Journal of Mathematical Extension
Vol. 16, No. 2, (2022) (5)1-10
URL: https://doi.org/10.30495/JME.2022.1440
ISSN: 1735-8299
Original Research Paper

Coposinormal Weighted Composition Operators on $H^2(\mathbb{D})$

T. Prasad
Cochin University of Science and Technology

Abstract. In this paper, we study coposinormal composition operators and posinormal weighted composition operators on the Hardy space $H^2(\mathbb{D})$. We show that if $W_{\psi,\varphi}$ is coposinormal on $H^2(\mathbb{D})$, then ψ never vanishes on $\mathbb{D}$ also we prove that φ is univalent. Moreover, we study the commutant of a coposinormal weighted composition operator.

AMS Subject Classification: MSC 47B20; 47B33; 47B38.
Keywords and Phrases: posinormal operator, composition operator, cyclic operator, Toeplitz operator, Hardy space.

1 Introduction and Preliminaries

Let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be hyponormal if $T^*T \geq TT^*$. An operator $T \in B(\mathcal{H})$ is said to be posinormal if there exists a positive operator $P \in B(\mathcal{H})$ such that $TT^* = T^*PT$, equivalently T is posinormal if $TT^* \leq \lambda^2 T^*T$ for some $\lambda \geq 0$ [14]. An operator T is coposinormal if T^* is posinormal. Hyponormal operators are necessarily posinormal, although they need not be coposinormal, e.g., the unilateral shift $U \in B(l^2)$ is hyponormal but not coposinormal(see [15, 16]).

Received: October 2019; Accepted: March 2020

In a recent paper, Le and Rhaly [13] studied coposinormality of Cesàro matrices.

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$. The Hardy space $H^2(\mathbb{D})$ (also written H^2) consists of all analytic functions on $\mathbb{D}$ having power series representations with square summable coefficients. The space $H^\infty(\mathbb{D})$ (also written H^∞) consists of all analytic and bounded functions on $\mathbb{D}$. If φ is an analytic map of $\mathbb{D}$ into itself, the composition operator C_φ on H^2 is defined by $C_\varphi f = f \circ \varphi$ where $f \in H^2$. The boundedness of C_φ for any analytic map φ of $\mathbb{D}$ into itself is a consequence of the Littlewood Subordination Theorem (see [5]).

Let φ be an holomorphic self map on the unit disc $\mathbb{D}$, and let ψ be an holomorphic map on $\mathbb{D}$. The weighted composition operator $W_{\psi,\varphi}$ on Hardy space H^2 induced by φ with weight ψ is given by

$$W_{\psi,\varphi} f = \psi(f \circ \varphi),$$

where $f \in H^2$. If ψ is bounded, then $W_{\psi,\varphi}$ is bounded. For $\psi \in H^\infty$, the multiplication operator on H^2 is given by $M_\psi f = \psi f$ for all $f \in H^2$. Remark that $W_{\psi,\varphi}$ can be written by $W_{\psi,\varphi} = M_\psi C_\varphi$. We refer [11] and [7] for more details of weighted composition operators.

For $z_0 \in \mathbb{D}$, the function K_{z_0} defined by $K_{z_0}(z) = \frac{1}{1-\bar{z}_0 z}$ is called the reproducing kernel for z_0 in H^2. It is well known that the linear span of the reproducing kernels $\{K_{z_0} : z_0 \in \mathbb{D}\}$ is dense in H^2. Cowen [4] gave an adjoint formula of a composition operator whose symbol is a linear fractional selfmap of $\mathbb{D}$. If $\varphi(z) = \frac{az+b}{cz+d}$ is a linear fractional selfmap of $\mathbb{D}$, then $C_\varphi^* = M_g C_\sigma M_h^*$, where $g(z) = \frac{1}{-\bar{b}z+\bar{d}}$, $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}$, and $h(z) = cz + d$. The function σ is called the Krein adjoint of φ, while g, h are called the Cowen auxillary functions of φ. It follows from [4] that σ is a self-map of $\mathbb{D}$ and $g \in H^\infty$. Note that

$$W_{\psi,\varphi}^* K_z = \overline{\psi(z)} K_{\varphi(z)} \tag{1}$$

when $W_{\psi,\varphi}$ is bounded on H^2 and $z \in \mathbb{D}$.

