Journal of Mathematical Extension
Vol. 6, No. 4, (2012), 11-19

Filter Regular Sequence and Generalized Local Cohomology with Respect to a Pair of Ideals

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Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Two notions of filter regular sequence and generalized local cohomology module with respect to a pair of ideals are introduced, and their properties are studied. Some vanishing and non-vanishing theorems are given for this generalized version of generalized local cohomology module.

AMS Subject Classification: 13D45; 13D07
Keywords and Phrases: Generalized local cohomology, filter regular sequence

1. Introduction

Throughout this paper, let R be a commutative Noetherian ring and I, J two ideals of R. Let M and N be two R-modules. For notations and terminologies not given in this paper, the reader is referred to [3], [4] and [7] if, necessary.
As a generalization of the usual local cohomology modules, in [7], the authors introduced the local cohomology modules with respect to a pair of ideals (I, J). To be more precise, let $W(I, J) = \{\mathfrak{p} \in \text{spec}(R) | I^n \subseteq \mathfrak{p} + J$ for some positive integer $n\}$. For an R-module M, the (I, J)-torsion submodule $\Gamma_{I,J}(M)$ of M, which consists of all elements x of M with $\text{Supp}(Rx) \subseteq W(I, J)$, is considered. Let i be an integer, the local cohomology functor $H^i_{I,J}$ with respect to (I, J) is defined to be the i-th right derived functor of $\Gamma_{I,J}$.

Received: January 2012; Accepted: October 2012

In this paper, we introduce a generalization of the notion of generalized local cohomology module, which we call a generalized local cohomology module with respect to a pair of ideals (I, J). Let $\widetilde{W}(I, J)$ denote the set of ideals $\mathfrak{a}$ of R such that $I^n \subseteq \mathfrak{a} + J$ for some integer n. For each integer $i \geqslant 0$, we define the functor $H^i_{I,J}(-,-) : \xi_R \times \xi_R \longrightarrow \xi_R$ by $H^i_{I,J}(M, N) = \varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} Ext^i_R(\frac{M}{aM}, N), M, N \in \xi_R$ (where ξ_R denotes the category of all R-modules and all R-homomorphisms). Then $H^i_{I,J}(-,-)$ is an additive, R-linear functor which is contravariant in the first variable and covariant in the second variable. This functor do indeed generalize all the functors described in [5], [6] and [7]. One of our main goals is to give criteria for the vanishing and non-vanishing of $H^i_{I,J}(M, N)$ by using (I, J)-grade$_N M$.
The organization of this paper is as follows.
We introduce the notion of filter regular sequence with respect to a pair of ideals (I, J). Some their characterizations are presented in Section 2. In Section 3, we define a generalization of generalized local cohomology modules and their basic properties are studied. In the final section we discuss the vanishing and non-vanishing of generalized local cohomology with respect to (I, J) by using the length of filter regular sequence with respect to (I, J).

2. Regular Sequences with Respect to Pair of Ideals

Throughout this note R is a Noetherian ring and I, J are two ideals of R and M is a finitely generated R-module. Let $W(I, J)$ denote the set of prime ideals $\mathfrak{p}$ of R such that $I^n \subseteq J + \mathfrak{p}$ for some integer n.

Definition 2.1. *Let $x_1, x_2, ..., x_n$ be a sequence of R. We say that $x_1, \ldots, x_t$ is an M-filter regular sequence with respect to (I, J) if and only if, $Supp(\frac{(x_1,\ldots,x_{i-1})M:x_i}{(x_1,\ldots,x_{i-1})M}) \subseteq W(I, J)$ for all $i = 1, \ldots, t$.*
Note that as a special case of the notion, if $J = 0$ then $x_1, x_2, \ldots, x_t$ is called an I-filter regular sequence with respect to M in sense of [2].
The following theorem gives an equivalent condition for the existence of M-filter regular sequence with respect to (I, J).

