac{d}{dt}\int_0^t g(t-\tau)|\psi(\tau) - \psi(t)|^2 d\tau\\
&\qquad + \frac{1}{2}\int_0^t g'(t-\tau)|\psi(\tau) - \psi(t)|^2 d\tau\\
&\qquad + \frac{1}{2}\frac{d}{dt}\left\{\int_0^t g(\tau)d\tau|\psi(t)|^2\right\} - \frac{1}{2}g(t)|\psi(t)|^2. \ \square
\end{aligned}$$

3 Spatial estimates

In this section we will investigate the spatial behavior of solutions for the problem (1)-(4). We begin with the following lemma.

Lemma 3.1. *Suppose that $m \le p$ and (7) holds, then there exists a positive constant $C > 0$ such that*

$$\int_{D_z} |u|^{m+2} dx' \le C \left(\|\nabla u\|_{2,D_z}^2 + \int_{D_z} |u|^{p+2} dx' \right).$$

Proof. If $\|u\|_{m+2,D_z} \le 1$, then by (7) we have

$$\|u\|_{m+2,D_z}^{m+2} \le \|u\|_{m+2,D_z}^2 \le B_z^2 \|\nabla u\|_{2,D_z}^2.$$

where B_z is the best embedding constant on D_z. If $\|u\|_{m+2,D_z} > 1$, then

$$\|u\|_{m+2,D_z}^{m+2} \le \|u\|_{m+2,D_z}^{p+2} \le B_z^{p+2} \|u\|_{p+2,D_z}^{p+2}.$$

Taking $C = \sup_z C_z$, with $C_z = B_z^2 + B_z^{p+2}$, completes the proof. □

For solutions of the problem (1)-(4), we introduce the energy function $\phi_\epsilon(z,T)$ given by

$$\begin{aligned} \phi_\epsilon(z,T) = \int_0^T \int_{D_z} \Big[(u_t + \epsilon u) \Big(& u_{x_n} + u_{x_n} |\nabla u|^p \\ & + u_{tx_n} |\nabla u_t|^p - (g * u_{x_n})(t) \Big) \Big] dx' dt, \end{aligned} \tag{8}$$

where $\epsilon > 0$ is a constant. By lemma 2.3 we have

$$\begin{aligned} &\int_{\Omega_z} \int_0^t g(t-\tau) \nabla u_t(t) \nabla u(\tau) d\tau dx \\ &\quad = -\frac{1}{2} \frac{d}{dt} \Big[(g \circ \nabla u)_{\Omega_z}(t) \Big] + \frac{1}{2} \frac{d}{dt} \left[\int_0^t g(\tau) d\tau \|\nabla u(t)\|_{2,\Omega_z}^2 \right] \\ &\quad + \frac{1}{2} (g' \circ \nabla u)_{\Omega_z}(t) - \frac{1}{2} g(t) \|\nabla u\|_{2,\Omega_z}^2, \end{aligned} \tag{9}$$

where

$$(g \circ v)_D(t) = \int_0^t g(t-\tau) \|v(t) - v(\tau)\|_{2,D}^2 d\tau.$$

Using the divergence theorem, (8), (9) and the initial-boundary conditions (3) and (4), we obtain