Let φ be an automorphism of $\mathbb{D}$ and let $\partial\mathbb{D}$ denote the unit circle. Then φ is of the form $\varphi(z) = \frac{az+\bar{b}}{bz+\bar{a}}$ for all $z \in \mathbb{D}$, where a and b in $\mathbb{C}$ with $|a|^2 - |b|^2 = 1$. When $b \neq 0$, it is easy to calculate that $\frac{iIm(a) \pm \sqrt{|b|^2-(Im(a))^2}}{b}$ are the fixed points of φ. If $|Im(a)| = |b|$, then φ is called parabolic, and we say that φ is hyperbolic if $|Im(a)| < |b|$. If

$|Im(a)| > |b|$, then φ is said to be elliptic. We note that φ is elliptic if and only if one of its fixed points is inside $\mathbb{D}$ and another is outside $\mathbb{D}$. In this sense, this type also includes the case when $b = 0$, i.e., when 0 and ∞ are the fixed points of φ. Remark that φ is parabolic if and only if it has only one fixed point lying on $\partial\mathbb{D}$, while φ is hyperbolic if and only if it has two fixed points lying on $\partial\mathbb{D}$.

Let φ be an analytic selfmap of $\mathbb{D}$. For each positive integer n, we write $\varphi_1 := \varphi$ and $\varphi_{n+1} := \varphi \circ \varphi_n$, which is called the iterate of φ for n. If φ is not an elliptic automorphism of $\mathbb{D}$, then for each $z \in \mathbb{D}$ there is a (unique) point w in the closure of $\mathbb{D}$ such that

$$w = \lim_{n\to\infty} \varphi_n(z).$$

The point w called the Denjoy-Wolff point of φ and characterized as follows: if $|w| < 1$, then $\varphi(w) = w$ and $|\varphi'(w)| < 1$; if $w \in \partial\mathbb{D}$, then $\varphi(w) = w$ and $0 < \varphi'(w) \leq 1$.

Let $d\theta$ denote the usual arc length measure on the unit circle $\partial\mathbb{D}$. For $h \in L^\infty(\partial\mathbb{D}, d\theta)$, the Toeplitz operator with symbol h, denoted T_h, is the operator on H^2 defined by $T_h(f) = P(hf)$, where P denotes the orthogonal projection of $L^2(\partial\mathbb{D}, d\theta)$ onto H^2.

In [7], Cowen and Ko characterized weighted composition operators on H^2. In [1], Bourdan and Narayan studied several properties of normal weighted composition operators on H^2. Normal and cohyponormal weighted composition studied in [6] by Cowen, Jung and Ko. In [10], Fatehi, Shaabani and Thompson has been studied quasinormal and hyponormal weighted composition operators on H^2 and Bergman space A^2_α with linear fractional compositional symbol. In this note, we focus on coposinormality of weighted composition operators on H^2.

2 Main Results

Throughout this section, $R(T)$ and $\ker(T)$ denote range and null space of $T \in B(\mathcal{H})$, respectively. Sadraoui [17, 18] studied hyponormality and cohyponormality of composition operators by using [8, Theorem 1]. The following result is due to [14].

Proposition 2.1. *([14, Theorem 2.1]) For $T \in B(\mathcal{H})$ the following statements are equivalent:*
(1) T is posinormal;
(2) $R(T) \subseteq R(T^*)$;
(3) $TT^* \leq \lambda^2 T^*T$ for some $\lambda \geq 0$; and
(4) there exists a $A \in B(\mathcal{H})$ such that $T = T^*A$.