Theorem 2.2. *Let M be a finitely generated module over a local ring R with maximal ideal $\mathfrak{m}$. Then the following conditions are equivalent:*

(i) $x_1, x_2, \ldots, x_t$ *is M-filter regular sequence with respect to (I, J);*

(ii) $x_i \notin \bigcup \mathfrak{p}_{\mathfrak{p} \in Ass\frac{M}{(x_1,x_2,,x_{i-1})M} - W(I,J)}$ *for $i = 1, \ldots, t$;*

(iii) $\frac{x_1}{1}, \frac{x_2}{1}, \ldots, \frac{x_t}{1}$ *is a poor $M_{\mathfrak{p}}$-sequence for all $\mathfrak{p} \in SuppM - W(I.J)$;*

(iv) *For all $i = 1, \ldots, t$, $x_1, x_2, \ldots, x_i$ is M-filter regular sequence with respect to (I, J) and $x_{i+1}, x_{i+2}, \ldots, x_t$ is $\frac{M}{(x_1,x_2,\ldots,x_i)M}$-filter regular sequence with respect to (I, J).*

Proof. $ii \Longrightarrow i$: Suppose the contrary and let $1 \leqslant i \leqslant n$ be such that $W(I, J) \not\supseteq \mathrm{Supp}(\frac{(x_1,\ldots,x_{i-1})M:_M x_i}{(x_1,\ldots,x_{i-1})M})$.
Then there is $\mathfrak{q} \in \mathrm{Supp}(\frac{(x_1,\ldots,x_{i-1})M:_M x_i}{(x_1,\ldots,x_{i-1})M}) - W(I, J)$. Thus there exist $\mathfrak{p} \subseteq \mathfrak{q}$, which $\mathfrak{p} \in \mathrm{Ass}(\frac{(x_1,\ldots,x_{i-1})M:_M x_i}{(x_1,\ldots,x_{i-1})M})$. Then there is $m \in (x_1, ..., x_{i-1})M :_M x_i$ such that $0 : m + (x_1, \ldots, x_{i-1})M = \mathfrak{p}$. Therefore

$$x_i \in \mathfrak{p} \subseteq \bigcup \mathfrak{q}_{\mathfrak{q} \in \mathrm{Ass}\frac{M}{(x_1,x_2,\ldots,x_i)M} - W(I,J)}.$$

This is contradiction and the proof is completed.

$i \Longrightarrow ii$: Suppose that contrary. Let $1 \leqslant i \leqslant t$ be such that

$$x_i \in \bigcup \mathfrak{p}_{\mathfrak{p} \in \mathrm{Ass}\frac{M}{(x_1,x_2,\ldots,x_i)M} - W(I,J)}.$$

Then there is $x_i \in \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}(\frac{M}{(x_1,x_2,\ldots,x_i)M}) - W(I, J)$.
Thus $\mathfrak{p} = (0 : (x_1, \ldots, x_{i-1})M + m)$ for some $m \in M$. So, $\mathfrak{p} \in \mathrm{Ass}(\frac{(x_1,\ldots,x_{i-1})M:_M x_i}{(x_1,\ldots,x_{i-1})M}) - W(I, J)$. This is a contradiction. Therefore $x_i \notin \bigcup \mathfrak{p}_{\mathfrak{p} \in \mathrm{Ass}\frac{M}{(x_1,x_2,\ldots,x_i)M} - W(I,J)}$ for all $i = 1, \ldots, t$ and the proof is completed.

$iii \Longrightarrow i$: Let $\mathrm{Supp}(\frac{(x_1,\ldots,x_{i-1})M:_M x_i}{(x_1,\ldots,x_{i-1})M}) \nsubseteq W(I, J)$, then there is $p \in \mathrm{Supp}(\frac{(x_1,\ldots,x_{i-1})M:_M x_i}{(x_1,\ldots,x_{i-1})M}) - W(I, J)$, hence $\mathfrak{p} \in \mathrm{Supp}(M) - W(I, J)$, it follows from (iii) that $(\frac{x_1}{1}, \ldots, \frac{x_{i-1}}{1})M_{\mathfrak{p}} = (\frac{x_1}{1}, \frac{x_2}{1}, \ldots, \frac{x_{i-1}}{1})M_{\mathfrak{p}} : \frac{x_i}{1}$. Thus

$\mathfrak{p} \notin \mathrm{Supp}(\frac{(x_1,\dots,x_{i-1}M:_M x_i}{(x_1,\dots,x_{i-1})M})$, which is a contradiction.
The equivalence of (iii) and (iv), and (i)$\Longrightarrow$(iii) are clear. $\square$