$$\begin{aligned}
\phi_\epsilon(z,T) &= \phi_\epsilon(0,T)\\
&+ \frac{1}{2}\|u_t\|^2_{2,\Omega_z} + \frac{a\epsilon}{2}\|u\|^2_{2,\Omega_z} + \epsilon(u,u_t)_{\Omega_z} + \frac{1}{p+2}\int_{\Omega_z}|\nabla u|^{p+2}dx\\
&+ \frac{1}{2}\left(1-\int_0^T g(\tau)d\tau\right)\|\nabla u\|^2_{2,\Omega_z} + \frac{1}{2}(g\circ\nabla u)_{\Omega_z}(T)\\
&+ (a-\epsilon)\int_0^T \|u_t\|^2_{2,\Omega_z}dt + \int_0^T\left(\frac{1}{2}g(t)+\epsilon\right)\|\nabla u\|^2_{2,\Omega_z}dt\\
&+ \int_0^T\int_{\Omega_z}|u_t|^{m+2}dxdt + \epsilon\int_0^T\int_{\Omega_z}uu_t|u_t|^m dxdt\\
&+ \epsilon\int_0^T\int_{\Omega_z}|\nabla u|^{p+2}dxdt + \int_0^T\int_{\Omega_z}|\nabla u_t|^{p+2}dxdt\\
&+ \epsilon\int_0^T\int_{\Omega_z}\nabla u.\nabla u_t|\nabla u_t|^p dxdt - \frac{1}{2}\int_0^T(g'\circ\nabla u)_{\Omega_z}(t)dt\\
&- \epsilon\int_0^T\int_{\Omega_z}\int_0^t g(t-\tau)\nabla u(t)\nabla u(\tau)d\tau dxdt.
\end{aligned} \tag{10}$$

To get an upper bound for $\phi_\epsilon(z,T)$ in (8), we use the Young's inequality to obtain

$$\begin{aligned}
&\int_{D_z}\int_0^t g(t-\tau)u(t)u_{x_n}(\tau)d\tau dx'\\
&\le \frac{1-l}{2}\|u\|^2_{2,D_z}\\
&\qquad + \frac{1}{2}\int_{D_z}\int_0^t g(t-\tau)|u_{x_n(\tau)} - u_{x_n}(t) + u_{x_n}(t)|^2 d\tau dx'\\
&\le \frac{1-l}{2}\|u\|^2_{2,D_z} + (g\circ\nabla u)_{D_z}(t) + (1-l)\|\nabla u\|^2_{2,D_z}.
\end{aligned} \tag{11}$$

Analogously,

$$\begin{aligned}
&\int_{D_z}\int_0^t g(t-\tau)u_t(t)u_{x_n}(\tau)d\tau dx'\\
&\quad \le \frac{1-l}{2}\|u_t\|^2_{2,D_z} + (g\circ\nabla u)_{D_z}(t) + (1-l)\|\nabla u\|^2_{2,D_z}.
\end{aligned} \tag{12}$$

Therefore, from (8),(11) and (12) one can write

$$
\begin{aligned}
&\phi_\epsilon(z,T)\\
&\quad\leq \int_0^T \int_{D_z} \Big\{(u_t+\epsilon u)\big(u_{x_n}+u_{x_n}|\nabla u|^p+u_{tx_n}|\nabla u_t|^p\big)\Big\}\,dx'dt\\
&\quad+(1+\epsilon)\int_0^T (g\circ\nabla u)_{D_z}(t)dt+(1-l)(1+\epsilon)\int_0^T \|\nabla u\|_{2,D_z}^2 dt\\
&\quad+\frac{1-l}{2}\int_0^T (\|u_t\|_{2,D_z}^2+\epsilon\|u\|_{2,D_z}^2)dt.
\end{aligned}
\tag{13}
$$

Using Young's inequality and lemma 2.1 we obtain

$$
\begin{aligned}
&\int_0^T \int_{D_z} u u_{x_n}|\nabla u|^p dx'dt\\
&\qquad\leq \frac{1}{p+2}\int_0^T \int_{D_z} |u|^{p+2}dx'dt\\
&\qquad\quad+\frac{p+1}{p+2}\int_0^T \int_{D_z} |\nabla u|^{p+2}dx'dt\\
&\qquad\leq \left(\frac{B_z^{p+2}+p+1}{p+2}\right)\int_0^T \int_{D_z} |\nabla u|^{p+2}dx'dt.
\end{aligned}
\tag{14}
$$