Let T_h denotes the Toeplitz operator on H^2 with symbol h. Let φ is a linear fractional transformation that maps the disk into the disk with $\varphi(1) = 1$, $\varphi'(1) = s$. Sadraoui [17, 18] proved that $\varphi(z) = \frac{(1+r+s)z+1-r-s}{(1+r-s)z+1+s-r}$, where $Re(r) \leq 0$ and $0 < s < 1$.

If $\varphi(z) = \frac{(1+r+s/2\sqrt{s})z+(1-r-s)/2\sqrt{s}}{(1+r-s/2\sqrt{s})z+(1+s-r)/2\sqrt{s}} = \frac{az+b}{cz+d}$ and $\psi(z) = \frac{\overline{a}z-\overline{c}}{-\overline{b}z+\overline{d}}$, then $A = C_{\psi^{-1}\circ\varphi} T_{-\overline{b}z+\overline{d}} T_{\overline{z}} T_{\overline{a}z-\overline{c}}$ is an operator on H^2(see [17, Page 27]).

Theorem 2.2. *Let $\varphi(z) = \frac{az+b}{cz+d}$ and $\psi(z) = \frac{\overline{a}z-\overline{c}}{-\overline{b}z+\overline{d}}$ are as above. Then C_φ is coposinormal.*

Proof. By [18, Theorem 2.4.3], $C_\varphi = AC_\varphi^*$. Then by a similar argument as in the proof of [8, Theorem 1], we obtain that $C_\varphi^* C_\varphi = C_\varphi A^* A C_\varphi^* = ||A||^2 C_\varphi C_\varphi^* - C_\varphi(||A||^2 - A^*A)C_\varphi^* \leq ||A||^2 C_\varphi C_\varphi^*$. Hence the operator C_φ is coposinormal. □

The following example considered by Fatehi, Shaabani and Thompson [10] to prove hyponormality $T_\psi C_\varphi$. This is a narrowed example of what Sadraoui [18] proved in section 2.5. Now we consider this example for the study posinormality of weighted composition operator $W_{\psi,\varphi}$ on H^2.

Example 2.3. For $0 < s < 1$, let
$\psi(z) = \frac{1}{1-(1-s)z}$,
$\varphi(z) = \frac{sz}{1-(1-s)z}$,
$\tau(z) = \frac{sz+1-s}{sz(1-s)+1-s+s^2}$,
$\sigma(z) = sz + 1 - s$,
and $\eta(z) = \frac{s}{sz(1-s)+1-s+s^2}$ (see Example 3.6 of [10]). From [10], we have $C_\sigma = (T_\psi C_\varphi)^*$ and $C_\sigma = T_\eta C_\tau T_\psi C_\varphi$. Let $A = (T_\eta C_\tau)^*$. Then, $W_{\psi,\varphi}^* = (T_\psi C_\varphi)^* = C_\sigma = T_\eta C_\tau T_\psi C_\varphi = A^* W_{\psi,\varphi}$. Hence by Proposition 2.1, it follows that $W_{\psi,\varphi}$ is posinormal on H^2.

Now we expand on Sadraoui's example to construct weights f so that $W_{f\psi,\varphi}$ is posinormal on H^2.

Theorem 2.4. *Suppose $\varphi(z)$, $\psi(z)$, $\tau(z)$, $\sigma(z)$, and $\eta(z)$ are as in Example 2.3. Let f be such that $f, \frac{1}{f} \in H^\infty$. Suppose further that there exist $g \in H^\infty$ such that $g \circ \sigma = f$. Then $W_{f\psi,\varphi}$ is posinormal on H^2.*