Remark 2.3. (i) *Let R be Noetherian let $\mathfrak{a}$ be an arbitrary ideal of $\widetilde{W}(I,J)$ and Supp$\frac{M}{\mathfrak{a}M} \nsubseteq W(I,J)$, it is straightforward to see that, any two maximal M-filter regular sequence with respect to (I,J) in $\mathfrak{a}$ have the same length. We denote the length of a maximal M-filter regular sequence with respect to (I,J) in $\mathfrak{a}$ by $g(\mathfrak{a},M)$.*
(ii) *Let $\widetilde{W}(I,J)$ denote the set of ideals $\mathfrak{a}$ of R such that $I^n \subseteq \mathfrak{a}+J$ for some integer n. We define a partial order on $\widetilde{W}(I,J)$ by letting $\mathfrak{a} \leqslant \mathfrak{b}$ if $\mathfrak{a} \supseteq \mathfrak{b}$ for $\mathfrak{a},\mathfrak{b} \in \widetilde{W}(I,J)$. $\widetilde{W}(I,J)$ is non-empty. We shall apply Zorn׳s lemma to this partially ordered set. Let φ be a non-empty totally ordered subset of $\widetilde{W}(I,J)$. Then $\cap_{\mathfrak{a}_i\in\varphi}\mathfrak{a}_i$ is in $\widetilde{W}(I,J)$. Thus J is an upper bound for φ in $\widetilde{W}(I,J)$, and so it follows from Zorn׳s lemma that $\widetilde{W}(I,J)$ has at least one maximal element.*

Definition 2.4. *We use the notation $g((I,J),M)$ to denote the length of a maximal M-filter regular sequence with respect to (I,J), as $g((I,J),M)$ $=$ inf$\{g(\mathfrak{a},M) \mid \mathfrak{a} \in \widetilde{W}(I,J)\} =$ inf$\{g(\mathfrak{a},M) \mid \mathfrak{a}$ is maximal element of directed set $\widetilde{W}(I,J)\}$.*
As an important special case of the previous remark we have, if Supp$(\frac{(x_1,\dots,x_{i-1})M:_M x_i}{(x_1,\dots,x_{i-1})M}) = \varnothing$, then $x_1,x_2,\dots,x_{i-1},x_i$ is poor M-regular sequence with respect to (I,J) and if, in addition, $(x_1,\dots,x_t)M \neq M$, we call $x_1,\dots,x_t$ an M-regular sequence.

Remark 2.5. *Let R be a Noetherian ring, M a finitely generated R-module, and $\mathfrak{a}$ an ideal such that $\mathfrak{a}M \neq M$. Then all maximal M-regular sequence in $\mathfrak{a}$ have the same length and the common length of the maximal M-regular sequence in $\mathfrak{a}$ called the grade of $\mathfrak{a}$ on N, denoted by grade$(\mathfrak{a},M)$.*

Definition 2.6. *Suppose that M is finitely generated R-module and that I and J are ideals of R. We define the grade of (I,J) on M, denoted by grade$((I,J),M)$, as grade$((I,J),M) =$ inf$\{$grade$(\mathfrak{a},M)|\mathfrak{a} \in \widetilde{W}(I,J)\} =$ inf$\{$grade$(\mathfrak{a},M)|\mathfrak{a}$ is maximal element of directed set$\widetilde{W}(I,J)\}$.*

3. Generalized Local Cohomology Modules Defined by a Pair of Ideals

In the present section, we recall definition and basic properties of generalized local cohomology modules defined by a pair of ideals that we shall use.

Let M and N be finitely generated R-module over a local ring $(R, \mathfrak{m})$ and let I and J be two ideals of R. For each integer $i \geqslant 0$, we define the $H^i_{I,J}(-,-) : \xi_R \times \xi_R \longrightarrow \xi_R$ by $H^i_{I,J}(-,-) : \xi_R \times \xi_R \longrightarrow \xi_R$ by $H^i_{I,J}(M,N) = \varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} \operatorname{Ext}^i_R(\frac{M}{aM}, N), M, N \in \xi_R$. Then $H^i_{I,J}(-,-)$ is an additive, R-linear functor which is contravariant in the first variable and covariant in the second variable.