Similarly, we have

$$
\begin{aligned}
\int_0^T \int_{D_z} u_t u_{x_n}|\nabla u|^p dx'dt \leq& \frac{B_z^{p+2}}{p+2}\int_0^T \int_{D_z} |\nabla u_t|^{p+2}dx'dt\\
&+\frac{p+1}{p+2}\int_0^T \int_{D_z} |\nabla u|^{p+2}dx'dt,
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\int_0^T \int_{D_z} u u_{tx_n}|\nabla u_t|^p dx'dt \leq& \frac{B_z^{p+2}}{p+2}\int_0^T \int_{D_z} |\nabla u|^{p+2}dx'dt\\
&+\frac{p+1}{p+2}\int_0^T \int_{D_z} |\nabla u_t|^{p+2}dx'dt,
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&\int_0^T \int_{D_z} u_t u_{tx_n}|\nabla u_t|^p dx'dt\\
&\qquad\leq \left(\frac{B_z^{p+2}+p+1}{p+2}\right)\int_0^T \int_{D_z} |\nabla u_t|^{p+2}dx'dt.
\end{aligned}
\tag{17}
$$

Now, using the estimates (13)-(17), Poincaré and Young's inequalities, one can find

$$
\begin{aligned}
|\phi_\epsilon(z,T)| \le{}& M_1 \int_0^T \int_{D_z} |\nabla u|^{p+2} dx' dt \\
&+ M_2 \int_0^T \int_{D_z} |\nabla u_t|^{p+2} dx' + M_3 \int_0^T \|\nabla u\|^2_{2,D_z} dt \\
&+ (1+\epsilon) \int_0^T (g \circ \nabla u)_{D_z} dt + \left(1 - \frac{l}{2}\right) \int_0^T \|u_t\|^2_{2,D_z} dt,
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
M_1 &= \frac{1}{p+2}\big[(1+\epsilon)(1+p) + 2\epsilon \mathcal{B}^{p+2}\big], \\
M_2 &= \frac{1}{p+2}\big[(1+\epsilon)(1+p) + 2\mathcal{B}^{p+2}\big], \\
M_3 &= \frac{1}{2}\big[\big(1+\epsilon(2-l)\big)\mathcal{B}^2 + (3-2l)\epsilon + 2(1-l)\big],
\end{aligned}
$$

and $\mathcal{B} = \sup_z B_z$. On the other hand, a differentiation with respect to z from (10), using the assumptions (5), (6) and the inequality $\epsilon(u,u_t)_D \ge -\frac{1}{4}\|u_t\|^2_{2,D} - \epsilon^2 \|u\|^2_{2,D}$, for $\epsilon \le \frac{a}{2}$, we get

$$
\begin{aligned}
\frac{\partial \phi_\epsilon}{\partial z} \ge{}& (a-\epsilon) \int_0^T \|u_t\|^2_{2,D_z} dt + \epsilon \int_0^T \|\nabla u\|^2_{2,D_z} dt \\
&+ \int_0^T \int_{D_z} |u_t|^{m+2} dx' dt + \epsilon \int_0^T \int_{D_z} u u_t |u_t|^m dx' dt \\
&+ \epsilon \int_0^T \int_{D_z} |\nabla u|^{p+2} dx' dt + \int_0^T \int_{D_z} |\nabla u_t|^{p+2} dx' dt \\
&- \epsilon \int_0^T \int_{D_z} \int_0^t g(t-\tau) \nabla u(t).\nabla u(\tau) d\tau dx' dt \\
&+ \epsilon \int_0^T \int_{D_z} \nabla u.\nabla u_t |\nabla u_t|^p dx' dt.
\end{aligned} \tag{19}
$$