Proof. Suppose $\psi(z) = \frac{1}{1-(1-s)z}$, $\varphi(z) = \frac{sz}{1-(1-s)z}$, $\tau(z) = \frac{sz+1-s}{sz(1-s)+1-s+s^2}$, $\sigma(z) = sz + 1 - s$, and $\eta(z) = \frac{s}{sz(1-s)+1-s+s^2}$. If $A = (T_\eta C_\tau T_g^* T_{\frac{1}{f}})^*$, then $A^* W_{f\psi,\varphi} = T_\eta C_\tau T_g^* T_{\frac{1}{f}} T_f T_\psi C_\varphi$. Since $g \circ \sigma = f$ and $C_\sigma^* = (T_\psi C_\varphi)$, it follows that $A^* W_{f\psi,\varphi} = C_\sigma T_f^* = (T_f C_\sigma^*)^* = W_{f\psi,\varphi}^*$. Therefore, $W_{f\psi,\varphi}$ is posinormal by Proposition 2.1. □

Example 2.5. Let $\varphi(z)$, $\psi(z)$, $\tau(z)$, $\sigma(z)$, and $\eta(z)$ are as in Example 2.3 with $s = \frac{1}{2}$. Take $f(z) = \frac{1}{z+2}$ and $g(z) = \frac{1}{2z+1}$. Then $W_{f\psi,\varphi}$ is posinormal. The inequality $||f(z)|| \geq ||g(z)||$ not true for all $z \in \mathbb{D}$ and so $W_{f\psi,\varphi}$ is not hyponormal by [10, Theorem 3.7].

Let φ be an analytic self-map of $\mathbb{D}$ and $\psi = K_a$ for some $a \in \mathbb{D}$. If $W_{\psi,\varphi}$ is hyponormal, then $|\varphi(0)| \leq |a|$ (see, [9, Proposition 3.4.]). Now we prove the following result in a similar manner for posinormal operator $W_{\psi,\varphi}$.

Theorem 2.6. *Let φ be an analytic self-map of $\mathbb{D}$ and $\psi = K_{z_0}$ for some $z_0 \in \mathbb{D}$. If $W_{\psi,\varphi}$ is posinormal, then $\lambda^2 - \lambda^2|\varphi(0)|^2 + |z_0|^2 - 1 \geq 0$ for some $\lambda \geq 0$.*

Proof. Suppose $W_{\psi,\varphi}$ is posinormal. Since $K_0 \equiv 1$, it follows that

$$\begin{aligned}
\frac{1}{1-|\varphi(0)|^2} &= \langle \overline{\psi(0)} K_{\varphi(0)}, \overline{\psi(0)} K_{\varphi(0)} \rangle \\
&= \langle W_{\psi,\varphi} W_{\psi,\varphi}^* K_0, K_0 \rangle \\
&\leq \lambda^2 \langle W_{\psi,\varphi}^* W_{\psi,\varphi} K_0, K_0 \rangle \\
&= \lambda^2 \langle \psi, \psi \rangle \\
&= \frac{\lambda^2}{1-|z_0|^2}.
\end{aligned}$$

This completes the proof. □

The following result is immediate.

Corollary 2.7. *Let φ be an analytic self-map of $\mathbb{D}$ and $\psi = K_{z_0}$ for some $z_0 \in \mathbb{D}$. If $W_{\psi,\varphi}$ is coposinormal, then $\lambda^2 - \lambda^2|z_0|^2 + |\varphi(0)|^2 - 1 \geq 0$ for some $\lambda \geq 0$.*

Let φ be a nonconstant analytic function on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and let $\psi \in H^\infty$ be not identically zero on $\mathbb{D}$. If $W_{\psi,\varphi}$ is cohyponormal on H^2, then ψ never vanishes on $\mathbb{D}$ and also φ is univalent(see [6]). Now we will prove similar result for coposinormal operators by the method of [1, Proposition 3],[6].