Theorem 3.1. *Let M be a fixed R-module. Then, for each $i \geqslant 0$, the functors $\varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} Ext^i_R(\frac{M}{aM}, -)$ and $\varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H^i_{\mathfrak{a}}(M, -)$ (from ξ_R to ξ_R) are naturally equivalent.*

Proof. We must first explain the construction of the functor $\varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H^i_{\mathfrak{a}}(M, -)$. Let $\mathfrak{a}, \mathfrak{b} \in \widetilde{W}(I,J)$ with $\mathfrak{a} \leqslant \mathfrak{b}$ ($\mathfrak{a} \supseteq \mathfrak{b}$). Also, let $n \geqslant 1$ be an integer. Then the natural homomorphism $\frac{M}{\mathfrak{b}^n M} \longrightarrow \frac{M}{\mathfrak{a}^n M}$ induces the homomorphism $\operatorname{Ext}^i_R(\frac{M}{\mathfrak{b}^n M}, N) \longrightarrow \operatorname{Ext}^i_R(\frac{M}{\mathfrak{a}^n M}, N)$ for any integer $i \geqslant 0$ and any R-module N. Also, if $n \leqslant m$, then the diagram

$$\begin{array}{ccc} \operatorname{Ext}^i_R(\frac{M}{\mathfrak{a}^n M}, N) & \longrightarrow & \operatorname{Ext}^i_R(\frac{M}{\mathfrak{b}^n M}, N) \\ \downarrow & & \downarrow \\ \operatorname{Ext}^i_R(\frac{M}{\mathfrak{a}^m M}, N) & \longrightarrow & \operatorname{Ext}^i_R(\frac{M}{\mathfrak{b}^m M}, N) \end{array}$$

commutes. Thus we have a homomorphism $\Pi^{\mathfrak{b}}_{\mathfrak{a}} : \varinjlim_n \operatorname{Ext}^i_R(\frac{M}{\mathfrak{a}^n M}, N) \longrightarrow \varinjlim_n \operatorname{Ext}^i_R(\frac{M}{\mathfrak{b}^n M}, N)$, that is $\Pi^{\mathfrak{b}}_{\mathfrak{a}} : H^i_{\mathfrak{a}}(M,N) \longrightarrow H^i_{\mathfrak{b}}(M,N)$.

It is easy to see that these homomorphisms together with the modules $H^i_{\mathfrak{a}}(M,N)$ form a direct system of R-modules and R-homomorphisms over the directed set $\widetilde{W}(I,J)$.

Since $\varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H^0_{\mathfrak{a}}(M,-)$ and $\varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} \operatorname{Hom}_R(\frac{M}{\mathfrak{a}M}, N)$ are naturally equivalent functors (from ξ_R, to ξ_R) and the sequences $\varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H^i_{\mathfrak{a}}(M,-)_{i\in\mathbb{Z}}$ and $\varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} \operatorname{Ext}^i_R(\frac{M}{\mathfrak{a}M}, -)_{i\in\mathbb{Z}}$ are negative

strongly connected sequences of functors, these two sequences are isomorphic.
In particular $\varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H^i_{\mathfrak{a}}(M,N) \cong \varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} \operatorname{Ext}^i_R(\frac{M}{\mathfrak{a}M},N) \cong H^i_{I,J}(M,N)$ for any integer $i \geqslant 0$ and any R-module N. □

In this part, we investigate some basic properties of generalized local cohomology modules defined by a pair of ideals. We first write a remark.

Remark 3.2. (i) *For an R-module M, we denote by $\Gamma_{I,J}(M)$ the set of elements x of M such that $I^n x \subseteq Jx$ for some integer n.*
(ii) *We say that M is (I,J)-torsion (respectively (I,J)-torsion-free) precisely when $\Gamma_{I,J}(M) = M$ (respectively $\Gamma_{I,J}(M) = 0$). It is clear that if $M = R$, then $H^i_{I,J}(M,N)$ is converted to $H^i_{I,J}(N)$. In addition, $H^i_{I,J}(N)$ coincides with $H^i_I(N)$ with the support in the closed subset $V(I)$ if $J = 0$.*

Lemma 3.3. Let M and N be finitely generated R-modules. Then

(i) *$\mathrm{Supp}N \subseteq W(I,J)$ if and only if $\Gamma_{I,J}(N) = N$.*

(ii) *$H^0_{I,J}(M,N) = \mathrm{Hom}(M,\Gamma_{I,J}(N))$.*

(iii) *If $\mathrm{Supp}M \cap \mathrm{Supp}N \subseteq W(I,J)$, then $H^i_{I,J}(M,N) = \mathrm{Ext}^i_R(M,N)$.*