Using Young 's inequality, lemma 3.1 and lemma 2.1, for any $\delta > 0$, we obtain

$$\begin{aligned}
&\int_0^T \int_{D_z} u u_t |u_t|^m dx dt \\
&\qquad \geq -c(\delta) \int_0^T \int_{D_z} |u_t|^{m+2} dx dt - \delta \int_0^T \int_{D_z} |\ u|^{m+2} dx dt \\
&\qquad \geq -c(\delta) \int_0^T \int_{D_z} |u_t|^{m+2} dx dt \\
&\qquad\quad - \delta C \int_0^T \left(\|\nabla u\|^2_{2,D_z} + \mathcal{B}^{p+2} \int_{D_z} |\nabla u|^{p+2} dx \right) dt.
\end{aligned} \tag{20}$$

Then, from (20), using Young's inequality for the last term in the right hand side of (19) and

$$\begin{aligned}
\int_{D_z} \int_0^t g(t-\tau) \nabla u(t) . \nabla u(\tau) d\tau dx = &\frac{1}{2} \left(\int_0^t g(\tau) d\tau \right) \|\nabla u\|^2_{D_z} \\
&+ \frac{1}{2} \int_0^t g(t-\tau) \|\nabla u(\tau)\|^2_{D_z} d\tau - \frac{1}{2} (g \circ \nabla u)_{D_z}(t),
\end{aligned}$$

the estimate (19) can be rewritten in the form

$$\begin{aligned}
\frac{\partial \phi_\epsilon}{\partial z} \geq &(a-\epsilon) \int_0^T \|u_t\|^2_{2,D_z} dt \\
&+ \frac{\epsilon}{2} \int_0^T \left(1 - \int_0^t g(\tau) d\tau - 2\delta C \right) \|\nabla u\|^2_{2,D_z} dt \\
&+ \big(1 - \epsilon c(\delta)\big) \int_0^T \int_{D_z} \left(|u_t|^{m+2} + |\nabla u_t|^{p+2} \right) dx' dt \\
&+ \epsilon \left(1 - \delta\big(1 + C\mathcal{B}^{p+2}\big)\right) \int_0^T \int_{D_z} |\nabla u|^{p+2} dx' dt \\
&+ \frac{\epsilon}{2} \int_0^T \left(\|\nabla u\|^2_{2,D_z} - \int_0^t g(t-\tau) \|\nabla u(\tau)\|^2_{2,D_z} d\tau \right) dt \\
&+ \frac{\epsilon}{2} \int_0^T (g \circ \nabla u)_{D_z}(t) dt.
\end{aligned}$$

The classical Young's inequality for convolutions asserts that

$$\|f_1 * f_2\|_r \leq \|f_1\|_q \|f_2\|_s \ ,$$

where $1 \leq r, s, q \leq \infty$ and $q^{-1} + s^{-1} = r^{-1} + 1$. If $r = 1$ then, for $s = q = 1$ we have

$$\begin{aligned}
&\int_0^T \int_0^t g(t-\tau)\|\nabla u(\tau)\|_{2,D_z}^2 d\tau dt \\
&\qquad\qquad = \|g * \|\nabla u\|_{2,D_z}^2\|_1 \leq \|g\|_1 \cdot \|\|\nabla u\|_{2,D_z}^2\|_1 \\
&\qquad\qquad = \int_0^T g(t)dt \int_0^T \|\nabla u\|_{2,D_z}^2 dt \\
&\qquad\qquad \leq (1-l)\int_0^T \|\nabla u\|_{2,D_z}^2 dt \leq \int_0^T \|\nabla u\|_{2,D_z}^2 dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial \phi_\epsilon}{\partial z} \geq & c\int_0^T \int_{D_z} \left(u_t^2 + |\nabla u|^2 + |u_t|^{m+2} + |\nabla u|^{p+2} + |\nabla u_t|^{p+2}\right) dt \\
& + c\int_0^T (g \circ \nabla u)_{D_z}(t)dt,
\end{aligned} \tag{21}$$

where

$$c = \min\left\{a - \epsilon, \frac{\epsilon}{2}(l - 2\delta C), 1 - \epsilon c(\delta), \epsilon\left(1 - \delta(1 + C\mathcal{B}^{p+2})\right)\right\},$$