Lemma 2.8. *Let φ be a nonconstant analytic function on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and let $\psi \in H^\infty$ be not identically zero on $\mathbb{D}$. If $W_{\psi,\varphi}$ is coposinormal on H^2, then ψ never vanishes on $\mathbb{D}$.*

Proof. Suppose that $W_{\psi,\varphi}$ is coposinormal. Then from [14], we have $\ker(W^*_{\psi,\varphi}) \subseteq \ker(W_{\psi,\varphi})$. If $f \in \ker(W_{\psi,\varphi})$, then $\psi.f \circ \varphi \equiv 0$ on $\mathbb{D}$. Since ψ is not identically zero on $\mathbb{D}$ and φ is nonconstant analytic function on $\mathbb{D}$, by the open mapping theorem, we have $f \equiv 0$ on $\mathbb{D}$. Thus, $\ker(W_{\psi,\varphi}) = \{0\}$. Since $\ker(W^*_{\psi,\varphi}) \subseteq \ker(W_{\psi,\varphi}) = \{0\}$ and $\ker(M^*_\psi) \subseteq \ker(W^*_{\psi,\varphi})$, we have

$$\ker(M^*_\psi) = \{0\}.$$

Therefore by [2, Theorem 2.19] and [2, Corollary 2.10], $\overline{R(M_\psi)} = H^2$. Hence it follows that ψ is cyclic on H^2 and hence ψ is an outer function by [12, Corollary 1.5]. In particular, ψ never vanishes on $\mathbb{D}$. □

Theorem 2.9. *Let φ be a nonconstant analytic function on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and let $\psi \in H^\infty$ be not identically zero on $\mathbb{D}$. If $W_{\psi,\varphi}$ is coposinormal on H^2, then φ is univalent.*

Proof. Suppose that $W_{\psi,\varphi}$ is coposinormal. Assume that there are distinct points z_1 and z_2 in $\mathbb{D}$ such that $\varphi(z_1) = \varphi(z_2)$. From Lemma 2.8, we obtain that $\psi(z_1) \neq 0$ and $\psi(z_2) \neq 0$. Set $h = \frac{K_{z_1}}{\overline{\psi(z_1)}} - \frac{K_{z_2}}{\overline{\psi(z_2)}}$. Then h is a nonzero vector on $H^2(\mathbb{D})$. By equation (1), we obtain that $W^*_{\psi,\varphi}h \equiv 0$. Since $W^*_{\psi,\varphi}$ is posinormal on $H^2(\mathbb{D})$, it holds $W_{\psi,\varphi}h = 0$. That is $\psi(z)h(\varphi(z)) = 0$ for each $z \in \mathbb{D}$. Since ψ never vanish on $\mathbb{D}$ and since φ is nonconstant analytic function on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, by the open mapping theorem $h \equiv 0$ on $\mathbb{D}$, which is a contradiction. This completes the proof. □

Cowen, Jung, and Ko[6] has been studed cyclic and commutant of cohyponormal weighted composition operators. Now we extend these results to coposinormal weighted composition operators by the method of [6].

Theorem 2.10. *Let φ be a nonconstant analytic function on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, not an elliptic automorphism of $\mathbb{D}$, with $\varphi(w) = w$ for some $w \in \mathbb{D}$, and let $\psi \in H^\infty \setminus \{0\}$. If $W_{\psi,\varphi}$ is coposinormal on $H^2(\mathbb{D})$, then $W^*_{\psi,\varphi}$ is cyclic.*

Proof. Let $f \in H^2$ such that $f \perp \bigvee_{n=0}^{\infty} (W^*_{\psi,\varphi})^n K_{z_0}$ for an arbitrary point $z_0 \in \mathbb{D}$ not equal to w. By [20, Lemma 1] and [19, Section 5.2, Proposition 1], notice that the sequence $\{\varphi_n(z_0)\}_{n=0}^{\infty}$ consists of points in $\mathbb{D}$ which converges to w. We have

$$0 = \langle f, (W^*_{\psi,\varphi})^n K_{z_0} \rangle = \langle (W_{\psi,\varphi})^n f, K_{z_0} \rangle.$$