Proof. (i) This is immediate by [7, 1.8].
(ii) $H^0_{I,J}(M,N) = \varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H^0_{\mathfrak{a}}(M,N) = \varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} \operatorname{Hom}(M,\Gamma_{\mathfrak{a}}(N)) = \operatorname{Hom}(M, \varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} \Gamma_{\mathfrak{a}}(N)) = \operatorname{Hom}(M,\Gamma_{I,J}(N))$.
(iii) There is a minimal injective resolution E^* of N such that $\operatorname{Supp}(E^i) \subseteq \operatorname{Supp}N$ for all $i \geqslant 0$. Since $\operatorname{Supp}(\operatorname{Hom}(M,E^i)) \subseteq \operatorname{Supp}M \cap \operatorname{Supp}N \subseteq W(I,J)$, so $\operatorname{Hom}(M,E^i)$ is (I,J)-torsion. Therefore, for all $i \geqslant 0$, $H^i_{I,J}(M,N) \cong \varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H^i_{\mathfrak{a}}(M,N) \cong \varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H^i\Gamma_{\mathfrak{a}}(\operatorname{Hom}(M,E^*)) \cong H^i \varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} \Gamma_{\mathfrak{a}}(\operatorname{Hom}(M,E^*)) \cong H^i\Gamma_{I,J}(\operatorname{Hom}(M,E^*)) \cong H^i\operatorname{Hom}(M,E^*) \cong Ext^i_R(M,N)$.
It is obvious that if $J = 0$, then $H^i_{I,J}(M,N)$ coincides with the generalized local cohomology module was introduced by Herzog in [6]. On the other hand, if J contains I then it is easy to see that $\Gamma_{I,J}(N) = N$ and $H^i_{I,J}(M,N) = \operatorname{Ext}^i_R(M,N)$. □

4. Vanishing and Non-Vanishing of $H^i_{I,J}(M,N)$

Lemma 4.1. *Suppose that I and J are ideals of R, M a non-zero finitely generated R-module of finite projective dimension, and N an R-module of finite krull dimension. Then $H^i_{I,J}(M,N)=0$ for all $i > pd(M)+dim(N)$.*

Proof. Suppose $\mathfrak{a} \in \widetilde{W}(I,J)$. Then, in view of [1], $H^i_{\mathfrak{a}}(M,N)=0$ for all $i > \mathrm{pd}(M)+\dim(N)$. The claim now follows immediately from Theorem 3.1. □

Remark 4.2. *Suppose that M and N are finitely generated R-modules and that $(0:M)N \neq N(M \otimes N \neq 0)$. Recall that the N-grade of M written $grade_N M$, is the length of any maximal N-sequence contained in $(0:M)$. Then $grade_N M$ is equal to the least integer r such that $Ext^r_R(M,N) \neq 0$.*
For any ideal $\mathfrak{a}$ of R for which $\mathfrak{a}N \neq N$, we define the grade of $\mathfrak{a}$ on N as $grade_N \frac{R}{\mathfrak{a}}$ $(grade(\mathfrak{a},N))$ in sense of Remark 2.5.

Definition 4.3. *Let I and J ideals of R, M and N finitely generated R-modules. We define N-grade of M with respect to $(I.J)$, denoted by (I,J)-$grade_N M$, as (I,J)-$grade_N M = inf\{grade_N \frac{M}{\mathfrak{a}M} | \mathfrak{a} \in \widetilde{W}(I,J)\} = inf\{grade_N \frac{M}{\mathfrak{a}M} | \mathfrak{a}$ is maximal element of directed set $\widetilde{W}(I,J)\}$.*

Note: If every $\mathfrak{a} \in \widetilde{W}(I,J)$, $\frac{M}{\mathfrak{a}M} \otimes N = 0$, then (I,J)-$\mathrm{grade}_N M = \infty$, otherwise we have (I,J)-$\mathrm{grade}_N M < \infty$.