in which, δ is chosen so small enough such that $\delta < \min\{\frac{l}{2C}, \frac{1}{1+C\mathcal{B}^{p+2}}\}$ and so whence δ is fixed, we choose $\epsilon > 0$ sufficiently small such that $\epsilon < \min\{\frac{a}{2}, \frac{1}{c(\delta)}\}$. Upon exploiting the inequalities (18) and (21) we arrive at

$$\frac{\partial \phi_\epsilon}{\partial z} - \frac{c}{\gamma}\phi_\epsilon \geq 0, \tag{22}$$

$$\frac{\partial \phi_\epsilon}{\partial z} + \frac{c}{\gamma}\phi_\epsilon \geq 0, \tag{23}$$

where $\gamma = \max\left\{M_1, M_2, M_3, 1+\epsilon, 1 - \frac{l}{2}\right\}$. Now, assume that for some $z_0 \geq 0$, $\phi_\epsilon(z_0, t)$ is positive, since $\frac{\partial \phi_\epsilon}{\partial z}$ is positive, then $\phi_\epsilon(z, t)$ remains positive for $z \geq z_0$. Then, from (22) we have

$$\phi_\epsilon(z, T) \geq \phi_\epsilon(z_0, T)\exp\left(\frac{c}{\gamma}(z - z_0)\right). \tag{24}$$

Next, we consider that for all z, $\phi_\epsilon(z,t)$ is negative. Then, we have form (23) that the inequality

$$-\phi_\epsilon(z,T) \leq -\phi_\epsilon(0,T)\exp(-\frac{c}{\gamma}z), \tag{25}$$

is satisfied. In fact, if $\phi_\epsilon(z,t) \to 0$ as $z \to \infty$, from (10) we have

$$\begin{aligned}
&-\phi_\epsilon(z,T)\\
&= \frac{1}{2}\|u_t\|^2_{2,R_z} + \frac{a\epsilon}{2}\|u\|^2_{2,R_z} + \epsilon(u,u_t)_{R_z}\\
&+ \frac{1}{p+2}\int_{R_z}|\nabla u|^{p+2}dx + \frac{1}{2}\left(1-\int_0^T g(\tau)d\tau\right)\|\nabla u\|^2_{2,R_z}\\
&+ \frac{1}{2}(g\circ\nabla u)_{R_z}(T) + (a-\epsilon)\int_0^T \|u_t\|^2_{2,R_z}dt\\
&+ \int_0^T\left(\frac{1}{2}g(t)+\epsilon\right)\|\nabla u\|^2_{2,R_z}dt + \int_0^T\int_{R_z}|u_t|^{m+2}dxdt\\
&+ \epsilon\int_0^T\int_{R_z}uu_t|u_t|^m dxdt + \epsilon\int_0^T\int_{R_z}|\nabla u|^{p+2}dxdt\\
&+ \int_0^T\int_{R_z}|\nabla u_t|^{p+2}dxdt + \epsilon\int_0^T\int_{R_z}\nabla u.\nabla u_t|\nabla u_t|^p dxdt\\
&- \epsilon\int_0^T\int_{R_z}\int_0^t g(t-\tau)\nabla u(t)\nabla u(\tau)d\tau dxdt\\
&- \frac{1}{2}\int_0^T(g'\circ\nabla u)_{R_z}(t)dt.
\end{aligned} \tag{26}$$

Summarily, we have proved the following theorem:

Theorem 3.2 *Let u be a nontrivial solution of the initial boundary value problem (1)-(4) which satisfies (5)-(6) such that*

$$0 \leq m \leq \min\left\{p, \frac{4}{n-2}\right\}, \quad n \geq 3.$$

Then, either the solution becomes exponentially unbounded in the form (24) or it satisfies the spatial decay estimate (25).