Since the equality $W^n_{\psi,\varphi} = W_{\psi.(\psi\circ\varphi).(\psi\circ\varphi_2)\ldots(\psi\circ\varphi_{n-1}),\varphi_n}$ holds for any positive integer n, the following equality

$$\psi(z_0)\psi(\varphi(z_0))\psi(\varphi_2(z_0))\ldots.\psi(\varphi_{n-1}(z_0))f(\varphi_n(z_0)) = 0$$

holds for any positive integer n. Since $W_{\psi,\varphi}$ is coposinormal, ψ never vanishes on $\mathbb{D}$ by Lemma 2.8. Hence we have $f(\varphi_n(z_0)) = 0$ for any positive integer n. Then by identity theorem we have $f \equiv 0$ and so $H^2 = \bigvee_{n=0}^{\infty} (W^*_{\psi,\varphi})^n K_{z_0}$. This completes the proof. □

Set of all operators which commute with a fixed operator T forms a weakly closed algebra which is called the commutant of T. Commutant of $T \in B(\mathcal{H})$ is denoted by $\{T\}'$. Next we study the commutant of a coposinormal weighted composition operators.

Theorem 2.11. *Let φ be a nonconstant analytic function on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ with the Denjoy-Wolff point $w \in \mathbb{D}$ and let $\psi \in H^\infty$ be not identically zero on $\mathbb{D}$. Suppose that $W_{\psi,\varphi}$ is coposinormal on $H^2(\mathbb{D})$. If ϕ is an analytic self map of $\mathbb{D}$ and $\tau \in H^\infty$ such that $\tau(w) \neq 0$ and $W_{\tau,\phi} \in \{W_{\psi,\varphi}\}'$, then w is a fixed point of ϕ.*

Proof. Suppose that $W_{\tau,\phi} \in \{W_{\psi,\varphi}\}'$. From the equalities $W^*_{\tau,\phi} W^*_{\psi,\varphi} K_w = \overline{\psi(w)\tau(\varphi(w))} K_{\phi(\varphi(w))}$ and $W^*_{\psi,\varphi} W^*_{\tau,\phi} K_w = \overline{\tau(w)\psi(\phi(w))} K_{\varphi(\phi(w))}$, it follows that

$$\overline{\psi(w)\tau(\varphi(w))}K_{\phi(\varphi(w))} = \overline{\tau(w)\psi(\phi(w))}K_{\varphi(\phi(w))}.$$

Since $\varphi(w) = w$ and $\tau(w) \neq 0$ for $w \in \mathbb{D}$, we obtain that

$$\overline{\psi(w)}K_{\phi(w)} = \overline{\psi(\phi(w))}K_{\varphi(\phi(w))}.$$

Thus, for all $z \in \mathbb{D}$, we have

$$\overline{\psi(w)} - \overline{\psi(w)\varphi(\phi(w))}z = \overline{\psi(\phi(w))} - \overline{\psi(\phi(w))\phi(w)}z.$$

Thus, $\psi(w) = \psi(\phi(w))$ and $\psi(w)\varphi(\phi(w)) = \psi(\phi(w))\varphi(\phi(w))$. From these equalities, we have $\psi(w)\varphi(\phi(w)) = \psi(w)\phi(w)$. Since $W_{\psi,\varphi}$ is coposinormal on $H^2(\mathbb{D})$, ψ never vanishes on $\mathbb{D}$ by Lemma 2.8. Hence, $\varphi(\phi(w)) = \phi(w) \in \mathbb{D}$. Now the iterates φ_n converges uniformly to w and φ_n converges uniformly to $\phi(w)$ by the Denjoy-Wolff Theorem and hence w is a fixed point of ϕ. □

Corollary 2.12. *Let φ be a nonconstant analytic function on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ with the Denjoy Wolff point $w \in D$ and let $\psi \in H^\infty$ be not identically zero on D. Suppose that $W_{\psi,\varphi}$ is coposinormal on $H^2(\mathbb{D})$. If ϕ is an analytic self map of $\mathbb{D}$ and $\tau \in H^\infty$ such that $\tau(z) \neq 0$ and $W_{\tau,\phi} \in \{W_{\psi,\varphi}\}'$, then $\{f \in H^2 : f(w) = 0\}$ is an invariant subspace for $W_{\tau,\phi}$.*