Theorem 4.4. *Suppose that M and N are finitely generated R-modules and that I and J are ideals of R. Also, let (I,J)-$grade_N M = t < \infty$. Then $H^i_{I,J}(M,N)=0$ for all $i<t$ and $H^t_{I,J}(M,N) \neq 0$.*

Proof. By Theorem 3.1, $H^i_{I,J}(M,N) \cong \varinjlim_{\mathfrak{a} \in \widetilde{W}(I,J)} \mathrm{Ext}^i_R(\frac{M}{\mathfrak{a}M},N)$ for all i. Let $i<t$. Then $i < \mathrm{grade}_N \frac{M}{\mathfrak{a}M}$ for all $\mathfrak{a} \in \widetilde{W}(I,J)$. This implies that $H^i_{I,J}(M,N)=0$. Next there is an ideal, b say, in $\widetilde{W}(I,J)$ for which $\mathrm{grade}_N \frac{M}{\mathfrak{b}M} = t$. Let $\mathfrak{a} \in \widetilde{W}(I,J)$ be such that $\mathfrak{b} \leqslant \mathfrak{a} (\mathfrak{a} \subseteq \mathfrak{b})$. Since $\mathrm{grade}_N \frac{M}{\mathfrak{a}M} \geqslant t$, there is an N-sequence $x_1, x_2, \ldots, x_t$ which is contained in ann$\frac{M}{\mathfrak{a}M}$. Consider the natural epimorphism $\varphi : \frac{M}{\mathfrak{a}M} \longrightarrow \frac{M}{\mathfrak{b}M}$. Let $A = \ker\varphi$ so that the sequence $0 \longrightarrow A \longrightarrow \frac{M}{\mathfrak{a}M} \longrightarrow \frac{M}{\mathfrak{b}M} \longrightarrow 0$ is exact. This induces the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_R^{t-1}(A, N) \longrightarrow \mathrm{Ext}_R^t(\frac{M}{\mathfrak{b}M}, N) \longrightarrow \mathrm{Ext}_R^t(\frac{M}{\mathfrak{a}M}, N).$$

It is clear that $(0 : \frac{M}{\mathfrak{a}M}) \subseteq (0 : A)$, and hence $x_1, x_2, \ldots, x_t$ is an N-sequence contained in $(0 : A)$. Thus $\mathrm{Ext}_R^{t-1}(A, N) = 0$. Therefore for every $\mathfrak{a}$ in $\widetilde{W}(I, J)$ with $\mathfrak{b} \leqslant \mathfrak{a}$, the map $\mathrm{Ext}_R^t(\frac{M}{\mathfrak{b}M}, N) \longrightarrow \mathrm{Ext}_R^t(\frac{M}{\mathfrak{a}M}, N)$ is monomorphism. Since $\mathrm{Ext}_R^t(\frac{M}{\mathfrak{b}M}, N) \neq 0$, it follows that $\varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} \mathrm{Ext}_R^i(\frac{M}{\mathfrak{a}M}, N) \neq 0$ and the proof is completed. □

Corollary 4.5. *Suppose that N is finitely generated R-module and that I and J are ideals of R. Then $inf\{i|H^i_{I,J}(N) \neq 0\} = inf\{depthN_{\mathfrak{p}}|\mathfrak{p} \in W(I, J)\}$.*

Proof. By Theorem 4.3, $\inf\{i|H^i_{i,J}(N) \neq 0\} = \mathrm{grade}((I, J), N)$. It is clear from the definition that $\mathrm{grade}((I, J), N) \leqslant \mathrm{grade}(\mathfrak{p}, M)$ for all $\mathfrak{p} \in W(I, J)$, and it follows from Theorem 2.2 that $\mathrm{grade}(\mathfrak{p}, N) \leqslant \mathrm{depth}N_{\mathfrak{p}}$. Furthermore, if $\mathrm{grade}((I, J), N) = \infty$, then $\mathfrak{a}N = N$ for all $\mathfrak{a} \in \widetilde{W}(I, J)$, so that $\mathrm{depth}M_{\mathfrak{p}} = \infty$ for all $p \in W(I, J)$. Thus suppose $N \neq \mathfrak{a}N$ for some $\mathfrak{a} \in \widetilde{W}(I, J)$ and choose a maximal N-filter regular sequence x in $\mathfrak{a}$. By Theorem 2.2, there exists $\mathfrak{p} \in \mathrm{Ass}\frac{M}{xM} - W(I, J)$, and $\mathfrak{a} \subseteq p$. Now Since $\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Ass}(\frac{M}{xM})_{\mathfrak{p}}$, it follows that the $\mathfrak{p}R_{\mathfrak{p}}$ consists of zero-divisors of $\frac{M_{\mathfrak{p}}}{xM_{\mathfrak{p}}}$. Therefore x is a maximal $M_{\mathfrak{p}}$-sequence, as required. □

This result coincides with [7 ,Theorem 4.1].

Acknowledgment
The author would like to thank the referees for their careful reading and many suggestions on the paper.

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