4 A bound for the total energy

To give an estimate for the amplitude term, $-\phi_\epsilon(0,t)$, let us consider $\eta(x,t)$ be a smooth function which satisfies the boundary conditions (2)-(3) and decays uniformly to zero as x_n tends to infinity. We have

$$\begin{aligned}-\phi_\epsilon(0,T) = \int_0^T \int_{D_0} (\eta_t + \epsilon\eta)\Big(& u_{x_n} + u_{x_n}|\nabla u|^p \\ &+ u_{tx_n}|\nabla u_t|^p - (g * u_{x_n})(t)\Big) dx' dt.\end{aligned}$$

The divergence theorem gives

$$\begin{aligned}-\phi_\epsilon(0,T) =& \int_\Omega (\eta_t + \epsilon\eta) u_t dx + \int_0^T \int_\Omega \Big[\eta_{tt} u_t + (a-\epsilon)\eta_t u_t + a\epsilon\eta u_t \\ &+ (\nabla\eta_t + \epsilon\nabla\eta).\big(\nabla u + \nabla u|\nabla u|^p \\ &+ \nabla u_t|\nabla u_t|^p\big) + (\eta_t + \epsilon\eta) u_t |u_t|^m\Big] dx dt \\ &- \int_0^T \int_\Omega \int_0^t g(t-\tau)\nabla\eta_t(t).\nabla u(\tau) d\tau dx dt \\ &- \epsilon \int_0^T \int_\Omega \int_0^t g(t-\tau)\nabla\eta(t).\nabla u(\tau) d\tau dx dt.\end{aligned}$$

Using Young's inequality and the conditions (5), (6), for arbitrary positive constants ε_i, we obtain

$$\begin{aligned}&-\phi_\epsilon(0,T) \\ &\le \frac{\varepsilon_1}{2}\int_\Omega u_t^2 dx + \int_0^T \int_\Omega \Big[\frac{\varepsilon_2}{2} u_t^2 + \Big(\frac{\varepsilon_3}{2} + \varepsilon_7(1-l)(1+\epsilon)\Big)|\nabla u|^2 \\ &+ \left(\frac{1+\epsilon}{p+2}\right)\big(\varepsilon_4|\nabla u|^{p+2} + \varepsilon_5|\nabla u_t|^{p+2}\big) + \frac{\varepsilon_6}{m+2}|u_t|^{m+2}\Big] dx dt \\ &+ \varepsilon_7(1+\epsilon)(g \circ \nabla u)_\Omega(t) dt + \psi(\eta),\end{aligned} \tag{27}$$

where

$$\psi(\eta) =\frac{1}{2\varepsilon_1}\int_\Omega(\eta_t+\epsilon\eta)^2dx+\int_0^T\int_\Omega\Big[\frac{1}{2\epsilon_2}\Big(a\epsilon\eta+(a-\epsilon)\eta_t-\eta_{tt}\Big)^2$$
$$+\left(\frac{1}{\varepsilon_3}+\frac{1-l}{\varepsilon_7}\right)|\nabla\eta_t|^2+\left(\frac{\epsilon^2}{\varepsilon_3}+\frac{\epsilon(1-l)}{\varepsilon_7}\right)|\nabla\eta|^2$$
$$+\left(\frac{p+1}{p+2}\right)\left(\varepsilon_4^{-\frac{1}{p+1}}+\varepsilon_5^{-\frac{1}{p+1}}\right)\left(|\nabla\eta_t|^{p+2}+\epsilon|\nabla\eta|^{p+2}\right)$$
$$+\varepsilon_6^{-\frac{1}{m+1}}\left(\frac{m+1}{m+2}\right)(\eta_t+\epsilon\eta)^{m+2}\Big]dxdt.$$