Proof. From Theorem 2.11, we have $\phi(w) = w$. Thus for all $f \in H^2$, $(W_{\tau,\phi}f)(w) = \tau(w)f(w)$. Hence, $\{f \in H^2 : f(w) = 0\}$ is a invariant subspace for $W_{\tau,\phi}$. □

Acknowledgements
The author would like to express sincere thanks to the referee for the helpful comments and suggestions.

References

[1] P. Bourdan and S. K. Narayan, Normal weighted composition operator on the Hardy space $H^2(D)$, *J. Math. Anal. Appl.*, 367(2010), 278-286.

[2] J. B. Conway, *A Course in Functional Analysis*,, Second Edition, Springer-Verlag, New York, 1990.

[3] C.C. Cowen, Composition operators on H^2, *J. Operator Theory.*, 9(1983), 77-106.

[4] C.C. Cowen, Linear fractional composition operator on H^2, *Integral Equations Operator Theory.*, 11(1988), 151-160.

[5] C.C. Cowen, B.D. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, 1995.

[6] C. C. Cowen, S. Jung, and E. Ko, Normal and cohyponormal weighted composition operators, *Operator Theory: Adv and Appl.*, 240(2014), 69-85.

[7] C.C. Cowen and E. Ko, Hermitian weighted composition operators on H^2, *Trans. Amer. Math. Soc.*, 362(2010), 5771-5805.

[8] R. G. Douglas, On Majorization, Factorization, and Range Inclusion of Operators on Hilbert Spaces, *Proc. Amer. Math. Soc.*, 17 (1966), 413-415.

[9] M. Fatehi, M. Shaabani, Normal, cohyponormal and normaloid weighted composition operators on the Hardy and weighted Bergman spaces, *J. Korean Math. Soc.*, 54 (2017), 599-612.

[10] M. Fatehi, M. Shaabani and D.Thompson, Quasinormal and hyponormal weighted composition operators on H^2 and A^2_α a with linear fractional compositional symbol, *Complex Anal. Oper. Theory.*, 12(2018), 1767-1778 .

[11] G. Gunatillake , *Weighted composition operators*, Ph. D Thesis, Purde Univ, 1992.

[12] A. Hanine, *Cyclic vectors in some spaces of analytic functions*, PhD diss., Aix-Marseille, 2013.

[13] T. Le and H. C. Rhaly, Coposinormality of the Cesàro matrices, *Arch. Math.*, 110(2018), 167-173.

[14] H. C. Rhaly, Jr, Posinormal operators, *J. Math. Soc. Japan.*, 46 (1994), 587-605.

[15] H. C. Rhaly, Jr, A Comment on coposinormal operators, *Le Mathematiche.*, 68(2013) 83-86.

[16] H. C. Rhaly Jr. and B. E. Rhoades, The weighted mean operator on $l^{\prime 2}$ with the weighted sequence $w_n = n + 1$ is hyponormal, *New Zeland Journal of Mathematics.*, 44 (2014), 103-106.

[17] H. Sadraoui, On cohyponormality of composition operators and hyponormality of Toeplitz operators, *ICASTOR Journal of Mathematical Sciences.*, 10 (2016) 27-33.

[18] H. Sadraoui, *Hyponormality of Toeplitz and Composition Operators*, Thesis, Purdue University, 1992.

[19] J. H. Shapiro, *Composition operators and classical function theory*, Springer, 1993.

[20] T. Worner, Commutents of certain composition operators, *Acta. Sci. Math(Szeged).*, 68(2002), 413-432.

Thankarajan Prasad
Assistant Professor in Mathematics
Department of Mathematics
Cochin University of Science and Technology
Cochin-682022, Kerala, India.
E-mail: prasadvalapil@gmail.com