In (27) we have used

$$\int_\Omega\int_0^t g(t-\tau)\nabla\eta(t)\nabla u(\tau)d\tau dx$$
$$\leq\varepsilon_7(1-l)\int_\Omega|\nabla u|^2dx+\varepsilon_7(g\circ\nabla u)_\Omega(t)+\frac{1-l}{\varepsilon_7}\int_\Omega|\nabla\eta|^2dx,$$

and the same inequality when η is replaced by η_t. On the other hand from (26) and the same way followed in Theorem 3.2, for $\epsilon\leq\frac{a}{2}$, one can see that

$$-\phi_\epsilon(z,T)\geq\frac14\|u_t\|^2_{R_z}+c\int_0^T(g\circ\nabla u)_{R_z}(t)dt$$
$$+c\int_0^T\int_{R_z}\Big(|u_t|^2+|\nabla u|^2 \tag{28}$$
$$+|u_t|^{m+2}+|\nabla u|^{p+2}+|\nabla u_t|^{p+2}\Big)dxdt.$$

Thus, by using (27), (28) and choosing $\varepsilon_1=\frac14$, $\varepsilon_2=\varepsilon_3=c$, $\varepsilon_4=\varepsilon_5=\frac{c(p+2)}{2(\epsilon+1)}$, $\varepsilon_6=\frac{c}{2}(m+2)$ and $\varepsilon_7=\frac{c}{2(1+\epsilon)}$ we find

$$-\phi_\epsilon(0,T)<2\psi(\eta). \tag{29}$$

Now, we select

$$\eta(x,t)=h(x',t)\exp(-\kappa x_n),$$

where the function h is defined in (2) and κ is an arbitrary positive constant. By some simple calculations, we find

$$|\nabla\eta|=\mathrm{H}\exp(-\kappa x_n),\quad|\nabla\eta_t|=\mathrm{H}^{\mathrm{t}}\exp(-\kappa x_n), \tag{30}$$

where

$$\mathrm{H} := (|\nabla' h|^2 + \kappa^2 h^2)^{\frac{1}{2}}, \quad \mathrm{H}^{\mathrm{t}} := (|\nabla' h_t|^2 + \kappa^2 h_t^2)^{\frac{1}{2}},$$

such that ∇' denotes the gradient operator in $\mathbb{R}^{n-1}$. Therefore, from (29) and (30) we can write

$$\begin{aligned}
-\phi_\epsilon(0,T)& \\
&\leq \frac{2}{\kappa}\int_{D_0}(h_t+\epsilon h)^2 dx' \\
&+\int_0^T\int_{D_0}\frac{1}{2\kappa c}\Big(a\epsilon h+(a-\epsilon)h_t-h_{tt}\Big)^2 dx'dt \\
&+\left(\frac{1+2(1-l)(1+\epsilon)}{2\kappa c}\right)\int_0^T\int_{D_0}\big(H^t\big)^2 dx'dt \\
&+\left(\frac{\epsilon^2+2\epsilon(1-l)(1+\epsilon)}{2\kappa c}\right)\int_0^T\int_{D_0}\big(H\big)^2 dx'dt \\
&+K_1\int_0^T\int_{D_0}\Big(\big(H^t\big)^{p+2}+\epsilon\big(H\big)^{p+2}\Big)dx'dt \\
&+K_2\int_0^T\int_{D_0}\big(h_t+\epsilon h\big)^{m+2}dx'dt.
\end{aligned} \tag{31}$$

where

$$K_1=\frac{2}{\kappa}\left(\frac{p+1}{(p+2)^2}\right)\left(\frac{c(p+2)}{2(\epsilon+1)}\right)^{-\frac{1}{p+1}},$$

and

$$K_2=\frac{1}{\kappa}\left(\frac{m+1}{(m+2)^2}\right)\left(\frac{c}{2}(m+2)\right)^{-\frac{1}{m+1}}.$$

The inequality (31) shows an upper bound for the total energy depending on the boundary conditions and the positive constant κ which can be chosen optimally.

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